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# SOME INEQUALITIES FOR RATIONAL FUNCTIONS WITH PRESCRIBED POLES

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**Abstract.** In this paper, we prove some inequalities for rational functions with prescribed poles and restricted zeros. Our results generalize many well known inequalities available in literature.

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## 1. INTRODUCTION

Let $\mathscr{P}_n$ represents the class of all complex polynomials $p(z)$ of degree at most $n$ and $p'(z)$ be the derivative of $p(z)$. Let $D_{k-}$ and $D_{k+}$ denote the regions inside and outside the disk $T_k = \{z : |z| = k, k > 0\}$, respectively. For a function $f$ defined on $T_1$ in complex plane, we write $\|f\| := \sup_{z \in T_1} |f(z)|$, the chebyshev norm of $f$ on $T_1$,

$$w(z) := \prod_{i=1}^{n}(z-a_i); \quad B(z) := \prod_{i=1}^{n}\left(\frac{1-\overline{a}_i z}{z-a_i}\right)$$

and

$$\mathscr{R}_n := R_n(a_1, a_2, ..., a_n) = \left\{\frac{p(z)}{w(z)} : p \in \mathscr{P}_n\right\}.$$

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Then $\mathscr{R}_n$ represents the class of all rational functions with a finite limit at infinity and with at most $n$ poles $a_1, a_2, ..., a_n$ outside the unit disk.

Note that $B(z) \in \mathscr{R}_n$ and $|B(z)| = 1$ for $|z| = 1$. Throughout this paper, we shall assume that all poles $a_1, a_2, ..., a_n$ lie in $D_{1+}$.

If $p \in \mathscr{P}_n$, then we have the well known inequality that relates the norm of a polynomial to that of its derivative due to Bernstein[4].

$$\|p'\| \leq n\|p\|. \tag{1}$$

Aziz[1] and Malik[8] have proved the following refinement of inequality (1).

If $p \in \mathscr{P}_n$ and $p^*(z) = z^n\overline{p(1/\bar{z})}$, then

$$\| \, |(p^*(z))'| + |p(z)| \, \| = n\|p\|. \tag{2}$$

The next result was conjectured by Erdös and later proved by Lax[5].

If $p \in \mathscr{P}_n$ and $p \neq 0$ for $z \in D_{1-}$, then we have

$$\|p'\| \leq \frac{n}{2}\|p\|. \tag{3}$$

Furthermore, Li , Mohapatra, Rodriguez[7](see also [2], [6]) obtained inequalities similar to inequalities (1) and (3) for rational functions.They replaced polynomial $p(z)$ by a rational function $r(z)$ with prescribed poles $a_1, a_2, ..., a_n$ and $z^n$ by a Blaschke product $B(z)$.In fact, they proved following generalization of inequality (3).

**Theorem 1.1.** Suppose $r \in \mathscr{R}_n$ and all zeros of $r$ lie in $T_1 \cup D_{1+}$, then for $z \in T_1$

$$|r'(z)| \leq \frac{1}{2}|B'(z)|.\|r(z)\|. \tag{4}$$

Equality in (4) holds for $r(z) = \alpha B(z) + \beta$ with $|\alpha| = |\beta| = 1$.

Aziz and Zargar[3] proved the following generalization of Theorem (1.1). In fact they proved:

**Theorem 1.2.** If $r \in \mathscr{R}_n$, and all zeros of $r$ lie in $T_k \cup D_{k+}$, then for $z \in T_1$ , we have

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \frac{n(k-1)}{(k+1)}\frac{|r(z)|^2}{\|r\|^2}\right\}\|r(z)\|. \tag{5}$$

Equality in (5) holds for $r(z) = \left(\frac{z+k}{z-a}\right)^n$, where $k \geq 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$.

Recently B. A. Zargar , M. H. Gulzar, Rubia Akhter[9] considered the moduli of all zeros of $r(z)$ instead of considering maximum modulus of zeros of $r(z)$ and proved the following result:

**Theorem 1.3.** Suppose $r(z) = \frac{p(z)}{w(z)} \in \mathscr{R}_n$ and all zeros of $r$ lie in $T_k \cup D_{k+}$, where $k \geq 1$, then for $z \in T_1$, we have

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| + 2\left(\sum_{j=1}^{m} \frac{1}{1+|b_j|} - \frac{n}{2}\right)\frac{|r(z)|^2}{\|r\|^2}\right\}\|r\|. \tag{6}$$

Equality in (6) holds for $r(z) = \frac{(z+k)^m}{(z-a)^n}$, where $k \geq 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$.

In the same paper, they also proved the following refinement of Theorem (1.2).

**Theorem 1.4.** Suppose $r(z) = \frac{p(z)}{w(z)} \in \mathscr{R}_n$, where $r$ has exactly $n$ poles at $a_1, a_2, ..., a_n$ and all zeros of $r$ lie in $T_k \cup D_{k+}$ , $k \geq 1$, then for $z \in T_1$, we have

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \frac{n(k+1)-2m}{(k+1)}\frac{|r(z)|^2}{\|r(z)\|^2}\right\}\|r(z)\|. \tag{7}$$

Equality in (7) holds for $r(z) = \frac{(z+k)^m}{(z-a)^n}$, where $k \geq 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$.

## 2. Preliminaries

For the proof of main results, we need following Lemmas. The first Lemma is due to Aziz and Zargar[3].

**Lemma 2.1.** If $z \in T_1$, then

$$Re\left(\frac{zw'(z)}{w(z)}\right) = \frac{n - |B'(z)|}{2}.$$

The following Lemma is due to Li, Mohapatra, Rodriguez[7].

**Lemma 2.2.** If $r \in \mathscr{R}_n$ and $r^*(z) = B(z)\overline{r(\frac{1}{\bar{z}})}$, then for $z \in T_1$, we have

$$|(r^*(z))'| + |r'(z)| \leq |B'(z)|\|r\|.$$

**Lemma 2.3.** Let $r \in \mathscr{R}_n$ and all zeros of $r$ lie in $T_k \cup D_{k+}$, $k \geq 1$, with a zero of multiplicity $s$ at origin, then for $z \in T_1$

$$Re\left(\frac{zr'(z)}{r(z)}\right) \leq \frac{|B'(z)|}{2} + \left(\sum_{j=1}^{m-s} \frac{1}{1+|b_j|} - \frac{n-2s}{2}\right).$$

where $m$ is the number of zeros of $r$.

**Proof of Lemma 2.3.** Let $r(z)=\frac{z^s h(z)}{w(z)}\in\mathscr{R}_n$, where $h(z)$is a polynomial of degree $m-s$ having all its zeros in $T_k\cup D_{k+}$,$k\geq 1$.

This gives

$$\frac{zr'(z)}{r(z)}=s+\frac{zh'(z)}{h(z)}-\frac{zw'(z)}{w(z)}.$$

Equivalently,

$$Re\left(\frac{zr'(z)}{r(z)}\right)=s+Re\left(\frac{zh'(z)}{h(z)}\right)-Re\left(\frac{zw'(z)}{w(z)}\right). \tag{8}$$

Now using the fact that $h(z)$ is a polynomial of degree $m-s$ having all its zeros in $T_k\cup D_{k+}$, $k\geq 1$. If $b_1,b_2,...,b_{m-s}$ are the zeros of $h(z)$, where $|b_j|\geq k>1,\ j=1,2,...,m-s,\ (m\leq n)$ then we can write

$$h(z)=\sum_{j=0}^{m-s}c_jz^j=c_{m-s}\prod_{j=1}^{m-s}(z-bj),\ |b_j|\geq k,\ j=1,2,...,m-s.$$

which implies

$$Re\left(\frac{zh'(z)}{h(z)}\right)=Re\left(\sum_{j=1}^{m-s}\frac{z}{z-b_j}\right). \tag{9}$$

Using this in inequality (8), we obtain

$$Re\left(\frac{zr'(z)}{r(z)}\right)=s+Re\left(\sum_{j=1}^{m-s}\frac{z}{z-b_j}\right)-Re\left(\frac{zw'(z)}{w(z)}\right). \tag{10}$$

For $z\in T_1$ , this gives with the help of lemma (2.1) that

$$\begin{aligned}Re\left(\frac{zr'(z)}{r(z)}\right)&=s+Re\left(\sum_{j=1}^{m-s}\frac{z}{z-b_j}\right)-\left(\frac{n-|B'(z)|}{2}\right)\\&\leq s+\sum_{j=1}^{m-s}\frac{1}{1+|b_j|}-\left(\frac{n-|B'(z)|}{2}\right)\\&=\frac{|B'(z)|}{2}+\left(\sum_{j=1}^{m-s}\frac{1}{1+|b_j|}-\frac{n-2s}{2}\right).\end{aligned}$$

this completely proves lemma (2.3).

**Lemma 2.4.** Suppose $r \in \mathscr{R}_n$ has exactly $n$ poles $a_1, a_2, ..., a_n$ and all zeros of $r$ lie in $T_k \cup D_{k+}$, $k \geq 1$, with a zero of multiplicity $s$ at origin, then for $z \in T_1$

$$Re\left(\frac{zr'(z)}{r(z)}\right) \leq \frac{|B'(z)|}{2} - \frac{1}{1+k}\left(\frac{n(k+1) - 2sk - 2m}{2}\right).$$

where $m$ indicates the number of zeros of $r$.

**Proof of Lemma 2.4.** Let $r(z) = \frac{z^s h(z)}{w(z)} \in \mathscr{R}_n$, where $h(z)$is a polynomial of degree $m - s$ having all its zeros in $T_k \cup D_{k+}$,$k \geq 1$.

This gives

$$\frac{zr'(z)}{r(z)} = s + \frac{zh'(z)}{h(z)} - \frac{zw'(z)}{w(z)}.$$

Equivalently,

$$Re\left(\frac{zr'(z)}{r(z)}\right) = s + Re\left(\frac{zh'(z)}{h(z)}\right) - Re\left(\frac{zw'(z)}{w(z)}\right). \tag{11}$$

Since $h(z)$ is a polynomial of degree $m - s$ having all its zeros in $T_k \cup D_{k+}$, $k \geq 1$. If $b_1, b_2, ..., b_{m-s}$ are the zeros of $h(z)$, where $|b_j| \geq k > 1$, $j = 1, 2, ..., m - s$, then we can write

$$h(z) = \sum_{j=0}^{m-s} c_j z^j = c_{m-s} \prod_{j=1}^{m-s} (z - bj), \ (m \leq n), \ |b_j| \geq k > 1, \ j = 1, 2, ...m - s.$$

This gives

$$\frac{zh'(z)}{h(z)} = \sum_{j=1}^{m-s} \frac{z}{z - b_j}.$$

Which implies

$$Re\left(\frac{zh'(z)}{h(z)}\right) = Re\left(\sum_{j=1}^{m-s} \frac{z}{z - b_j}\right). \tag{12}$$

Now it can be easily verified that for $z \in T_1$ and $|b| \geq k > 1$

$$Re\left(\frac{z}{z - b}\right) \leq \frac{1}{1+k}.$$

Using this in inequality (12), we get for $z \in T_1$

$$Re\left(\frac{zh'(z)}{h(z)}\right) \leq \frac{m - s}{1+k}. \tag{13}$$

Inequality (11) in conjuction with Lemma (2.1) and inequality (13) yields for $z \in T_1$

$$\begin{aligned} Re\left(\frac{zr'(z)}{r(z)}\right) &\leq s+\frac{m-s}{1+k}-\left(\frac{n-|B'(z)|}{2}\right) \\ &= \frac{sk+m}{1+k}-\left(\frac{n-|B'(z)|}{2}\right) \\ &= \frac{1}{2}\left\{|B'(z)|-\frac{n(k+1)-2(sk+m)}{1+k}\right\}. \end{aligned}$$

which completely proves lemma (2.4).

## 3. Main Results

In this paper, we first present the following result which provides the generalization of Theorem (1.3). In fact we prove:

**Theorem 3.1.** Suppose $r \in \mathscr{R}_n$ and all zeros of $r$ lie in $T_k \cup D_{k^+}$ , $k \geq 1$ with $s$ fold zeros at origin. If $r(z) = \frac{z^s h(z)}{w(z)}$, where $h(z) = \sum_{j=0}^{m-s} c_j z^j$, $(m \leq n)$, then for $z \in T_1$, we have

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)|+2\left(\sum_{j=1}^{m-s}\frac{1}{1+|b_j|}-\frac{n-2s}{2}\right)\frac{|r(z)|^2}{\|r\|^2}\right\}\|r\|. \tag{14}$$

where $m$ indicates the number of zeros of $r$.

Equality in (14) holds for $r(z) = \frac{z^s(z+k)^{m-s}}{(z-a)^n}$ where $a > 1$, $k \geq 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$.

**Proof of Theorem 3.1.** We have

$$r^*(z) = B(z)\overline{r\left(\frac{1}{\bar{z}}\right)}$$

Now

$$(r^*(z))' = B'(z)\overline{r(\frac{1}{\bar{z}})} - B(z)\overline{r(\frac{1}{\bar{z}})}' . \frac{1}{z^2}.$$

This implies for $z \in T_1$,

$$|(r^*(z))'| = \left||B'(z)|r(z) - z(r'(z))\right|.$$

Hence for $z \in T_1$ [ see[7], p.529] , we have by using Lemma (2.3)

$$\begin{aligned}\left|\frac{z(r^*(z))'}{r(z)}\right|^2 &= \left||B'(z)| - \frac{z(r'(z))}{r(z)}\right|^2 \\ &= |B'(z)|^2 + \left|\frac{z(r'(z))}{r(z)}\right|^2 - 2|B'(z)|Re\left(\frac{z(r'(z))}{r(z)}\right) \\ &\geq |B'(z)|^2 + \left|\frac{z(r'(z))}{r(z)}\right|^2 - 2|B'(z)|\left\{\frac{|B'(z)|}{2} + \left(\sum_{j=1}^{m-s}\frac{1}{1+|b_j|} - \frac{n-2s}{2}\right)\right\} \\ &= \left|\frac{zr'(z)}{r(z)}\right|^2 - 2\left(\sum_{j=1}^{m-s}\frac{1}{1+|b_j|} - \frac{n-2s}{2}\right)|B'(z)|.\end{aligned}$$

This gives for $z \in T_1$

$$\left\{|(r'(z))|^2 - 2\left(\sum_{j=1}^{m-s}\frac{1}{1+|b_j|} - \frac{n-2s}{2}\right)|B'(z)||r(z)|^2\right\}^{\frac{1}{2}} \leq |(r^*((z))'|. \tag{15}$$

By using lemma (2.2), we obtain for $z \in T_1$ that

$$\begin{aligned}|r'(z)| + \left\{|r'(z)|^2 - 2\left(\sum_{j=1}^{m-s}\frac{1}{1+|b_j|} - \frac{n-2s}{2}\right)|B'(z)||r(z)|^2\right\}^{\frac{1}{2}} &\leq |(r^*(z))'| + |r'(z)| \\ &\leq |B'(z)|\|r\|.\end{aligned}$$

Equivalently

$$\begin{aligned}|(r'(z)|^2 - 2\left(\sum_{j=1}^{m-s}\frac{1}{1+|b_j|} - \frac{n-2s}{2}\right)|B'(z)||r(z)|^2 &\leq \left\{B'(z)|\|r\| - |(r'(z))|\right\}^2 \\ &= |B'(z)|^2\|r\|^2 + |r(z)|^2 \\ &\quad - 2|B'(z)|\|r\||r(z)|.\end{aligned}$$

that is,

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| + 2\left(\sum_{j=1}^{m-s}\frac{1}{1+|b_j|} - \frac{n-2s}{2}\right)\frac{|r(z)|^2}{\|r\|^2}\right\}\|r\|.$$

which is the desired result.

**Remark 3.2.** By taking $s = 0$ in Theorem (3.1), it reduces to Theorem (1.3).

If $r(z)$ has exactly $n$ zeros in $T_k \cup D_{k^+}$, then we get the following result:

**Corollary 3.3.** Suppose $r \in \mathscr{R}_n$ and $r$ has all its zeros in $T_k \cup D_{k^+}$, $k \geq 1$ with $s$ - fold zeros at origin, then for $z \in T_1$

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \left(\sum_{j=1}^{n-s}\frac{|b_j|-1}{|b_j|+1} - s\right)\frac{|r(z)|^2}{\|r\|^2}\right\}\|r\|. \tag{16}$$

Equality in (16) holds for $r(z) = \frac{z^s(z+k)^{n-s}}{(z-a)^n}$ where $a > 1$, $k \geq 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$.

Now we prove the following result which provides the generalization of Theorem (1.4).

**Theorem 3.4.** Suppose $r \in \mathscr{R}_n$, where $r$ has exactly $n$ poles at $a_1, a_2, ..., a_n$ and all zeros of $r$ lie in $T_k \cup D_{k^+}$, $k \geq 1$ with $s$ - fold zeros at origin. If $r(z) = \frac{z^s h(z)}{w(z)}$, where $h(z) = \sum_{j=0}^{m-s} c_j z^j$, $(m \leq n)$, then for $z \in T_1$

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \frac{n(k+1)-2(sk+m)}{k+1}\frac{|r(z)|^2}{\|r(z)\|^2}\right\}\|r(z)\|. \tag{17}$$

where $m$ indicates the number of zeros of $r$.

Equality in (17) holds for $r(z) = \frac{z^s(z+k)^{n-s}}{(z-a)^n}$ where $a > 1$, $k \geq 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$.

**Proof of Theorem 3.4.** We have

$$r^*(z) = B(z)\overline{r\left(\frac{1}{\overline{z}}\right)}$$

Now

$$(r^*(z))' = B'(z)\overline{r(\frac{1}{\overline{z}})} - B(z)\overline{r(\frac{1}{\overline{z}})}' . \frac{1}{z^2}.$$

This implies for $z \in T_1$

$$|(r^*(z))'| = \left||B'(z)|r(z) - z(r'(z))\right|.$$

Hence for $z \in T_1$ [see[7], p.529] , we have by using Lemma (2.4)

$$\begin{aligned}\left|\frac{z(r^*(z))'}{r(z)}\right|^2 &= \left||B'(z)| - \frac{z(r'(z))}{r(z)}\right|^2\\ &= |B'(z)|^2 + \left|\frac{z(r'(z))}{r(z)}\right|^2 - 2|B'(z)|Re\left(\frac{z(r'(z))}{r(z)}\right)\\ &\geq |B'(z)|^2 + \left|\frac{z(r'(z))}{r(z)}\right|^2 - |B'(z)|\left\{|B'(z)| - \left(\frac{n(k+1)-2(sk+m)}{1+k}\right)\right\}\\ &= \left|\frac{z(r'(z)}{r(z)}\right|^2 + \frac{n(k+1)-2(sk+m)}{1+k}|B'(z)|.\end{aligned}$$

that is,

$$\left\{|r'(z)|^2 + \frac{n(k+1)-2(sk+m)}{1+k}|r(z)|^2|B'(z)|\right\}^{\frac{1}{2}} \leq |(r^*(z))'|. \tag{18}$$

This gives with the help of lemma (2.2)

$$|r'(z)| + \left\{|r'(z)|^2 + \frac{n(k+1)-2(sk+m)}{k+1}|B'(z)||r(z)|^2\right\}^{\frac{1}{2}} \leq |B'(z)|\|r(z)\|.$$

or equivalently,

$$\begin{aligned}|r'(z)|^2 + \frac{n(k+1)-2(sk+m)}{k+1}|B'(z)||r(z)|^2 &\leq \{|B'(z)|\|r(z)\| - |r'(z)|\}^2\\ &= |B'(z)|^2\|r(z)\|^2 - 2|B'(z)|\|r(z)\||r'(z)|\\ &+ |r'(z)|^2.\end{aligned}$$

which on simplification yields

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \frac{n(k+1)-2(sk+m)}{k+1}\frac{|r(z)|^2}{\|r(z)\|^2}\right\}\|r(z)\|.$$

This completes the proof of theorem 3.4.

**Remark 3.5.** By taking $s = 0$ in Theorem (3.4), it reduces to Theorem (1.4) .

If $r(z)$ has exactly $n$ zeros, then we have the following result:

**Corollary 3.6.** Suppose $r \in \mathscr{R}_n$, where $r$ has exactly $n$ poles at $a_1, a_2, ..., a_n$ and all zeros of $r$ lie in $T_k \cup D_{k^+}$, $k \geq 1$ with $s$ fold zeros at origin, then for $z \in T_1$

$$|r'(z)| \leq \frac{1}{2}\left\{|B'(z)| - \frac{n(k-1)-2sk}{k+1}\frac{|r(z)|^2}{\|r(z)\|^2}\right\}\|r(z)\|. \tag{19}$$

Equality in (19) holds for $r(z) = \frac{z^s(z+k)^{n-s}}{(z-a)^n}$ where $a > 1$, $k \geq 1$ and $B(z) = \left(\frac{1-az}{z-a}\right)^n$ evaluated at $z = 1$.

**Remark 3.7.** If we take $s = 0$ in Corollary (3.6), it reduces to Theorem (1.2).

## Conflict of Interests

The author(s) declare that there is no conflict of interests.

## References

[1] A. Aziz, Q. G. Mohammad, Simple proof of a Theorem of Erdös and Lax, Proc. Amer. Math. Soc. 80 (1980), 119-122.

[2] A. Aziz, W. M. Shah, Some refinements of Bernstein type inequalities for rational functions. Glas. Mat. 32(52) (1997), 29-37.

[3] A. Aziz, B. A. Zargar, Some properties of rational functions with prescribed poles, Canad. Math. Bull, 42(4) (1999), 417-426.

[4] S. N. Bernstein, Sur l'ordre de la meilleure approximation des fonctions continues par des polynômes de degré donné, Mem. Acad. R. Belg. 4 (1912), 1-103.

[5] P. D. Lax, Proof of a conjecture of Erdös on the derivative of a polynomial, Bull Amer. Math. Soc. 50 (1994), 509-513.

[6] X. Li, A comparison inequality for rational functions, Proc. Amer. Math. Soc. 139 (2011), 1659-1665.

[7] X. Li , R. N. Mohapatra, R. S. Rodriguez , Bernstein type inequalities for rational functions with prescribed poles, J. Lond. Math. Soc. 1 (1995), 523-531.

[8] M. A. Malik, An integral mean estimate for the polynomials, Proc. Amer. Math. Soc. 91 (1984), 281-284.

[9] B. A. Zargar, M. H. Gulzar, R. Akhter, Inequalities for the rational functions with prescribed poles and restricted zeros, Adv. Inequal. Appl. 2021 (2021), Article ID 1.