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INEQUALITIES ON TIME SCALES VIA DIAMOND-F_h INTEGRAL

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Abstract. In this article, new improvements and generalizations of Hermite-Hadamard integral inequalities for F_h-convex functions on time scales are established. In addition, examples and applications are given to further support the results.

Keywords: F_h-convex; Hermite-Hadamard; time scales; dynamic model.

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1. Introduction

Inequalities play important roles in many areas of Mathematics, of interest is in the theory of differential and integral equations, as well as difference equations. They form basic tools for constructing analytic proofs of many Mathematical Theorems. Of particular important inequalities are Cauchy-Schwartz's Inequality, Hölder's Inequality, Minkowski's inequality and Jensen's Inequality for convex functions. These inequalities are simple and flexible to be applicable in various Mathematical settings (see Mitrinovic, Pecaric and Fink [12]). For example, the discrete version of Jensen's inequality

$$f\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i x_i\right) \leq \frac{1}{P_n}\sum_{i=1}^{n} p_i f(x_i), \tag{1.1}$$

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provided $f:[a,b]\to R$ is a convex function on $[a,b]$, $x_i\in[a,b]$, and $p_i\geq 0$ with $P_n=\sum_{i=1}^{n}p_i>0$, was extended and generalized to the class of ϕ_{h-s} convex functions on classical intervals by Olanipekun, Mogbademu and Omotoyinbo [13]. Some applications to Schurs inequality were also established. A result of Olanipekun, Mogbademu and Omotoyinbo [13] is given as follows.

Theorem 1.1 Olanipekun, Mogbademu and Omotoyinbo [13] Let $f\in SX(\phi_{h-s},I)$ and $\sum_{i=1}^{n}t_i=T_n=1,\ t_i\in(0,1),\ i=1,2,...,n$. Then

$$f\left(\sum_{i=1}^{n}t_i\phi(x_i)\right)\leq\left[\frac{h(t_i)}{t_i}\right]^{-s}\sum_{i=1}^{n}f(\phi(x_i)).$$

By setting $h(t)=1$ and $s=1$ in Theorem 1.1, we recover (1.1).

Over the years, many authors have studied and obtained generalizations, variations, refinements and applications of these inequalities for functions convex on time scales (see Agarwal, Bohner and Peterson [1], Ammi, Ferreica and Torres [2], Ferreica, Ammi and Torres [11], Özkan, Sarikaya and Yildirim [14], Tuna and Kutukcu [16], Wong, Yeh, Yu and Hong [17], Wong, Yeh and Lian [18]).

First, we present an introduction to the concept of time scales theory. Time scale $\mathbb{T}$ is any arbitrary closed subset of $\mathbb{R}$ and time scale calculus gives unification and extension of classical results. For example, when $\mathbb{T}=\mathbb{R}$, the time scale yields results containing ordinary derivative or integral and when $\mathbb{T}=\mathbb{Z}$, the time scale integral becomes sum. For a detailed introduction to the concept of time scales, see Bohner and Peterson [3].

The theory of time scales was introduced to unify and extend traditional concepts of the theory of difference and differential equations Agarwal, Bohner and Peterson [1], Bohner and Peterson [3]. It has undergone tremendous expansions and developments on various aspects by several authors over the past three decades (see Agarwal, Bohner and Peterson [1]-Ferreica, Ammi and Torres [11], Özkan, Sarikaya and Yildirim [14]-Wong, Yeh and Lian [18]).

In the theory of time scales, the concepts of the delta(Δ) and nabla(∇) calculus with applications have been introduced and employed by several authors to establish variants of the classical Hölder's inequality, Minkowski's Inequality and Jensen's Inequality for convex functions on time scales (see Agarwal, Bohner and Peterson [1], Özkan, Sarikaya and Yildirim [14], Wong, Yeh, Yu and Hong [17], Wong, Yeh and Lian [18]).

Likewise, the diamond-α combined dynamic calculus on time scales, which is essentially a convex combination of the delta and nabla calculi, was developed, see Sheng, Fadag, Henderson and Davis [15]. This new combined dynamic calculus has generated a lot of interest among mathematicians, particularly in the study of the theory of certain integral inequalities for convex functions on time scales. We present some results of Ammi, Ferreica and Torres [2] and Ferreica, Ammi and Torres [11].

Theorem 1.2 Ammi, Ferreica and Torres [2](Hölder's Inequality). For continuous functions $f, g : [a,b] \to R$, we have:

$$\int_a^b |f(t)g(t)| \diamond_\alpha t \leq \left(\int_a^b |f(t)|^p \diamond_\alpha t\right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q \diamond_\alpha t\right)^{\frac{1}{q}},$$

where $p > 1$, and $q = \frac{p}{p-1}$.

Theorem 1.3 Ferreica, Ammi and Torres [11](Minkowski's inequality). For continuous functions $f, g : [a,b] \to R$, we have:

$$\left(\int_a^b |f(t)+g(t)|^p \diamond_\alpha t\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(t)|^p \diamond_\alpha t\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p \diamond_\alpha t\right)^{\frac{1}{p}},$$

where $p > 1$.

Theorem 1.4 Ferreica, Ammi and Torres [11](Jensen's inequality). Let c and d be real numbers. Suppose $g : [a,b] \to (c,d)$ is continuous and $F : (c,d) \to R$ is convex, then,

$$F\left(\frac{\int_a^b g(s) \diamond_\alpha s}{b-a}\right) \leq \frac{\int_a^b F(g(s)) \diamond_\alpha s}{b-a}.$$

More recently, the following new concepts of generalized class of convex functions, called F_h-convex functions with a new generalized diamond-F_h dynamic calculus on time scales were introduced by the authors Fagbemigun and Mogbademu [6] and subsequently employed in different directions (see Fagbemigun, Mogbademu and Olaleru [4]-Fagbemigun and Mogbademu [10]). These concepts are particularly useful in what follows:

Definition 1.1. Fagbemigun and Mogbademu [6] Let $\mathbb{T}$ be a time scale and let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. A mapping $f : I_{\mathbb{T}} \to \mathbb{R}$ is said to be F_h-convex on time scales if

$$f(\lambda x+(1-\lambda)y) \leq \left(\frac{\lambda}{h(\lambda)}\right)^{s} f(x)+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^{s} f(y), \tag{1.2}$$

for $s \in [0,1], 0 \leq \lambda \leq 1$ and $x, y \in I_{\mathbb{T}}$.

Remark 1.1. Some special cases of Definition 1.1 are discussed as follows.

(i) If $s = 1$ and $h(\lambda) = 1$ in (1.2), then $f \in SX(I_{\mathbb{T}})$, i.e, f is convex on time scales (see Ferreica, Ammi and Torres [11]).

(ii) If $s = 1, h(\lambda) = 1$, where $\lambda = \frac{1}{2}$, then $f \in J(I_{\mathbb{T}})$ is mid-point convex on time scales (see Fagbemigun and Mogbademu [6]).

(iii) When $s = 0$, $f \in P(I_{\mathbb{T}})$ is P-convex on time scales (see Fagbemigun and Mogbademu [6]).

(iv) If $s = 1$ and $h(\lambda) = 2\sqrt{\lambda(1-\lambda)}$, then $f \in MT(I_{\mathbb{T}})$ is MT-convex on time scales (see Fagbemigun and Mogbademu [6]).

(v) Choosing $h(\lambda) = \lambda^{\frac{s}{s+1}}$ gives h-convexity on time scales, that is, $f \in SX(h, I_{\mathbb{T}})$ (see Fagbemigun and Mogbademu [6]).

(v) If $\mathbb{T} = \mathbb{R}$, then f is convex on classical intervals (see Mitrinovic, Pecarić and Fink [12]).

Definition 1.2. Fagbemigun and Mogbademu [6] Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a real valued function, with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be *diamond-F_h differentiable* on $\mathbb{T}^k_k$ in the sense of Δ and ∇, for all $t \in \mathbb{T}^k_k$, we write

$$f^{\diamond_{F_h}}(t) = \left(\frac{\lambda}{h(\lambda)}\right)^{s} f^{\Delta}(t)+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^{s} f^{\nabla}(t), \tag{1.3}$$

where $s \in [0,1]$, *and* $0 \leq \lambda \leq 1$.

If f is defined in $t \in \mathbb{T}$ such that for any $\varepsilon > 0$, there is a neighbourhood U of m and $n \in U$, with $\mu_{mn} = \sigma(m) - n$ and $\nu_{mn} = \rho(m) - n$, we have,

$$\left|\left(\frac{\lambda}{h(\lambda)}\right)^{s}[f(\sigma(m))-f(n)]\nu_{mn}+\left(\frac{1-\lambda}{h(1-\lambda)}\right)^{s}[f(\rho(m))-f(n)]\mu_{mn}-f^{\diamond_{F_h}}(t)\mu_{mn}\nu_{mn}\right|$$

$$< \varepsilon|\mu_{mn}\nu_{mn}|, \qquad s \in [0,1] \textit{ and } \lambda \in [0,1].$$

Remark 1.2. We observe that:

(i) $f^{\diamond_{F_h}}(t)$ reduces to the diamond-α derivative for $F_h = \alpha, s = 1$ and $h(\lambda) = 1$ (see Sheng, Fadag, Henderson and Davis [15]). Thus every diamond-α differentiable function on $\mathbb{T}$ is diamond-F_h differentiable but the converse is not true, see Fagbemigun and Mogbademu [6].

(ii) If f is diamond-F_h differentiable for $0 \le s \le 1$, and $0 \le \lambda \le 1$, then f is both Δ and ∇ differentiable.

(iii) For $F_h = 1, s = 1$ and $h(\lambda) = 1$, the diamond-F_h derivative reduces to the standard Δ derivative or the standard ∇ derivative for $F_h = 0$, $s = 1$ and $h(\lambda) = 1$ while it representes a 'weighted dynamic derivative' for $F_h \in (0,1)$, $s = 1$ and $h(\lambda) = 1$.

(iv) The combined dynamic derivative (1.3) gives a centralized derivative formula on any uniformly discrete time scale $\mathbb{T}$ when $F_h = \frac{1}{2}$, $s = 1$ and $h(\lambda) = 1$.

(v) When $\mathbb{T} = \mathbb{R}$, then $f^{\Delta}(t) = f^{\nabla}(t) = f'(t)$ and $f^{\diamond_{F_h}}(t)$ becomes the ordinary differential calculus (see Mitrinovic, Pecaric and Fink [12]).

Definition 1.3. Fagbemigun and Mogbademu [6] Let $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a real valued function, with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. The *diamond-F_h integral* of a function $f : \mathbb{T} \to \mathbb{R}$ from a to b, where $a, b \in \mathbb{T}$ is given by;

$$\int_a^b f(t) \diamond_{F_h} t = \left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b f(t)\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b f(t)\nabla t, \tag{1.4}$$

where $s \in [0,1]$, $0 \le \lambda \le 1$, provided that f has a delta and nabla integral on $[a,b]_{\mathbb{T}}$ or $I_{\mathbb{T}}$.

Now, we extend to the diamond-F_h dynamics, some important and useful inequalities, for the generalized class of F_h-convex functions (1.2) on time scales.

2. Inequalities on Time Scales via Diamond-F_h Integral

Here, we prove Hölder's inequality using diamond-F_h integrals (1.4). Then, we obtain the Cauchy-Schwartz's and Minkowski's inequalities as corollaries. In section 2.2, we prove Jensen's diamond-F_h integral inequality on time scales.

First, we recall the following Lemma from Fagbemigun and Mogbademu [6].

Lemma 2.1. Fagbemigun and Mogbademu [6] Let $a,b \in \mathbb{T},\ \alpha,\ \beta \in \mathbb{R}$ and f,g be diamond-F_h integrable.

(i) If $f(t) \geq 0$ for all $t \in I_{\mathbb{T}}$, then $\int_a^b f(t) \diamond_{F_h} t \geq 0$.

(ii) $\int_a^b \alpha f(t) \diamond_{F_h} t + \int_a^b \beta g(t) \diamond_{F_h} t, = \alpha \int_a^b f(t) \diamond_{F_h} t + \beta \int_a^b g(t) \diamond_{F_h} t$

for all $t \in I_{\mathbb{T}}$.

2.1. Hölder, Cauchy-Schwartz and Minkowski Inequalities.

Theorem 2.1. (Hölder's Inequality) Let $a,b \in \mathbb{T}$ with $a < b$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. Suppose $f,g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ are continuous and integrable functions on $I_{\mathbb{T}}$, then for $p > 1$, define $q = \frac{p}{p-1}$, $\int_a^b f(t)g(t) \diamond_{F_h} t$ exists and

$$\left| \int_a^b f(t)g(t) \diamond_{F_h} t \right| \leq \left(\int_a^b |f(t)|^p \diamond_{F_h} t \right)^{\frac{1}{p}} \left(\int_a^b |g(t)|^q \diamond_{F_h} t \right)^{\frac{1}{q}}. \tag{2.1}$$

Proof. Just applying Young's inequality Wong, Yeh, Yu and Hong [17], the result follows easily.

Theorem 2.2.(Cauchy-Schwartz's Inequality) Let $a,b \in \mathbb{T}$ with $a < b$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$. Further, let $f,g : [a,b]_{\mathbb{T}} \to \mathbb{R}$ be $\diamond_{F_h}$ integrable on $\mathbb{I}_{\mathbb{T}}$, then $\int_a^b f(t)g(t) \diamond_{F_h} t$ exists and

$$\left| \int_a^b f(t)g(t) \diamond_{F_h} t \right| \leq \left(\int_a^b |f(t)|^2 \diamond_{F_h} t \right)^{\frac{1}{2}} \left(\int_a^b |g(t)|^2 \diamond_{F_h} t \right)^{\frac{1}{2}}.$$

Proof. Just let $p = 2$ in Theorem 2.1, result follows.

Remark 2.1

(i) In the particular case $F_h = \alpha$, Theorem 2.1 gives Theorem 1.2, i.e, diamond-α Hölder's inequality on time scales (see Ammi, Ferreica and Torres [2], Ferreica, Ammi and Torres [11]).

(ii) If $F_h = 1$, then the delta Hölder's inequality is obtained (see Wong, Yeh, Yu and Hong [17], Wong, Yeh and Lian [18]).

(iii) The case $F_h = 0$ gives nabla Hölder's inequality (see Özkan, Sarikaya and Yildirim [14]).

(iv) The classical Hölder's inequality is obtained if $\mathbb{T} = \mathbb{R}$ in Theorem 2.1 (see Mitrinovic, Pecaric and Fink [12]).

Note that Theorem 2.1 can alternatively be proved based on the application of Jensen's inequality (see Ammi, Ferreica and Torres [2]).

We can now prove Minkowski's inequality using Theorem 2.1.

Theorem 2.3 (Minkowski's inequality) For $p>1$, let $f,g:[a,b]_{\mathbb{T}}\to\mathbb{R}$ be continuous functions on $I_{\mathbb{T}}$, $h:\mathbb{J}_{\mathbb{T}}\subset\mathbb{T}\to\mathbb{R}$ be a non zero non negative function with the property that $h(t)>0$ for all $t\geq 0$. Then $\int_a^b f(t)g(t)\diamond_{F_h}t$ exists and

$$\left(\left|\int_a^b (f+g)(t)\diamond_{F_h}t\right|^p\right)^{\frac{1}{p}} \leq \left(\int_a^b |f(t)|^p\diamond_{F_h}t\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p\diamond_{F_h}t\right)^{\frac{1}{p}}.$$

Proof. Observe that

$$\int_a^b |(f+g)(t)|^p\diamond_{F_h}t = \int_a^b |f(t)+g(t)|^{p-1}|f(t)+g(t)|\diamond_{F_h}t,$$

$$\leq \int_a^b |f(t)||f(t)+g(t)|^{p-1}\diamond_{F_h}t + \int_a^b |g(t)||f(t)+g(t)|^{p-1}\diamond_{F_h}t, \tag{2.2}$$

by the triangle inequality.

Then, applying Hölder's inequality (2.1) with $q=\frac{p}{p-1}$ to inequality (2.2) to get

$$\begin{aligned}
\int_a^b |(f+g)(t)|^p\diamond_{F_h}t &\leq \left(\int_a^b |f(t)|^p\diamond_{F_h}t\right)^{\frac{1}{p}}\left(\int_a^b |f(t)+g(t)|^{(p-1)q}\diamond_{F_h}t\right)^{\frac{1}{q}} \\
&+ \left(\int_a^b |g(t)|^p\diamond_{F_h}t\right)^{\frac{1}{p}}\left(\int_a^b |f(t)+g(t)|^{(p-1)q}\diamond_{F_h}t\right)^{\frac{1}{q}} \\
&= \left(\int_a^b |f(t)+g(t)|^{(p-1)q}\diamond_{F_h}t\right)^{\frac{1}{q}} \\
&\times \left(\left(\int_a^b |f(t)|^p\diamond_{F_h}t\right)^{\frac{1}{p}} + \left(\int_a^b |g(t)|^p\diamond_{F_h}t\right)^{\frac{1}{p}}\right). \quad (2.3)
\end{aligned}$$

Dividing both sides of inequality (2.3) by $\left(\int_a^b |(f+g)(t)|^p\diamond_{F_h}t\right)^{\frac{1}{q}}$ gives Minkowski's inequality.

Remark 2.2

(i) If $F_h=\alpha$, we obtain Theorem 1.3, which is the Minkowski's inequality for diamond-α integral on time scales (see Özkan, Sarikaya and Yildirim [14], Wong, Yeh, Yu and Hong [17], Wong, Yeh and Lian [18]).

(ii) $F_h = 1$ gives delta Minkowski's inequality on time scales (see Özkan, Sarikaya and Yildirim [14], Wong, Yeh, Yu and Hong [17], Wong, Yeh and Lian [18]).

(iii) The case $F_h = 0$ gives nabla Minkowski's inequality on time scales (see Özkan, Sarikaya and Yildirim [14]).

(iv) The classical Minkowski's inequality is obtained if $\mathbb{T} = \mathbb{R}$ in Theorem 2.3 (see Mitrinovic, Pecaric and Fink [12]).

2.2. Jensen's Inequality. Jensen' s inequality is of great interest in the theory of differential and difference equations, as well as other areas of Mathematics. The original Jensen's inequality as seen in Mitrinovic, Pecaric and Fink Mitrinovic, Pecaric and Fink [12], is stated as follows:

Theorem 2.4 Mitrinovic, Pecaric and Fink [12] If $g \in C([a,b],(c,d))$ and $f \in C((c,d),\mathbb{R})$ is convex, then

$$f\left(\frac{\int_a^b g(s)ds}{b-a}\right) \leq \frac{\int_a^b f(g(s))ds}{b-a}. \tag{2.4}$$

Recently, Agarwal, Bohner and Peterson [1] extended the Jensen's inequality (2.4) to time scales via the delta integral.

Theorem 2.5 Agarwal, Bohner and Peterson [1] If $g \in C_{rd}([a,b],(c,d))$ and $f \in C((c,d),\mathbb{R})$ are convex, then,

$$f\left(\frac{\int_a^b g(s)\Delta s}{b-a}\right) \leq \frac{\int_a^b f(g(s))\Delta s}{b-a}. \tag{2.5}$$

Under similar hypothesis, Özkan, Sarikaya and Yildirim [14] replaced the delta integral in (2.5) by its corresponding nabla integral and obtained a complete analogous result.

An extended version of Theorem 2.5 was established by Wong, Yeh, Yu and Hong [17] via the delta integral.

Theorem 2.6 Wong, Yeh, Yu and Hong [17] Let $g \in C_{rd}([a,b],(c,d))$ and $h \in C_{rd}([a,b],\mathbb{R})$ with

$$\int_a^b |h(s)|\Delta s > 0,$$

where $a,b \in \mathbb{T}$ with $a < b$, and $c,d \in R$. If $f \in C_{rd}((c,d),R)$ is convex, then

$$f\left(\frac{\int_a^b |h(s)|g(s)\Delta s}{\int_a^b |h(s)|\Delta s}\right) \leq \frac{\int_a^b |h(s)|f(g(s))\Delta s}{\int_a^b |h(s)|\Delta s}. \tag{2.6}$$

The inequality (2.6) gives Theorem 2.5 of Agarwal, Bohner and Peterson [1] if $h = 1$.

With the recent introduction of the diamond-α derivatives and integrals on time scales, Ammi, Ferreica and Torres [2] extended Jensen's inequality to an arbitrary time scale via diamond-α integral by proving the following (see also Ferreica, Ammi and Torres [11]).

Theorem 2.7. Ammi, Ferreica and Torres [2] Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and $c, d \in R$. If $g \in C([a,b]_{\mathbb{T}}, (c,d))$ and $f \in C((c,d), \mathbb{R})$ is convex, then

$$f\left(\frac{\int_a^b g(s) \diamond_\alpha s}{b-a}\right) \leq \frac{\int_a^b f(g(s)) \diamond_\alpha s}{b-a}. \tag{2.7}$$

According to Ammi, Ferreica and Torres [2],

(i) the particular case $\alpha = 1$ reduces the inequality (2.7) to Theorem 2.5 of Agarwal, Bohner and Peterson [1].

(ii) if $\mathbb{T} = R$, then Theorem 2.7 gives the classical Jensen's inequality (2.4).

Furthermore, an extended Jensen's inequality on time scales via the diamond-α integral was established.

Theorem 2.8. Ammi, Ferreica and Torres [2] Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, and $c, d \in R$. If $g \in C([a,b]_{\mathbb{T}}, (c,d))$ and $h \in C([a,b]_{\mathbb{T}}, R)$ with

$$\int_a^b |h(s)| \diamond_\alpha s > 0.$$

If $f \in C((c,d), R)$ is convex, then,

$$f\left(\frac{\int_a^b |h(s)| g(s) \diamond_\alpha s}{\int_a^b |h(s)| \diamond_\alpha s}\right) \leq \frac{\int_a^b |h(s)| f(g(s)) \diamond_\alpha s}{\int_a^b |h(s)| \diamond_\alpha s}. \tag{2.8}$$

It was pointed out in Ammi, Ferreica and Torres [2] that Theorem 2.8 is the same as Theorem 3.13 of Özkan, Sarikaya and Yildirim [14]. However, a different approach was employed by Ammi, Ferreica and Torres [2] in proving the Theorem 2.8 than that proposed by Özkan, Sarikaya and Yildirim [14]. In the particular case $h = 1$, Theorem 2.8 reduces to Theorem 2.7 and to Theorem 2.6 for $\alpha = 1$.

We present here Jensen's Inequality via the diamond-F_h integral on time scales as follows.

Theorem 2.9. Let $a, b \in \mathbb{T}$ with $a < b$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. Let $g \in C(I_{\mathbb{T}}, (a,b))$ and $f \in C((a,b), I_{\mathbb{R}})$ be F_h-convex on an interval $I_{\mathbb{T}} \subset \mathbb{R}$, if $s \geq 1$, then

$$f\left(\frac{\int_a^b g(t) \diamond_{F_h} t}{b-a}\right) \leq \frac{\int_a^b f(g(t)) \diamond_{F_h} t}{b-a}. \tag{2.9}$$

Proof. Since f is F_h-convex, we have

$$\begin{aligned}
f\left(\frac{\int_a^b g(t) \diamond_{F_h} t}{b-a}\right) &= f\left(\frac{\left(\frac{\lambda}{h(\lambda)}\right)^s}{b-a}\int_a^b g(t)\Delta t + \frac{\left(\frac{1-\lambda}{h(1-\lambda)}\right)^s}{b-a}\int_a^b g(t)\nabla t\right) \\
&= f\left(\frac{\lambda \cdot \frac{\lambda^{s-1}}{(h(\lambda))^s}}{b-a}\int_a^b g(t)\Delta t + \frac{(1-\lambda)\cdot\frac{(1-\lambda)^{s-1}}{(h(1-\lambda))^s}}{b-a}\int_a^b g(t)\nabla t\right) \\
&\leq \left(\frac{\lambda}{h(\lambda)}\right)^s f\left(\frac{\frac{\lambda^{s-1}}{(h(\lambda))^s}}{b-a}\int_a^b g(t)\Delta t\right) + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s f\left(\frac{\frac{(1-\lambda)^{s-1}}{(h(1-\lambda))^s}}{b-a}\int_a^b g(t)\nabla t\right)
\end{aligned}$$

Now, using Jensen's inequality as in Theorem 1.1, we get

$$\begin{aligned}
f\left(\frac{\int_a^b g(t) \diamond_{F_h} t}{b-a}\right) &\leq \left(\frac{\lambda}{h(\lambda)}\right)^s \frac{\frac{\lambda^{s-1}}{(h(\lambda))^s}}{b-a}\int_a^b f(g(t))\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \frac{\frac{(1-\lambda)^{s-1}}{(h(1-\lambda))^s}}{b-a}\int_a^b f(g(t))\nabla t \\
&= \frac{1}{b-a}\left[\left(\frac{\lambda}{h(\lambda)}\right)^s \frac{\lambda^{s-1}}{(h(\lambda))^s}\int_a^b f(g(t))\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \frac{(1-\lambda)^{s-1}}{(h(1-\lambda))^s}\int_a^b f(g(t))\nabla t\right] \\
&\leq \frac{1}{b-a}\left[\left(\frac{\lambda}{h(\lambda)}\right)^s \int_a^b f(g(t))\Delta t + \left(\frac{1-\lambda}{h(1-\lambda)}\right)^s \int_a^b f(g(t))\nabla t\right] \\
&= \frac{1}{b-a}\int_a^b f(g(t)) \diamond_{F_h} t.
\end{aligned}$$

Remark 2.3

(i) If $F_h = \alpha$ in inequality (2.9), Theorem 2.9 reduces to Theorem 2.7 (see Ammi, Ferreica and Torres [2]).

(ii) In the case $F_h = 1$, it reduces to Theorem 2.5 (see Agarwal, Bohner and Peterson [1]).

(iii) If $\mathbb{T} = \mathbb{R}$ in the above Theorem, we obtain the classical Jensen's inequality, that is, Theorem 2.4 (see Mitrinovic, Pecaric and Fink [12]).

(iv) If $\mathbb{T} = \mathbb{Z}$ in Theorem 2.9, we recover Theorem 1.1. (see Olanipekun, Mogbademu and Omotoyinbo [13]). However, if $\mathbb{T} = \mathbb{Z}$ and $f(x) = -lnx$, then, one gets the well-known arithmetic-geometric mean inequality (see Mitrinovic, Pecaric and Fink [12]).

Theorem 2.10 (Generalized Jensen's Inequality): Let $\mathbb{T}$ be a time scale, $a, b \in \mathbb{T}$ with $a < b$, $h : \mathbb{J}_{\mathbb{T}} \subset \mathbb{T} \to \mathbb{R}$ be a non zero non negative function with the property that $h(t) > 0$ for all $t \geq 0$, where $\mathbb{J}_{\mathbb{T}}$ is a F_h-convex subset of the real $\mathbb{T}$. $g \in C(I_{\mathbb{T}}, (c,d))$ and $k \in C(I_{\mathbb{T}}, \mathbb{R})$ with $\int_a^b |k(t)| \diamond_{F_h} t > 0$. If $f \in C((c,d), \mathbb{R})$ is F_h-convex, then

$$f\left(\frac{\int_a^b |k(t)| g(t) \diamond_{F_h} t}{\int_a^b |k(t)| \diamond_{F_h} t}\right) \leq \frac{\int_a^b |k(t)| f(g(t)) \diamond_{F_h} t}{\int_a^b |k(t)| \diamond_{F_h} t}. \tag{2.10}$$

Proof. Since f is F_h-convex, the result follows from Theorem 2.9 and Lemma 2.1(i).

Remark 2.4

(i) If $k(t) = 1$ Theorem 2.10, we recover Theorem 2.9.

(ii) If $F_h = \alpha$, we obtain Theorem 2.8 (see Ammi, Ferreica and Torres [2]).

(iii) The case $F_h = 0$ gives Theorem 3.6 of Özkan, Sarikaya and Yildirim [14] on time scales.

(iv) In the particular case $F_h = 1$ in (2.10), it reduces to Theorem 2.6 (see Wong, Yeh, Yu and Hong [17]).

(v) For $k(t) = 1$ and $F_h = 1$, we get Theorem 2.5 (see Agarwal, Bohner and Peterson [1]).

(vi) The case $\mathbb{T} = \mathbb{R}$ and $k(t) = 1$ gives Theorem 2.4 (see Mitrinovic, Pecaric and Fink [12]).

(vii) When $\mathbb{T} = \mathbb{Z}$ and $k(t) = 1$, Theorem 1.1 is obtained (see Olanipekun, Mogbademu and Omotoyinbo [13]).

3. Conclusion

We have shown that the new diamond-F_h integral on time scales satisfies properties such as Holder and Minkowski-type inequalities. We trust that our diamond-F_h integral on time scales is interesting and useful, and will lead to subsequent investigations with important applications. For example, it is well known that inequalities play a major role in the development of other areas of mathematics such as the calculus of variations. Holder and Minkowski-type inequalities can be used to find explicitly the extremizers for some classes of variational problems on time scales.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

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