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SHARP CONTRA-HARMONIC MEAN BOUNDS FOR THE SÁNDOR-YANG MEANS

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Abstract. In this paper, we present the best possible two Sándor-Yang means bounds by the one-parameter contra-harmonic mean. As applications, we find new bounds for the second Seiffert and Neuman-Sándor means.

Keywords: Sándor-Yang mean; contra-harmonic mean; one-parameter mean

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1. INTRODUCTION

Let $p \in [0,1]$, $x, y > 0$ with $x \neq y$ and $M(x,y)$ be a one-parameter symmetric bivariate mean. Then the one-parameter mean $M(x,y;p)$, arithmetic mean $A(x,y)$, quadratic mean $Q(x,y)$, contra-harmonic mean $C(x,y)$, Neuman-Sándor mean $NS(x,y)$ and second Seiffert mean $T(x,y)$

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are respectively defined by

$$
\begin{aligned}
&M(x,y;p) = M[px+(1-p)y, py+(1-p)x]\\
&A(x,y) = \frac{x+y}{2}, Q(x,y) = \sqrt{\frac{x^2+y^2}{2}}, C(x,y) = \frac{x^2+y^2}{x+y},\\
&NS(x,y) = \frac{x-y}{2sinh^{-1}\left(\frac{x-y}{x+y}\right)}, T(x,y) = \frac{x-y}{2\tan^{-1}\left(\frac{x-y}{x+y}\right)}.
\end{aligned}
\tag{1.1}
$$

It is well known that inequalities

$$
A(x,y) < NS(x,y) < T(x,y) < Q(x,y) < C(x,y) \tag{1.2}
$$

hold for all $x,y>0$ with $x \neq y$, and the one-parameter mean $M(x,y;p)$ is continuous and strictly increasing with respect to $p \in [0,1]$ for fixed $x,y>0$ with $x \neq y$.

In[1], Yang introduced the Sándor-Yang mean $R_{AQ}(x,y)$ and $R_{QA}(x,y)$ as follows:

$$
R_{AQ}(x,y) = Q(x,y)\, e^{A(x,y)/T(x,y)-1}, \tag{1.3}
$$

$$
R_{QA}(x,y) = A(x,y)\, e^{Q(x,y)/NS(x,y)-1}. \tag{1.4}
$$

Recently, the bivariate means bounds and inequalities have been attracted attention of many scholars. In particular, many remarkable inequalities involving the Sándor-Yang mean can been found in the literature[4, 5, 6, 7, 8, 9, 10, 11].

Neuman[2] proved that the inequalities

$$
A(x,y) < R_{AQ}(x,y) < R_{QA}(x,y) < Q(x,y) \tag{1.5}
$$

for all $x,y>0$ with $x \neq y$.

Xu and Qian[3] found that $p_1 \leq 1/2 + \sqrt{2e^{\pi/2-2}-1}/2, q_1 \geq 1/2 + \sqrt{3}/6$, $p_2 \leq 1/2 + \sqrt{\left(3+2\sqrt{2}\right)^{\sqrt{2}} - e^2}/(2e)$ and $q_2 \geq 1/2 + \sqrt{6}/6$ are the best possible constants such that the double inequalities

$$
Q(x,y;p_1) < R_{AQ}(x,y) < Q(x,y;q_1), Q(x,y;p_2) < R_{QA}(x,y) < Q(x,y;q_2)
$$

for all $x,y>0$ with $x \neq y$.

From (1.1), (1.2) and (1.5) we clearly see that the function $r \mapsto C(x,y;r)$ is strictly increasing on $[1/2,1]$ and

$$C(x,y;1/2) = A(x,y) < R_{AQ}(x,y) < R_{QA}(x,y) < C(x,y) = C(x,y;1) \tag{1.6}$$

for all $x,y>0$ with $x \neq y$.

Motivated by inequalities (1.6), it is natural to ask "what are the best possible parameters $\lambda_1, \lambda_2, \mu_1, \mu_2 \in [1/2,1]$ such that the double inequalities

$$C(x,y;\lambda_1) < R_{AQ}(x,y) < C(x,y;\mu_1), C(x,y;\lambda_2) < R_{QA}(x,y) < C(x,y;\mu_2)$$

for all $x,y>0$ with $x \neq y$?" the main purpose of this paper is to answer this question.

2. Lemmas

In order to prove the desired theorems we need following eight Lemmas, which we present in this section.

Lemmas 2.1. (See [12, Theorem 1.25]) For $-\infty < a < b < +\infty$, let $f,g : [a,b] \to \mathbb{R}$ be continuous on $[a,b]$ and differentiable on (a,b), and $g'(x) \neq 0$ on (a,b). If $f'(x)/g'(x)$ is increasing (decreasing) on (a,b), then so are $\frac{f(x)-f(a)}{g(x)-g(a)}$ and $\frac{f(x)-f(b)}{g(x)-g(b)}$. If $f'(x)/g'(x)$ is strictly monotone, then the monotonicity in the conclusion is also strict.

Lemmas 2.2. (See [13, Lemma 1.1]) Suppose that the power series $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ have the radius of convergence $r>0$ and $a_n, b_n > 0$ for all $n \geq 0$. If the sequence $\{a_n/b_n\}$ is (strictly) increasing (decreasing) for all $n \geq 0$, then the function $f(x)/g(x)$ is also (strictly) increasing (decreasing) on $(0,r)$.

Lemmas 2.3. The function

$$f(t) = e^{t\cot(t)-1}$$

is strictly decreasing from $(0,\pi/4)$ onto $\left(e^{\pi/4-1},1\right)$.

Proof Simple computations yields

$$f(0) = 1, f\left(\frac{\pi}{4}\right) = e^{\pi/4-1} \tag{2.1}$$

$$\log f(t) = t\cot(t) - 1,$$

$$\frac{f'(t)}{f(t)} = \frac{t}{\sin^2(t)}\left[\frac{\sin(2t)}{2t} - 1\right] \tag{2.2}$$

Since the function $t \mapsto \sin(t)/t$ is strictly decreasing from $(0, \pi/2)$ onto $(2/\pi, 1)$, hence (2.2) lead to the conclusion that

$$f'(t) < 0 \tag{2.3}$$

for $t \in (0, \pi/4)$.

Therefore, Lemma2.3 follows easily from (2.1) and (2.3).

Lemmas 2.4. The function

$$g(t) = \operatorname{sech}(t)\, e^{t\coth(t)-1}$$

is strictly decreasing from $\left(0, \log\left(1+\sqrt{2}\right)\right)$ onto $\left(\left(1+\sqrt{2}\right)^{\sqrt{2}}/(2e), 1\right)$.

Proof Straightforward computations yields

$$g(0^+) = 1, g\left(\log\left(1+\sqrt{2}\right)\right) = \frac{\left(1+\sqrt{2}\right)^{\sqrt{2}}}{\sqrt{2}e}, \tag{2.4}$$

$$\log g(t) = t\coth(t) - \log[\cosh(t)] - 1,$$

$$\frac{g'(t)}{g(t)} = \frac{t}{\sinh^2(t)}\left[\frac{\tanh(t)}{t} - 1\right] \tag{2.5}$$

It is not difficult to verify that the function $t \mapsto \tanh(t)/t$ is strictly decreasing from $\left(0, \log\left(1+\sqrt{2}\right)\right)$ onto $\left(\sqrt{2}/\left(2\log\left(1+\sqrt{2}\right)\right), 1\right)$, hence equation (2.5) lead to the conclusion that

$$g'(t) < 0 \tag{2.6}$$

for $t \in \left(0, \log\left(1+\sqrt{2}\right)\right)$.

Therefore, part (2) follows easily from (2.4) and (2.6).

Lemmas 2.5. The function

$$h(t) = \frac{\tan(t) - t}{2\sin(t)\tan^2(t)}$$

is strictly decreasing from $(0, \pi/4)$ onto $\left(\sqrt{2}(1-\pi/4)/2, 1/6\right)$.

Proof Let $h_1(x) = \tan(t) - t$ and $h_2(x) = 2\sin(t)\tan^2(t)$.Then elaborated computations lead to

$$h(x) = \frac{h_1(x)}{h_2(x)} = \frac{h_1(x) - h_1(0)}{h_2(x) - h_2(0)}, \tag{2.7}$$

$$\frac{h'_1(x)}{h'_2(x)} = \frac{\cos(t)}{2[2 + \cos^2(t)]},$$

and

$$\left[\frac{h'_1(x)}{h'_2(x)}\right]' = -\frac{\sin(t)\left[1 + \sin^2(t)\right]}{2[2 + \cos^2(t)]^2} < 0 \tag{2.8}$$

for $(0, \pi/4)$.

It follows from (2.8) imply that the function $h'_1(x)/h'_2(x)$ is strictly decreasing on $(0, \pi/4)$.

Note that

$$h\left(0^+\right) = \lim_{t\to 0^+} \frac{h'_1(x)}{h'_2(x)} = \frac{1}{6}, h\left(\frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\left(1 - \frac{\pi}{4}\right). \tag{2.9}$$

Therefore, Lemma 2.5 follows easily from (2.7), (2.9) and Lemma 2.1 together with the monotonicity of $h'_1(x)/h'_2(x)$.

Lemmas 2.6. The function

$$k(t) = \frac{\sinh(2t) - 2t}{\sinh(3t) - 3\sinh(t)}$$

is strictly decreasing from $\left(0, \log\left(1 + \sqrt{2}\right)\right)$ onto $\left(\left(\sqrt{2} - \log\left(1 + \sqrt{2}\right)\right)/2, 1/3\right)$.

Proof Making use of power series expansion we get

$$k(t) = \frac{\sum\limits_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!} t^{2n+1} - 2t}{\sum\limits_{n=0}^{\infty} \frac{3^{2n+1}}{(2n+1)!} t^{2n+1} - 3\sum\limits_{n=0}^{\infty} \frac{1}{(2n+1)!} t^{2n+1}} = \frac{\sum\limits_{n=0}^{\infty} \frac{2^{2n+3}}{(2n+3)!} t^{2n}}{\sum\limits_{n=0}^{\infty} \frac{3^{2n+3}-3}{(2n+3)!} t^{2n}} \tag{2.10}$$

Let

$$a_n = \frac{2^{2n+3}}{(2n+3)!}, b_n = \frac{3^{2n+3} - 3}{(2n+3)!}. \tag{2.11}$$

Then

$$a_n > 0, b_n > 0, \tag{2.12}$$

and

$$\frac{a_{n+1}}{b_{n+1}}-\frac{a_n}{b_n}=-\frac{2^{2n+3}\left(5\times 3^{2n+1}+1\right)}{\left(3^{2n+2}-1\right)\left(3^{2n+4}-1\right)}<0 \tag{2.13}$$

for all $n\geq 0$.

Note that

$$k\left(0^{+}\right)=\frac{a_0}{b_0}=\frac{1}{3},k\left[\log\left(1+\sqrt{2}\right)\right]=\frac{\sqrt{2}-\log\left(1+\sqrt{2}\right)}{2}. \tag{2.14}$$

Therefore, Lemma 2.6 follows from Lemma 2.2 and (2.10)-(2.14).

Lemmas 2.7. The function

$$F\left(t\right)=\frac{\sec\left(t\right)e^{t\cot(t)-1}-1}{\tan^2\left(t\right)}$$

is strictly decreasing from $(0,\pi/4)$ onto $\left(\sqrt{2}e^{\pi/4-1},1/6\right)$.

Proof Let $F_1\left(t\right)=\sec\left(t\right)e^{t\cot(t)-1}-1$ and $F_2\left(t\right)=\tan^2\left(t\right)$. Then simple computations lead to

$$F\left(t\right)=\frac{F_1\left(t\right)}{F_2\left(t\right)}=\frac{F_1\left(t\right)-F_1\left(0^{+}\right)}{F_2\left(t\right)-F_2\left(0\right)}, \tag{2.15}$$

$$\begin{aligned}\frac{F'_1\left(t\right)}{F'_2\left(t\right)}&=e^{t\cot(t)-1}\frac{\cos\left(t\right)\left[\sin\left(t\right)-t\cos\left(t\right)\right]}{2\sin^3\left(t\right)}\\&=e^{t\cot(t)-1}\frac{\tan\left(t\right)-t}{2\sin\left(t\right)\tan^2\left(t\right)}=f\left(t\right)h\left(t\right)\end{aligned} \tag{2.16}$$

where the functions $f\left(t\right)$ and $h\left(t\right)$ are defined as in Lemma 2.3 and 2.5, respectively.

Note that

$$F\left(0\right)=\lim_{t\to 0^{+}}\frac{F'_1\left(t\right)}{F'_2\left(t\right)}=\lim_{t\to 0^{+}}f\left(t\right)\lim_{t\to 0^{+}}h\left(t\right)=\frac{1}{6}, \tag{2.17}$$

and

$$F\left(\frac{\pi}{4}\right)=\sqrt{2}e^{\pi/4-1}-1. \tag{2.18}$$

Therefore, Lemma 2.7 follows from Lemma 2.1, 2.3 and 2.5 together with (2.15)-(2.18).

Lemmas 2.8. The function

$$G\left(t\right)=\frac{e^{t\coth(t)-1}-1}{\sinh^2\left(t\right)}$$

is strictly decreasing from $\left(0, \log\left(1+\sqrt{2}\right)\right)$ onto $\left(\left(1+\sqrt{2}\right)^{\sqrt{2}}/e-1, 1/3\right)$.

Proof Let $G_1(t) = e^{t\coth(t)-1} - 1$ and $G_2(t) = \sinh^2(t)$.Then elaborated computations lead to

$$G(t) = \frac{G_1(t)}{G_2(t)} = \frac{G_1(t) - G_1(0^+)}{G_2(t) - G_2(0)}, \tag{2.19}$$

$$\begin{aligned} \frac{G'_1(t)}{G'_2(t)} &= e^{t\coth(t)-1} \frac{\sinh(t)\cosh(t) - t}{2\sinh^3(t)\cosh(t)} \\ &= \operatorname{sech}(t)\, e^{t\coth(t)-1} \frac{\sinh(2t) - 2t}{\sinh(3t) - 3\sinh(t)} = g(t)\, k(t) \end{aligned} \tag{2.20}$$

where the functions $g(t)$ and $k(t)$ are defined as in Lemma 2.4 and 2.6, respectively.

Note that

$$G(0) = \lim_{t\to0^+} \frac{G'_1(t)}{G'_2(t)} = \lim_{t\to0^+} g(t) \lim_{t\to0^+} k(t) = \frac{1}{3}, \tag{2.21}$$

and

$$G\left[\log\left(1+\sqrt{2}\right)\right] = \left(1+\sqrt{2}\right)^{\sqrt{2}}/e. \tag{2.22}$$

Therefore, Lemma 2.8 follows from Lemma 2.1, 2.4 and 2.6 together with (2.19)-(2.22).

3. Main Results

Theorem 3.1. Let $\lambda_1, \mu_1 \in [1/2, 1]$. Then the double inequality

$$C(x, y; \lambda_1) < R_{AQ}(a, b) < C(x, y; \mu_1)$$

holds for all $x, y > 0$ with $x \neq y$ if and only if $\lambda_1 \leq 1/2 + \sqrt{\sqrt{2}e^{\pi/4-1} - 1}/2 = 0.6878\cdots$ and $\mu_1 \geq 1/2 + \sqrt{6}/12 = 0.7041\cdots$.

Proof Since $R_{AQ}(x, y)$ and $C(x, y)$ are symmetric and homogenous of degree 1, we assume that $x > y > 0$.Let $v = (x-y)/(x+y) \in (0, 1)$,$t = \tan^{-1}(v) \in (0, \pi/4)$ and $p \in [1/2, 1]$.Then from (1.1) and (1.3) we have

$$C(x,y;p)-R_{AQ}(x,y)=C[px+(1-p)y,py+(1-p)x]-R_{AQ}(x,y)$$
$$=A(x,y)\left[1+(2p-1)^2v^2\right]-A(x,y)\sqrt{1+v^2}e^{\frac{\tan^{-1}(v)}{v}-1}$$
$$=A(x,y)\tan^2(t)\left[(2p-1)^2-\frac{\sec(t)\,e^{t\cot(t)-1}-1}{\tan^2(t)}\right]. \tag{3.1}$$

Therefore, Theorem 3.1 follows easily from (3.1) and Lemma 2.7.

Theorem 3.2. Let $\lambda_2,\mu_2\in[1/2,1]$. Then the double inequality

$$C(x,y;\lambda_2)<R_{QA}(a,b)<C(x,y;\mu_2)$$

holds for all $x,y>0$ with $x\neq y$ if and only if $\lambda_2\leq 1/2+\sqrt{\left(1+\sqrt{2}\right)^{\sqrt{2}}/e-1}/2$ and $\mu_2\geq 1/2+\sqrt{3}/6=0.7886\cdots$.

Proof Since $R_{QA}(x,y)$ and $C(x,y)$ are symmetric and homogenous of degree 1, we assume that $x>y>0$.Let $v=(x-y)/(x+y)\in(0,1)$,$t=\sinh^{-1}(v)\in\left(0,\log\left(1+\sqrt{2}\right)\right)$ and $q\in[1/2,1]$.Then from (1.4) one has

$$C(x,y;q)-R_{QA}(x,y)=C[qx+(1-q)y,qy+(1-q)x]-R_{QA}(x,y)$$
$$=A(x,y)\left[1+(2q-1)^2v^2\right]-A(x,y)\,e^{\frac{\sqrt{1+v^2}\sinh^{-1}(v)}{v}-1}$$
$$=A(x,y)\sinh^2(t)\left[(2q-1)^2-\frac{e^{t\coth(t)-1}-1}{\sinh^2(t)}\right]. \tag{3.2}$$

Therefore, Theorem 3.2 follows easily from (3.2) and Lemma 2.8.

As an application, then from Theorems 3.1 and 3.2 we get the following Corollary 3.3 immediately.

Corollary 3.3. Let

$$\alpha(x,y;\theta)=\log C(x,y;\theta)-\log Q(x,y)+1,$$

$$\beta(x,y;\theta)=\log C(x,y;\theta)-\log A(x,y)+1,$$

The double inequalities

$$\frac{A(x,y)}{\alpha(x,y;\mu_1)}<T(x,y)<\frac{A(x,y)}{\alpha(x,y;\lambda_1)},$$

$$\frac{Q(x,y)}{\beta(x,y;\mu_2)} < NS(x,y) < \frac{Q(x,y)}{\beta(x,y;\lambda_2)}$$

hold for all $x,y>0$ with $x\neq y,\lambda_1=1/2+\sqrt{\sqrt{2}e^{\pi/4-1}-1}/2$ and $\mu_1=1/2+\sqrt{6}/12,\lambda_2=1/2+\sqrt{\left(1+\sqrt{2}\right)^{\sqrt{2}}/e-1}/2$ and $\mu_2=1/2+\sqrt{3}/6$.

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CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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