Available online at http://scik.org
Adv. Inequal. Appl. 2022, 2022:1
https://doi.org/10.28919/aia/6451
ISSN: 2050-7461

SOME FIXED POINT RESULTS IN TVS- CONE METRIC SPACES WITH APPLICATION IN TVS A-CONE METRIC SPACE

MAMTA PATEL[1,*], SANJAY SHARMA[2]

[1]Department of Mathematics, Delhi Public School, Durg (C.G.) India

[2]Department of Mathematics, Bhilai Institute of Technology, Durg (C.G.) India

Abstract. In the year 2011, Cho et al. [2] has presented the new idea of c-distance in cone metric spaces. At that point after Dubey, A.K. et al. [4] demonstrated a few after effects of fixed point results using c-distance with contractive conditions in cone metric spaces. As of late, Tiwari, S.K. et al. [17] has demonstrated the expansion of fixed point hypothesis for contraction mappings applying c-distance. The motivation behind this paper is to build up a speculation of the outcomes demonstrated by Tiwari, S.K. et. al. [17] for contraction mappings applying c-distance. Moreover, we prove a theorem for such mapping applying c-distance in tvs $\mathscr{A}$-cone metric space as an application that extends the results of M.Abbas,et.al. [14]. The outcomes here sum up and expand a portion of the notable outcomes present in the literature [1,5,6,7,8,9,10].

Keywords: complete metric space; tvs-cone metric Space; tvs $\mathscr{A}$-cone metric space; c-distance; fixed point; contraction mapping.

2010 AMS Subject Classification: 47H10, 54H25.

1. Introduction

The generalization of classical metric space was presented by Huang and Zhang [11] supplanting an ordered Banach Space for the real numbers set , as an idea of cone metric space and

*Corresponding author
E-mail address: mamtapatel2u@gmail.com
Received July 02, 2021

indicated some fixed point results with contractive mappings on cone metric spaces. Afterward, numerous authors studied cone metric spaces and deduced some fixed point theorems for cone metric spaces. The c-distance is given by Cho et.al. [2]. Wang and Gao [18] introduced the w-distance as cone version which was characterized by Kada et.al.[13]. Recently in 2017 Fadail et al.[7,8,9,10] studied the results of fixed point theorems on T-Reich contraction type based on the idea of c-distance. Followed by the result, Tiwari et al. proved and generalized the extended unique fixed theorems applying the c-distance in these spaces. M.,Abbas,et.al [14] introduced the theory of a $\mathscr{A}$ cone metric space which is a generalization of S metric space. Our aim is to prove a fixed point theorem in tvs-$\mathscr{A}$ cone metric space applying c-distance as an application. Our results generalizes and extends the fixed point results.

2. Preliminaries

The following definitions and notations will be required in continuation. In this paper, we have denoted the set of real numbers by $\mathscr{R}$, a real Banach Space by $\mathscr{E}$ and a real Banach Algebra by $\mathscr{A}$.

Definition 2.1([3]): Consider $\mathscr{E}$ be a real Banach space and $\mathscr{P} \subset \mathscr{E}$ where θ the zero element in $\mathscr{E}$, then $\mathscr{P}$ is called a cone iff:

(a) $\mathscr{P}$ is closed, non-empty , $\mathscr{P} \neq \{\theta\}$,

(b) For all non-negative real numbers a,b and $s,t \in \mathscr{P}$ then $as+bt \in \mathscr{P}$,

(c) $s \in \mathscr{P}$,$-s \in \mathscr{P} \Rightarrow s=\theta$ iff $\mathscr{P} \cap (-\mathscr{P}) = \theta$.

Given that the cone $\mathscr{P}$ is a subset of $\mathscr{E}$, a partial ordering $\leq$,defined by $s \leq t$ iff $t-s \in \mathscr{P}$ on $\mathscr{E}$ in which $s \ll t$ stands for $t-s \in \mathscr{P}$. $\mathscr{P}$ is solid if *int* $\mathscr{P} \neq \phi$. If $s,t \in \mathscr{E}$, then there is a least number $K>0$ such that ,

$\theta \leq s \leq t \Rightarrow \| s \| \leqslant K \| t \|$

Then cone $\mathscr{P}$ is called normal and number K which satisfies the given condition is called a normal constant.

Definition 2.2([3]): Suppose a vector valued mapping $d_1 : X \times X \to \mathscr{E}$, where X be a non-empty set satisfies

(a) For every $s,t \in X$ $d_1(s,t) \geq 0$ and $d(s,t) = 0$ iff $s = t$,

(b) For every $s,t \in X$, $d_1(s,t) = d_1(t,s)$,

(c) For every $s,t,w \in X$, $d_1(s,t) \leq d_1(s,t) + d_1(w,t)$.

Then d_1 is said to be a tvs-cone metric on X, and the pair (X,d_1) is tvs-cone metric space.

Definition 2.3([2]): Let (X,d_1) be a tvs-cone metric space over Banach space $\mathscr{E}$ and$\{s_n\}_{n\geq 1}$be a sequence in X.Then

(a) For every $c \in \mathscr{E}$ with $\theta \ll c$,$\{s_n\}_{n\geq 1}$ converges to X,then there exists a natural number N such that $d_1(s_n,s) \ll c \; \forall \; n \geq N$. Let us represent this by $s_n \to s \, as \, n \to \infty$.

(b) For every $c \in \mathscr{E}$ with $\theta \ll c$, $\{s_n\}_{n\geq 1}$ is a Cauchy Sequence if there is a natural number N such that $d_1(s_n,s_m) \ll c \; \forall \; n,m \geq N$.We denote this by $s_n \to s \; as \, n \to \infty$.

(c)(X,d_1) is a complete cone metric space if each Cauchy sequence in X is convergent.

Lemma 2.4([16]):

(1) If $\mathscr{P}$ be a cone in a real Banach space $\mathscr{E}$ and $\alpha \leq \lambda\alpha$, where $\alpha \in \mathscr{P}$ and $\theta \leq \lambda < 1$, then $\alpha = \theta$.

(2) If $c \in int \; \mathscr{P}, \theta \leq \alpha_n$ and $\alpha_n \to \theta$, then there exists a positive integer N such that $a_n \ll c \; \forall \; n \geq N$.

Now, we define c-distance on a tvs-cone metric space (X,d_1) as it is a generalization of w-distance given by Kada et al.[3].

Definition 2.5([2]): A function $q_1 : X \times X \to \mathscr{E}$, where (X,d_1) be a tvs-cone metric space.Then q_1 is called a c-distance on X if it satisfies:

(a) If $s,t \in X$, then $q_1(s,t) \geq \theta$,

(b) If $s,t,w \in X$, then $q_1(s,t) \leq q_1(s,t) + q_1(t,w)$,

(c) For all $s \in X$ and $n \geq 1$, if $q_1(s,t_n) \leq u$ for any $u = u_s \in \mathscr{P}$, then $q_1(s,t) \leq u$ whenever $\{t_n\}$ in X converges to a point $t \in X$,

(d) For all $c \in \mathscr{E}$ with $\theta \ll c$, there exists $e \in \mathscr{E}$ with $\theta \in e$ such that $q_1(w,s) \ll e$ and

$q_1(w,t) \ll e \Rightarrow q_1(s,t) \ll c$.

Lemma 2.6([2]): Let (X,d_1) be a tvs-cone metric space and $\{s_n\}$ and $\{t_n\}$ be two sequences in X, and $s,t,w \in X$. q_1 is a c-distance on X. If u_n converges to 0 in $\mathscr{P}$. The following conditions hold:

(a) If $u_n \geq (s_n,t)$ and $u_n \geq (s_n,w)$, then $t = w$,

(b) If $u_n \geq (s_n,t_n)$ and $u_n \geq (s_n,w)$, then $\{t_n\}$ converges to z,

(c) If $u_n \geq (s_n,s_m)$ for $m > n > n_0$, then $\{s_n\}$ is a Cauchy sequence in X,

(d) If $u_n \geq (t,s_n)$ then $\{s_n\}$ is a Cauchy sequence in X.

Definition2.7([11]): Let X be a non-empty set . A function $d_1 : X^n \to \mathscr{A}$ is a tvs $\mathscr{A}$-cone metric on X if,

(a) $\theta \leq d_1(s_1,s_2,..,s_{n-1},t)$ for all $s,t \in X$, and $d_1(s_1,s_2,..,s_{n-1},s_n) = \theta$ if and only if $s_1 = s_2 = .. = s_n$,

(b) $d_1(s_1,s_2,..,s_{n-1},t_n) = d_1(t_1,t_2,..,t_{n-1},s_n)$ for all $s,t \in X$,

(c) $d_1(s_1,s_2,..,s_{n-1},t) \leq d_1(s_1,s_2,..,s_{n-1},t) + d_1(w_1,w_2,..,w_{n-1},t)$ for all $s,t,w \in X$.

Then (X,d_1) is called a tvs $\mathscr{A}$- cone metric space.

3. Main Results

The following results generalizes and extend the results of [17].

Theorem 3(a): Let us consider (X,d_1) be tvs-cone metric space, $\mathscr{P}$ be a solid cone and q_1 be a c-distance on X . If $\mathbb{T} : X \to X$ be continuous and satisfies the contractive condition:

(i) $q_1(\mathbb{T}s,\mathbb{T}t) \leq a_1[q_1(s,t) + q_1(s,\mathbb{T}s)] + a_2[q_1(s,t) + q_1(t,\mathbb{T}t)] + a_3[q_1(s,\mathbb{T}t) + q_1(t,\mathbb{T}s)] + a_4[q_1(s,t) + q_1(s,\mathbb{T}t)] + a_5[q_1(s,t) + q_1(t,\mathbb{T}s)]$

(ii) $q_1(\mathbb{T}t,\mathbb{T}s) \leq a_1[q_1(t,s) + q_1(\mathbb{T}s,s)] + a_2[q_1(t,s) + q_1(\mathbb{T}t,t)] + a_3[q_1(\mathbb{T}t,s) + q_1(\mathbb{T}s,t)] + a_4[q_1(t,s) + q_1(\mathbb{T}t,s)] + a_5[q_1(t,s) + q_1(\mathbb{T}s,t)]$

for all $s,t \in X$, in which a_1,a_2,a_3,a_4,a_5 are non-negative real numbers such that $2(a_1 + a_2 + a_3 + a_4 + a_5) < 1$, then $\mathbb{T}$ has a fixed point $s^\star \in X$. The sequence $\{\mathbb{T}^n s\}$ converges to the fixed point. And if $u = \mathbb{T}u$ then $(u,u) = \theta$. The fixed point is unique.

Proof. Choose $s_o \in X$. Set $s_1 = \mathbb{T}s_0, s_2 = \mathbb{T}s_1 = \mathbb{T}^2 s_0, ----, s_{n+1} = \mathbb{T}s_n = \mathbb{T}^n s_0$. Then we have

$q_1(s_n, s_{n+1}) \leq q_1(\mathbb{T}s_{n-1}, \mathbb{T}s_n)$

$\leq a_1[q_1(s_{n-1}, s_n) + q_1(s_{n-1}, \mathbb{T}s_{n-1})] + a_2[q_1(s_{n-1,}, s_n) + q_1(s_{n,}, \mathbb{T}s_n)] + a_3[q_1(s_{n-1}, \mathbb{T}s_n) + q_1(s_{n,}, \mathbb{T}s_{n-1})] + a_4[q_1(s_{n-1}, s_n) + q_1(s_{n-1,}, \mathbb{T}s_n)] + a_5[q_1(s_{n-1}, s_n) + q_1(s_n, \mathbb{T}s_{n-1})]$

$\leq a_1[q_1(s_{n-1}, s_n) + q_1(s_{n-1}, s_n)] + a_2[q_1(s_{n-1,}, s_n) + q_1(s_n, s_{n+1})] + a_3[q_1(s_{n-1}, s_{n+1}) + q_1(s_n, s_n)] + a_4[q_1(s_{n-1}, s_n) + q_1(s_{n-1}, s_n)] + a_5[q_1(s_{n-1}, s_n) + q_1(s_n, s_n)]$

$\therefore q_1(s_n, s_{n+1}) \leq (2a_1 + a_2 + a_3 + 2a_4 + a_5) q_1(s_{n-1}, s_n) + (a_2 + a_3 + a_5) q_1(s_n, s_{n+1})$

$q_1(s_{n,}, s_{n+1}) \leq \frac{2a_1+a_2+a_3+2a_4+a_5}{1-(a_2+a_3+a_4)} = hq_1(s_{n-1}, s_n) ---------- (1)$

Similarly

$q_1(s_{n+1}, s_n) \leq q_1(\mathbb{T}s_n, \mathbb{T}s_{n-1})$

$\leq a_1[q_1(s_n, s_{n-1}) + q_1(\mathbb{T}s_{n-1}, s_{n-1})] + a_2[q_1(s_{n,}, s_{n-1}) + q_1(\mathbb{T}s_{n,}, s_n)] + a_3[q_1(\mathbb{T}s_n, s_{n-1}) + q_1(\mathbb{T}s_{n-1}, s_n)] + a_4[q_1(s_n, s_{n-1}) + q_1(\mathbb{T}s_{n,}, s_{n-1})] + a_5[q_1(s_n, s_{n-1}) + q_1(\mathbb{T}s_{n-1}, s_n)]$

$\leq a_1[q_1(s_n, s_{n-1}) + q_1(s_n, s_{n-1})] + a_2[q_1(s_{n,}, s_{n-1}) + q_1(s_{n+1}, s_n)] + a_3[q_1(s_{n+1}, s_{n-1}) + q_1(s_n, s_n)] + a_4[q_1(s_n, s_{n-1}) + q_1(s_{n+1}, s_{n-1})] + a_5[q_1(s_n, s_{n-1}) + q_1(s_n, s_n)]$

$\therefore q_1(s_{n+1}, s_n) \leq (2a_1 + a_2 + a_3 + 2a_4 + a_5) q_1(s_n, s_{n-1}) + (a_2 + a_3 + a_5) q_1(s_{n+1}, s_n)$

$q_1(s_{n+1,}, s_n) \leq \frac{2a_1+a_2+a_3+2a_4+a_5}{1-(a_2+a_3+a_4)} = hq_1(s_n, s_{n-1}) ----- (2)$

Denote $u_n = q_1(s_n, s_{n+1}) + q_1(s_{n+1}, s_n)$

Adding equations (1) and (2), we get

$u_n \leq (2a_1 + a_2 + a_3 + 2a_4 + a_5) u_{n-1} + (a_2 + a_3 + a_5) u_n$

i.e

$u_n \leq hu_{n-1}$,Where $h = \frac{2a_1+a_2+a_3+2a_4+a_5}{1-a_2+a_3+a_4)} < 1$.

Suppose $m > n \geq 1$.Then we get

$q_1(s_{n,}, s_m) \leq q_1(s_n, s_{n+1}) + q_1(s_{n+1}, s_{n+2}) + ---- + q_1(s_{n-1}, s_m)$

$\leq (h^n + h^{n+1} + ---- + h^{n-1}) q_1(s_0, s_1) + q_1(s_1, s_0)$

$= u_n \leq \frac{h^n}{1-h}[q_1(s_0, s_1) + q_1(s_1, s_0)] \to \infty, h \to \infty$

So, from Lemma 2.6 it is clear that $\{s_n\}$ is a Cauchy sequence in X. We have X as complete, there exists $s^\star \in X$, such that $s_n \to s^\star \mathbb{T}$ is continuous, then $s^\star = lim\, s_{n+1} = lim\, \mathbb{T}(s_n) =$

$\mathbb{T}(lim s_n) = \mathbb{T}(s^\star)$.

Therefore $s^\star$ is a fixed point of $\mathbb{T}$. Suppose that $u = \mathbb{T}u$,

Then we have

$q_1(u,u) \leq q_1(\mathbb{T}u,\mathbb{T}u) \leq a_1[q_1(u,u) + q_1(u,\mathbb{T}u)] + a_2[q_1(u,u) + q_1(u,\mathbb{T}u)] + a_3[q_1(u,\mathbb{T}u) + q_1(u,\mathbb{T}u)] + a_4[q_1(u,u) + q_1(u,\mathbb{T}u)] + a_5[q_1(u,u) + q_1(u,\mathbb{T}u)]$

$= 2(a_1 + a_2 + a_3 + a_4 + a_5)q_1(u,u)$

Since $2(a_1 + a_2 + a_3 + a_4 + a_5) < 1$ then by lemma 2.6 we have $q_1(u,u) = \theta$.

Proving the uniqueness of the fixed point:

Let there be another fixed point t^* of $\mathbb{T}$, then we have

$q_1(s^*,t^*) \leq q_1(\mathbb{T}s^*,\mathbb{T}t^*)$

$\leq a_1q_1(s^*,t^*) + a_2q_1(s^*,\mathbb{T}s^*) + a_3[q_1(s^*,\mathbb{T}s^*) + q_1(s^*,\mathbb{T}s^*)] + a_4[q_1(s^*,t^*) + q_1(s^*,\mathbb{T}t^*)] + a_5[q_1(s^*,t^*) + q_1(t^*,\mathbb{T}s^*)]$

$= (a_1 + a_2 + 2a_3 + 2a_4 + 2a_5)q_1(s^*,t^*)$

$\leq 2(a_1 + a_2 + a_3 + a_4 + a_5)q_1(s^*,t^*)$

Since $2(a_1 + a_2 + a_3 + a_4 + a_5) < 1$, by lemma 2.6 we get $q_1(s^*,t^*) = \theta$ and also $(s^*,s^*) = \theta$, we get $s^* = t^*$.

Theorem 3(b): Let (X,d_1) be tvs-cone metric space, with cone $\mathscr{P}$ and let q_1 be a c-distance on X. Suppose that $\mathbb{T} : X \to X$ be continuous and satisfies the contractive condition:

(i) $q_1(\mathbb{T}s,\mathbb{T}t) \leq a_1q_1(s,t) + a_2q_1(s,\mathbb{T}s) + a_3[q_1(s,\mathbb{T}s) + q_1(t,\mathbb{T}t)] + a_4[q_1(s,\mathbb{T}t) + q_1(t,\mathbb{T}s)] + a_5[q_1(s,t) + q_1(s,\mathbb{T}s)] + a_6[q_1(s,t) + q_1(t,\mathbb{T}t)]$

(ii) $q_1(\mathbb{T}t,\mathbb{T}s) \leq a_1[q_1(t,s) + a_2q_1(\mathbb{T}s,s)] + a_3[q_1(\mathbb{T}s,s) + q_1(\mathbb{T}t,t)] + a_4[q_1(\mathbb{T}t,s) + q_1(\mathbb{T}s,t)] + a_5[q_1(t,s) + q_1(\mathbb{T}s,s)] + a_6[q_1(t,s) + q_1(\mathbb{T}t,t)]$

for all $s,t \in X$, in which a_1,a_2,a_3,a_4,a_5 are non-negative real numbers such that $2(a_1 + a_2 + a_3 + a_4 + a_5) < 1$, then $\mathbb{T}$ has a fixed point $s^\star \in X$. The sequence $\{\mathbb{T}^n s\}$ converges to the fixed point. And if $u = \mathbb{T}u$ then $(u,u) = \theta$. The fixed point is unique.

$q_1(s_n,s_{n+1}) \leq q_1(\mathbb{T}s_{n-1},\mathbb{T}s_n)$

$\leq a_1q_1(s_{n-1},s_n) + a_2q_1(s_{n-1},s_n)] + a_3[q_1(s_{n-1,},\mathbb{T}s_{n-1}) + q_1(s_{n,},\mathbb{T}s_n)] + a_4[q_1(s_{n-1},\mathbb{T}s_n) + q_1(s_{n,},\mathbb{T}s_{n-1})] + a_5[q_1(s_{n-1},s_n) + q_1(s_{n-1,},\mathbb{T}s_{n-1})] + a_6[q_1(s_{n-1},s_n) + q_1(s_n,\mathbb{T}s_n)]$

$\leq\ a_1q_1(s_{n-1},s_n)\ +\ a_2q_1(s_{n-1},s_n)]\ +\ a_3[q_1(s_{n-1,},s_n)\ +\ q_1(s_n,s_{n+1})]\ +\ a_4[q_1(s_{n-1},s_{n+1})\ + q_1(s_n,s_n)]+a_5[q_1(s_{n-1},s_n)+q_1(s_{n-1},s_n)]+a_6[q_1(s_{n-1},s_n)+q_1(s_n,s_{n+1})]$

$\therefore q_1(s_n,s_{n+1})\leq (\ a_1+a_2+a_3+a_4+2a_5+a_6)q_1(s_{n-1},s_n)+(a_3+a_4+a_6)q_1(s_n,s_{n+1})$

$q_1(s_{n,},s_{n+1})\leq \frac{a_1+a_2+a_3+a_4+2a_5+a_6}{1-(a_3+a_4+a_6)} = hq_1(s_{n-1},s_n)---------(3)$

Similarly

$q_1(s_{n+1},s_n)\leq q_1(\mathbb{T}s_n,\mathbb{T}s_{n-1})$

$\leq\ a_1q_1(s_n,s_{n-1})\ +\ a_2q_1(s_n,s_{n-1})\ +\ a_3[q_1(\mathbb{T}s_{n-1,},s_{n-1})\ +\ q_1(\mathbb{T}s_{n,},s_n)]\ +\ a_4[q_1(\mathbb{T}s_n,s_{n-1})\ + q_1(\mathbb{T}s_{n-1},s_n)]+a_5[q_1(s_n,s_{n-1})+q_1(\mathbb{T}s_{n-1,},s_{n-1})]+a_6[q_1(s_n,s_{n-1})+q_1(\mathbb{T}s_n,s_n)]$

$\leq\ a_1q_1(s_n,s_{n-1})\ +\ a_2q_1(s_n,s_{n-1})\ +\ a_3[q_1(s_{n,},s_{n-1})\ +\ q_1(s_{n+1},s_n)]\ +\ a_4[q_1(s_{n+1},s_{n-1})\ + q_1(s_n,s_n)]+a_5[q_1(s_n,s_{n-1})+q_1(s_n,s_{n-1})]+a_6[q_1(s_n,s_{n-1})+q_1(s_{n+1},s_n)]$

$\therefore q_1(s_{n+1},s_n)\leq (a_1+a_2+a_3+a_4+2a_5+a_6)q_1(s_n,s_{n-1})+(a_3+a_4+a_6)q_1(s_{n+1},s_n)$

$q_1(s_{n,},s_{n+1})\leq \frac{a_1+a_2+a_3+a_4+2a_5+a_6}{1-(a_3+a_4+a_6)} = hq_1(s_{n-1},s_n)---------(4)$

Denote $u_n = q_1(s_n,s_{n+1})+q_1(s_{n+1},s_n)$

Adding equations(3) and (4), we get

$u_n\leq\ (a_1+a_2+a_3+a_4+2a_5+a_6)u_{n-1}+(a_3+a_4+a_6)u_n$

i.e. $u_n\leq\ hu_{n-1}$,

Where $h=\frac{a_1+a_2+a_3+a_4+2a_5+a_6}{1-(a_3+a_4+a_6)}<1$

Let $m>n\geq 1$, then it follows that

$q_1(s_n,s_m)\leq q_1(s_n,s_{n+1})+q_1(s_{n+1},s_{n+2})+---+q_1(s_{n-1},s_m)$

$\leq (h^n+h^{n+1}+-----+h^{n-1})q_1(s_0,s_1)+q_1(s_1,s_0)=u_n$

$\leq \frac{h^n}{1-h}[q_1(s_0,s_1)+q_1(s_1,s_0)] \Rightarrow u_n\rightarrow\infty, h\rightarrow\infty$

Thus, from Lemma 2.6 it is clear that $\{s_n\}$ is a Cauchy sequence in X. Since we have X complete, there exists $s^\star\in X$ Such that $s_n\rightarrow s^\star\mathbb{T}$ is continuous, then $s^\star = lim\, s_{n+1} = lim\,\mathbb{T}(s_n) = \mathbb{T}(lim\, s_n) = \mathbb{T}(s^\star)$. Therefore $s^\star$ is a fixed point of $\mathbb{T}$.

Suppose that $u=\mathbb{T}u$,

Then we have

$q_1(u,u)\leq q_1(\mathbb{T}u,\mathbb{T}u)$

$q_1(u,u)\ \leq a_1q_1(u,u)+a_2q_1(u,\mathbb{T}u)+a_3[q_1(u,\mathbb{T}u)+q_1(u,\mathbb{T}u)]+a_4[q_1(u,\mathbb{T}u)+q_1(u,\mathbb{T}u)]+ a_5[q_1(u,u)+q_1(u,\mathbb{T}u)]+a_6[q_1(u,u)+q_1(u,\mathbb{T}u)]$

$= (a_1+a_2+2a_3+2a_4+2a_5+2a_6)q_1(u,u)$ Since $2(a_1+a_2+2a_3+2a_4+2a_5+2a_6) < 1$ then by lemma 2.6 we have $q_1(u,u) = \theta$.

Proving the uniqueness of the fixed point:

Let there be another fixed point y^* of $\mathbb{T}$, then we have

$q_1(s^*,t^*) \leq q_1(\mathbb{T}s^*,\mathbb{T}t^*)$

$\leq a_1q_1(s^*,t^*)+a_2q_1(s^*,\mathbb{T}s^*)+a_3[q_1(s^*,\mathbb{T}s^*)+q_1(s^*,\mathbb{T}s^*)]+a_4[q_1(s^*,\mathbb{T}s^*)+q_1(s^*,\mathbb{T}s^*)]+$

$a_5[q_1(s^*,s^*)+q_1(s^*,\mathbb{T}s^*)]+]$ $a_6[q_1(s^*,s^*)+q_1(s^*,\mathbb{T}s^*)$

$= (a_1+a_2+2a_3+2a_4+2a_5+2a_6)q_1(s^*,t^*)$

$\leq 2(a_1+a_2+a_3+a_4+a_5+a_6)q_1(s^*,t^*)$

Since $2(a_1+a_2+a_3+a_4+a_5+a_6) < 1$ then by lemma 2.6 we have $q_1(s^*,t^*) = \theta$ and also we have $(s^*,t^*) = \theta$

hence we get $s^* = t^*$.

Example: Let $\mathscr{E} = \mathscr{R}$ *and* $\mathscr{P} = (s \in \mathscr{E}; s \geq 0\}$ *Let* $X = [0,1]$,and define a mapping $f : X \to X$ such that $f(s) = s^2, s = \frac{1}{1000}, t = \frac{9}{1000}, a_1 = \frac{1}{100}, a_2 = \frac{2}{100}, a_3 = \frac{3}{100}, a_4 = \frac{4}{100}, a_5 = \frac{5}{100}, a_6 = \frac{6}{100}$. Let $d(fs, ft) = d(s-t)$.

Solution: $q_1(\mathbb{T}s,\mathbb{T}t) = |s^2-t^2|$

$= |s-t||s+t|$

$= |s-t||\frac{1}{1000}+\frac{9}{1000}|$

$= |s-t| \times 0.03 = a_3|s-t|$

$= a_3q_1(s,t)$

$\leq a_1q_1(s,t) + a_2q_1(s,\mathbb{T}s) + a_3[q_1(s,\mathbb{T}s) + q_1(t,\mathbb{T}t)] + a_4[q_1(s,\mathbb{T}t) + q_1(t,\mathbb{T}s)]+a_5[q_1(s,t) + q_1(s,\mathbb{T}s)] + a_6[q_1(s,t)+q_1(t,\mathbb{T}t)]$.

4. APPLICATION:

Here we prove fixed point result for tvs $\mathscr{A}$-cone metric space using c-distance.

Theorem 4: Let (X,d_1) be tvs $\mathscr{A}$-cone metric space, $\mathscr{P}$ be a solid cone and q_1 be a c-distance on X . Suppose that $\mathbb{T} : X \to X$ be continuous and satisfies the contractive condition:

(i) $q_1(\mathbb{T}s,\mathbb{T}s,....,\mathbb{T}s,\mathbb{T}s) \leq a_1[q_1(s,s,....,s,t) + q_1(s,s,....,s,\mathbb{T}t)] + a_2[q_1(s,s,....,s,t) + q_1(t,t,....t,\mathbb{T}t)] + a_3[q_1(s,s,....,s,\mathbb{T}t) + q_1(t,t,....,t,\mathbb{T}s)] + a_4[q_1(s,s,....,s,t) + q_1(s,s,....,s,\mathbb{T}t)]+a_5[q_1(s,s,....,s,t)+q_1(t,t,....t,\mathbb{T}s)]$

(ii) $q_1(\mathbb{T}t,\mathbb{T}t,....,\mathbb{T}t,\mathbb{T}s) \leq a_1[q_1(t,t,....t,s) + q_1(\mathbb{T}s,\mathbb{T}s,....,\mathbb{T}s,s)] + a_2[q_1(t,t,....,t,s)+ q_1(\mathbb{T}t,\mathbb{T}t,....,\mathbb{T}t,t)] + a_3[q_1(\mathbb{T}t,\mathbb{T}t,....,\mathbb{T}t,s) + q_1(\mathbb{T}s,\mathbb{T}s,....,\mathbb{T}s,t)] + a_4[q_1(t,t,....t,s)+q_1(\mathbb{T}t,\mathbb{T}t,....,\mathbb{T}t,s)]+a_5[q_1(t,t,....,t,s)+q_1(\mathbb{T}s,\mathbb{T}s,....,\mathbb{T}s,t)]$

for all $s,t \in X$, where a_1,a_2,a_3,a_4,a_5 are non negative real numbers such that $2(a_1+a_2+a_3+a_4+a_5) < 1$.

Then $\mathbb{T}$ has a fixed point $s^\star \in X$ iterative sequence $\{\mathbb{T}^n s\}$converges to the fixed point. If $u=\mathbb{T}u$. Then $(u,u)=\theta$. The fixed point is unique.

Choose $s_o \in X$.

Set $s_1=\mathbb{T}s_0, s_2=\mathbb{T}s_1=\mathbb{T}^2 s_0,----,s_{n+1}=\mathbb{T}s_n=\mathbb{T}^n s_0$.

Then we have

$q_1(s_n,s_n,...,s_n s_{n+1}) \leq q_1(\mathbb{T}s_{n-1},\mathbb{T}s_{n-1},..,\mathbb{T}s_{n-1},\mathbb{T}s_n)$
$\leq a_1[q_1(s_{n-1},s_{n-1},..,s_{n-1},s_n)+q_1(s_{n-1},s_{n-1},..,s_{n-1},\mathbb{T}s_{n-1})]+a_2[q_1(s_{n-1,},s_{n-1},..,s_{n-1},s_n)+ q_1(s_{n,},s_n,..,s_n,\mathbb{T}s_n)]+ a_3[q_1(s_{n-1},s_{n-1},..,s_{n-1},\mathbb{T}s_n) + q_1(s_{n,},s_n,..,s_n,\mathbb{T}s_{n-1})] + a_4[q_1(s_{n-1},s_{n-1},..,s_{n-1},s_n) + q_1(s_{n-1,},s_{n-1},..,s_{n-1},\mathbb{T}s_n)] + a_5[q_1(s_{n-1},s_{n-1},..,s_{n-1},s_n) + q_1(s_n,s_n,..,s_n,\mathbb{T}s_{n-1})]$
$\leq a_1[q_1(s_{n-1},s_{n-1},..,s_{n-1},s_n) + q_1(s_{n-1},s_{n-1},..,s_{n-1},s_n)] + a_2[q_1(s_{n-1,},s_{n-1},..,s_{n-1},s_n) + q_1(s_n,s_n,..,s_n,s_{n+1})] + a_3[q_1(s_{n-1},s_{n-1},..,s_{n-1},s_{n+1}) + q_1(s_n,s_n,..,s_n,s_n)] + a_4[q_1(s_{n-1},s_{n-1},..,s_{n-1},s_n) + q_1(s_{n-1},s_{n-1},..,s_{n-1},s_n)] + a_5[q_1(s_{n-1},s_{n-1},..,s_{n-1},s_n) + q_1(s_n,s_n,..,s_n,s_n)]$

$\therefore\ q_1(s_n,s_n,..,s_n,s_{n+1}) \leq (\ 2a_1+a_2+a_3+2a_4+a_5)q_1(s_{n-1},s_{n-1},..,s_{n-1},s_n)+\ (a_2+a_3+a_5)q_1(s_n,s_n,..,s_n,s_{n+1})$

$$q_1(s_{n,},s_{n,},..,s_{n,},s_{n+1}) \leq \frac{2a_1+a_2+a_3+2a_4+a_5}{1-(a_2+a_3+a_4)} = hq_1(s_{n-1},s_{n-1},..,s_{n-1},s_n)--(5)$$

Similarly

$q_1(s_{n+1},s_{n+1},..,s_{n+1},s_n) \leq q_1(\mathbb{T}s_n,\mathbb{T}s_n,..,\mathbb{T}s_n,\mathbb{T}s_{n-1})$
$\leq a_1[q_1(s_n,s_n,..,s_n,s_{n-1}) + q_1(\mathbb{T}s_{n-1},\mathbb{T}s_{n-1},..,\mathbb{T}s_{n-1},s_{n-1})] + a_2[q_1(s_{n,},s_n,..,s_n,s_{n-1}) + q_1(\mathbb{T}s_{n,},\mathbb{T}s_n,..,\mathbb{T}s_n,s_n)] + a_3[q_1(\mathbb{T}s_n,\mathbb{T}s_n,..,\mathbb{T}s_n,s_{n-1}) + q_1(\mathbb{T}s_{n-1},\mathbb{T}s_{n-1},..,\mathbb{T}s_{n-1},s_n)] + a_4[q_1(s_n,s_n,..,s_n,s_{n-1}) + q_1(\mathbb{T}s_{n,},\mathbb{T}s_n,..,\mathbb{T}s_n,s_{n-1})] + a_5[q_1(s_n,s_n,..,s_n,s_{n-1}) + q_1(\mathbb{T}s_{n-1},\mathbb{T}s_{n-1},..,\mathbb{T}s_{n-1},s_n)]$

$\leq a_1[q_1(s_n,s_n,..,s_n,s_{n-1}) + q_1(s_n,s_n,..,s_n,s_{n-1})] + a_2[q_1(s_{n,},s_n,..,s_n,s_{n-1}) + q_1(s_{n+1},s_n)] + a_3[q_1(s_{n+1},s_{n+1},..,s_{n+1},s_{n-1}) + q_1(s_n,s_n,..,s_n,s_n)] + a_4[q_1(s_n,s_n,..,s_n,s_{n-1}) + q_1(s_{n+1},s_{n+1},..,s_{n+1},s_{n-1})] + a_5[q_1(s_n,s_n,..,s_n,s_{n-1}) + q_1(s_n,s_n,..,s_n,s_n)]$

$\therefore q_1(s_{n+1},s_{n+1},..,s_{n+1},s_n) \leq (2a_1+a_2+a_3+2a_4+a_5)q_1(s_n,s_n,..,s_n,s_{n-1}) + (a_2+a_3+a_5)q_1(s_{n+1},s_{n+1},..,s_{n+1},s_n)$

$q_1(s_{n+1},s_{n+1},..,s_{n+1},s_n) \leq \frac{2a_1+a_2+a_3+2a_4+a_5}{1-(a_2+a_3+a_4)} = hq_1(s_n,s_n,..,s_n,s_{n-1}) - -(6)$

Denote $u_n = q_1(s_n,s_n,..,s_n,s_{n+1}) + q_1(s_{n+1},s_{n+1},..,s_{n+1},s_n)$

Adding equations (5) and (6), we get

$u_n \leq (2a_1+a_2+a_3+2a_4+a_5)u_{n-1} + (a_2+a_3+a_5)u_n$

i.e

$u_n \leq hu_{n-1}$,Where $h = \frac{2a_1+a_2+a_3+2a_4+a_5}{1-a_2+a_3+a_4)} < 1$.

Let $m > n \geq 1$.

Then it follows that

$q_1(s_n,s_n,..,s_n,s_m) \leq q_1(s_n,,..,s_n,s_{n+1}) + q_1(s_{n+1},s_{n+1},..,s_{n+1},s_{n+2}) + - - - - + q_1(s_{n-1},s_{n-1},..,s_{n-1},s_m)$

$\leq (h^n + h^{n+1} + - - - - + h^{n-1})q_1(s_0,s_0,..,s_0,s_1) + q_1(s_1,s_1,..,s_1,s_0)$

$\leq \frac{h^n}{1-h}[q_1(s_0,s_0,..,s_0,s_1) + q_1(s_1,s_1,..,s_1,s_0)] = u_n \to \infty, h \to \infty$

Thus, from Lemma 2.6 it is clear that $\{s_n\}$ is a Cauchy sequence in X. Since we have X complete, there exists $s^\star \in X$

Such that $s_n \to s^\star \mathbb{T}$ is continuous, then $s^\star = lim s_{n+1} = lim \mathbb{T}(s_n) = \mathbb{T}(s^\star)$.

Therefore $s^\star$ is a fixed point of $\mathbb{T}$. Suppose that $u = \mathbb{T}u$,

$q_1(u,u,...,u,u) \leq q_1(\mathbb{T}u,\mathbb{T}u,..,\mathbb{T}u,\mathbb{T}u)$

Then we have

$\leq a_1[q_1(u,u,..,u,u) + q_1(u,u,..,u,\mathbb{T}u)] + a_2[q_1(u,u,..,u,u) + q_1(u,u,..,u,\mathbb{T}u)] + a_3[q_1(u,u,..,u,\mathbb{T}u) + q_1(u,u,..,u,\mathbb{T}u)] + a_4[q_1(u,u,..,u,u) + q_1(u,u,..,u,\mathbb{T}u)] + a_5[q_1(u,u,..,u,u) +$

$q_1(u,u,..,u,\mathbb{T}u)]$

$= 2(a_1+a_2+a_3+a_4+a_5)q_1(u,u,..,u,u)$

Since $2(a_1+a_2+a_3+a_4+a_5) < 1$ then by lemma 2.6 we have $q_1(u,u,..,u,u) = \theta$.

Proving the uniqueness of the fixed point:

Let there be another fixed point t^* of $\mathbb{T}$, then we have

$q_1(s^*,s^*,..,s^*,t^*) \leq q_1(\mathbb{T}s^*,s^*,..,s^*,\mathbb{T}t^*)$

$\leq a_1q_1(s^*,s^*,..,s^*,t^*)+a_2q_1(s^*,s^*,..,s^*,\mathbb{T}s^*)+a_3[q_1(s^*,s^*,..,s^*,\mathbb{T}s^*)+$
$q_1(s^*,s^*,..,s^*,\mathbb{T}x^*)] + a_4[q_1(s^*s^*,..,s^*,t^*) + q_1(s^*s^*,..,s^*,\mathbb{T}t^*)] + a_5[q_1(s^*s^*,..,s^*,t^*) +$
$q_1(t^*,t^*,..,t^*,\mathbb{T}s^*)]$

$= (a_1+a_2+2a_3+2a_4+2a_5)q_1(s^*,s^*,..,s^*,t^*)$

$\leq 2(a_1+a_2+a_3+a_4+a_5)q_1(s^*,s^*,..,s^*,t^*)$

Since $2(a_1+a_2+a_3+a_4+a_5) < 1$ then by lemma 2.6 we have $q_1(s^*,s^*,..,s^*,t^*) = \theta$ and also we have $(s^*,s^*,..,s^*,s^*) = \theta$ hence we get $s^* = t^*$.

Therefore the fixed point is unique.

5. Conclusion:

In this paper, we proved unique fixed point theorem in tvs-cone metric space with conclusions. The outcomes here sums up and generalizes the ongoing aftereffects of Tiwari,S.K et.al.,[17] utilizing c-distance with contractive conditions. And as an application the result has been proved in tvs $\mathscr{A}$-cone metric space which generalises and extends the further scope of our outcomes.

Conflict of Interests

The author(s) declare that there is no conflict of interests.

References

[1] S.M. Abusalim, M.S.M. Noorani, Generalized distance in cone metric spaces and tripled coincidence point and common tripled fixed point theorems. Far East J. Math. Sci. 91 (2014), 65-87.

[2] Y.J. Cho, R. Saadati, S. Wang, Common fixed point theorems on generalized distance in ordered cone metric spaces, Comput. Math. Appl. 61 (2011), 1254-1260.

[3] A.K. Dubey, R. Verma, R.P. Dubey, Cone metric spaces and Fixed point theorems of contractive mapping for c-distance. Int. J. Math Appl. 3 (2015), 83-88.

[4] A.K. Dubey, U. Mishra, Some fixed point results single valued mapping for c-distance in Tvs-cone metric spaces. Filomat 30 (2016), 2915-2934.

[5] Z.M. Fadail, A.G.B. Ahmad, Z. Golubovic, Fixed point theorems of single valued mapping for c-distance in cone metric spaces, Abstr. Appl. Anal. 2012 (2012), Article ID 826815.

[6] Z.M. Fadail, A.G.B. Ahmad, L. Paunovic, New Fixed point results of single valued mapping for c-distance in cone metric spaces, Abstr. Appl. Anal. 2012 (2012), Article ID 639713.

[7] Z.M. Fadail, A.G.B. Ahmad, S. Radenovic, Common Fixed point and fixed point results under c-distance in cone metric spaces, Appl. Math. Inform. Sci. Lett. 1 (2013), 47-52.

[8] Z.M. Fadail, A.G. Bin Ahmad, Common coupled fixed point theorems of single-valued mapping for c-distance in cone metric spaces, Abstr. Appl. Anal. 2012 (2012), 901792.

[9] Z.M. Adail, A.G.B. Ahmad, New coupled Coincidence point and common Fixed point results in cone metric spaces with c-distance. Far East J. Math. Sci. 77 (2013), 65-84.

[10] Z.M. Fadail, A.G.B. Ahmad, Coupled Fixed point theorems of single valued mapping for c-distance in cone metric spaces. J. Appl. Math. 2012 (2012), 246516.

[11] J. Fernandez, S. Saelee, K. Saxena, N. Malviya, P. Kumam, The A-cone metric space over Banach Algebra with applications, Cogent Math. Stat. 4 (2017), 1282690.

[12] L. Huang, X. Zhang, Cone metric spaces and fixed point theorems of Contractive Mappings, J. Math. Anal. Appl. 332 (2007), 1468-1476.

[13] O. Kada, T. Suzuki, W. Takahashi, Non convex minimization theorems and fixed point theorems in complete metric spaces, Math. Japon. 44 (1996), 381-391.

[14] M. Abbas, B. Ali, Y.I. Suleiman, Generalized coupled common fixed point results in partially ordered A-metric spaces, Fixed Point Theory Appl. 2015 (2015), 64.

[15] M. Đorđević, D. Đorić, Z. Kadelburg, S. Radenović, D. Spasić, Fixed point results under c-distance in tvs-cone metric spaces, Fixed Point Theory Appl. 2011 (2011) 29.

[16] W. Sintunavarat, Y.J. Cho, P. Kumam, Common fixed point theorems for c-distance in ordered cone metric spaces, Comput. Math. Appl. 62 (2011), 1969-1978.

[17] S.K. Tiwari, D. Kaushik, G.K. Sahu, Cone metric spaces and extension of fixed point theorem for contraction mappings applying c-distance, IOSR J. Math. 14 (2018), 9-13.

[18] S. Wang, B. Gao, Distance in cone metric spaces and common fixed point theorems, Appl. Math. Lett. 24 (2011), 17-1739.