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ON AN OPERATOR PRESERVING INEQUALITIES BETWEEN POLYNOMIALS

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Abstract. Let $\mathscr{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n and $\mathcal{B}_n$ a family of operators that maps $\mathscr{P}_n$ into itself. In this paper, we consider a problem of investigating the dependence of

$$\left| B[P(Rz)] - \alpha B[P(rz)] + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} B[P(rz)] \right|$$

on the maximum and minimum modulus of $|P(z)|$ on $|z| = k$ for arbitrary real or complex numbers $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq k$ and establish certain sharp operator preserving inequalities between polynomials, from which a variety of interesting results follows as special cases.

Keywords: Polynomials; Inequalities in the complex domain; $\mathcal{B}_n$-operator.

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1. Introduction

Let $\mathscr{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n. A famous result known as Bernstein's inequality (for reference, see [8, p.531], [10, p.508] or [11] states that if $P \in \mathscr{P}_n$, then

$$(1) \qquad \underset{|z|=1}{Max} |P'(z)| \leq n \underset{|z|=1}{Max} |P(z)|,$$

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whereas concerning the maximum modulus of $P(z)$ on the circle $|z| = R > 1$, we have

$$\underset{|z|=R}{Max}|P(z)| \leq R^n \underset{|z|=1}{Max}|P(z)|, \quad R \geq 1. \tag{2}$$

(for reference, see [7, p.442] or [8, vol.I, p.137]).

If we restrict ourselves to the class of polynomials $P \in \mathscr{P}_n$ having no zero in $|z| < 1$, then inequalities (1) and (2) can be respectively replaced by

$$\underset{|z|=1}{Max}|P'(z)| \leq \frac{n}{2}\underset{|z|=1}{Max}|P(z)|, \tag{3}$$

and

$$\underset{|z|=R}{Max}|P(z)| \leq \frac{R^n+1}{2}\underset{|z|=1}{Max}|P(z)|, \quad R \geq 1. \tag{4}$$

Inequality (3) was conjectured by Erdös and later verified by Lax [5], whereas inequality (4) is due to Ankey and Ravilin [1]. Aziz and Dawood [2] further improved inequalities (3) and (4) under the same hypothesis and proved that,

$$\underset{|z|=1}{Max}|P'(z)| \leq \frac{n}{2}\left\{\underset{|z|=1}{Max}|P(z)| - \underset{|z|=1}{Min}|P(z)|\right\}, \tag{5}$$

$$\underset{|z|=R}{Max}|P(z)| \leq \frac{R^n+1}{2}\underset{|z|=1}{Max}|P(z)| - \frac{R^n-1}{2}\underset{|z|=1}{Min}|P(z)|, \quad R \geq 1. \tag{6}$$

As a compact generalization of Inequalities (1) and (2), Aziz and Rather [3] have shown that if $P \in \mathscr{P}_n$ then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > 1$ and $|z| \geq 1$,

$$\begin{aligned}&\left|P(Rz) - \alpha P(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}P(z)\right| \\ &\qquad \leq |z|^n \left|R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right| \underset{|z|=1}{Max}|P(z)|.\end{aligned} \tag{7}$$

The result is sharp and equality in (7) holds for the polynomial $P(z) = az^n$, $a \neq 0$.

As a corresponding compact generalization of Inequalities (3) and (4), they [3] have also shown that if $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for all $\alpha, \beta \in \mathbb{C}$ with

$|\alpha| \leq 1, |\beta| \leq 1,\ R > 1$ and $|z| \geq 1$,

$$\left|P(Rz) - \alpha P(z) + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}P(z)\right|$$

$$\leq \frac{1}{2}\left[\left|R^n - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right||z|^n\right.$$

$$\left.+\left|1 - \alpha + \beta\left\{\left(\frac{R+1}{2}\right)^n - |\alpha|\right\}\right|\right]\underset{|z|=1}{Max}|P(z)|. \tag{8}$$

The result is best possible and equality in (8) holds for $P(z) = az^n + b$, $|a| = |b|$.

Q. I. Rahman [9] (see also Rahman and Schmeisser [10, p. 538]) introduced a class $\mathcal{B}_n$ of operators B that carries a polynomial $P \in \mathscr{P}_n$ into

$$B[P(z)] = \lambda_0 P(z) + \lambda_1\left(\frac{nz}{2}\right)\frac{P'(z)}{1!} + \lambda_2\left(\frac{nz}{2}\right)^2\frac{P''(z)}{2!}, \tag{9}$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$U(z) = \lambda_0 + n\lambda_1 z + \frac{n(n-1)}{2}\lambda_2 z^2 \tag{10}$$

lie in half plane $|z| \leq |z - n/2|$.

As a generalization of the inequalities (1) and (3), Q. I. Rahman [9, inequalities 5.2 and 5.3] proved that if $P \in \mathscr{P}_n$, then

$$|B[P(z)]| \leq |B[z^n]|\underset{|z|=1}{Max}|P(z)|, \quad \text{for} \quad |z| \geq 1, \tag{11}$$

and if $P \in \mathscr{P}_n$, $P(z) \neq 0$ in $|z| < 1$, then

$$|B[P(z)]| \leq \frac{1}{2}\left\{|B[z^n]| + |\lambda_0|\right\}\underset{|z|=1}{Max}|P(z)|, \quad \text{for} \quad |z| \geq 1, \tag{12}$$

where $B \in \mathcal{B}_n$.

1. Preliminaries

For the proof of our results, we need the following Lemmas.

Lemma 1.1. *If $P \in \mathscr{P}_n$ and $P(z)$ have all its zeros in $|z| \leq k$ where $k \geq 0$, then for every $R \geq r$, $Rr \geq k^2$ and $|z| = 1$, we have*

$$|P(Rz)| \geq \left(\frac{R+k}{r+k}\right)^n |P(rz)|.$$

The above is due to Aziz and Zargar [4]. The next lemma follows from Corollary 18.3 of [6, p. 86].

Lemma 1.2. *If $P \in \mathscr{P}_n$ and $P(z)$ has all zeros in $|z| \leq k$, where $k > 0$ then all the zeros of $B[P(z)]$ also lie in $|z| \leq k$.*

Lemma 1.3. *If $P \in \mathscr{P}_n$ and $P(z)$ have no zero in $|z| < k$, where $k > 0$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$,*

$$\begin{aligned} &\left|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta) B[P(rz)]\right| \\ &\qquad \leq k^n \left|B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta) B[Q(rz/k^2)]\right|, \end{aligned} \tag{13}$$

where $Q(z) = z^n \overline{P(1/\overline{z})}$ and

$$\Phi_k(R, r, \alpha, \beta) = \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} - \alpha. \tag{14}$$

Proof. By hypothesis, the polynomial $P(z)$ does not vanish in $|z| < k$. Therefore, all the zeros of polynomial $Q(z/k^2)$ lie in $|z| < k$. As

$$|k^n Q(z/k^2)| = |P(z)| \quad \text{for} \quad |z| = k,$$

applying Theorem 2.1 to $P(z)$ with $F(z)$ replaced by $k^n Q(z/k^2)$, we get for arbitrary real or complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$,

$$|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta) B[P(rz)]| \leq k^n \left|B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta) B[Q(rz/k^2)]\right|,$$

This proves Lemma 1.3.

□

Lemma 1.4. *If $P \in \mathscr{P}_n$ and $Q(z) = z^n\overline{P(1/\overline{z})}$ then for $\alpha, \beta \in \mathbb{C}$,with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq k,\ k \leq 1$ and $|z| \geq 1$,*

$$\left|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]\right| + k^n \left|B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)]\right|$$

$$\leq \left\{|\lambda_0|\left|1 + \Phi_k(R, r, \alpha, \beta)\right| + \frac{|B[z^n]|}{k^n}\left|R^n + r^n\Phi_k(R, r, \alpha, \beta)\right|\right\} \underset{|z|=k}{Max} |P(z)|, \tag{15}$$

where $\Phi_k(R, r, \alpha, \beta)$ is given as (14).

Proof. Let $M = Max_{|z|=k}|P(z)|$, then by Rouche's theorem, the polynomial $F(z) = P(z) - \mu M$ does not vanish in $|z| < k$ for every $\mu \in \mathbb{C}$ with $|\mu| > 1$. Applying Lemma 1.3 to polynomial $F(z)$, we get for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,

$$|B[F(Rz)] + \Phi_k(R, r, \alpha, \beta)B[F(rz)]| \leq k^n \left|B[H(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[H(rz/k^2)]\right|,$$

where $H(z) = z^n\overline{F(1/\overline{z})} = Q(z) - \overline{\mu}Mz^n$. Replacing $F(z)$ by $P(z) - \mu M$ and $H(z)$ by $Q(z) - \overline{\mu}Mz^n$, we have for $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,

$$\left|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)] - \mu\lambda_0\left(1 + \Phi_k(R, r, \alpha, \beta)\right)M\right|$$

$$\leq k^n\Bigg|B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)]$$

$$- \frac{\overline{\mu}}{k^{2n}}\left(R^n + r^n\Phi_k(R, r, \alpha, \beta)\right)MB[z^n]\Bigg| \tag{16}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$.

Now choosing argument of μ in the right hand side of inequality (16) such that

$$k^n\left|B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)] - \frac{\overline{\mu}}{k^{2n}}\left(R^n + r^n\Phi_k(R, r, \alpha, \beta)\right)MB[z^n]\right|$$

$$= \frac{|\overline{\mu}|}{k^n}\left|R^n + r^n\Phi_k(R, r, \alpha, \beta)\right||B[z^n]|M - k^n\left|B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)]\right|$$

which is possible by applying Corollary 2.3 to polynomial $Q(z/k^2)$, and using the fact $Max_{|z|=k}|Q(z/k^2)| = M/k^n$, we get for $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,

$$\left|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta)B[P(rz)]\right| - |\mu\lambda_0|\left|\left(1 + \Phi_k(R, r, \alpha, \beta)\right)M\right|$$

$$\leq \frac{|\overline{\mu}|}{k^n}\left|R^n + r^n\Phi_k(R, r, \alpha, \beta)\right||B[z^n]|M - k^n\left|B[Q(Rz/k^2)] + \Phi_k(R, r, \alpha, \beta)B[Q(rz/k^2)]\right|$$

Equivalently for $|\alpha| \leq 1, |\beta| \leq 1$ and $|z| \geq 1$,

$$\left|B[P(Rz)] + \Phi_k(R,r,\alpha,\beta)B[P(rz)]\right| + k^n \left|B[Q(Rz/k^2)] + \Phi_k(R,r,\alpha,\beta)B[Q(rz/k^2)]\right|$$
$$\leq |\mu| \left\{ |\lambda_0| \left|1 + \Phi_k(R,r,\alpha,\beta)\right| + \frac{1}{k^n} \left|R^n + r^n \Phi_k(R,r,\alpha,\beta)\right| |B[z^n]| \right\} M$$

Letting $|\mu| \to 1$, we get the conclusion of Lemma 1.4 and this completes proof of Lemma 1.4. □

2. Main results

Theorem 2.1. *If $F \in \mathscr{P}_n$ and $F(z)$ has all its zeros in the disk $|z| \leq k$ where $k > 0$ and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \quad for \quad |z| = k,$$

then for $|\alpha| \leq 1, |\beta| \leq 1,\ R > r \geq k$ and $|z| \geq 1$,

$$|B[P(Rz)] + \Phi_k(R,r,\alpha,\beta)B[P(rz)]| \leq |B[F(Rz)] + \Phi_k(R,r,\alpha,\beta)B[F(rz)]|, \tag{17}$$

where

$$\Phi_k(R,r,\alpha,\beta) = \beta \left\{ \left(\frac{R+k}{k+r}\right)^n - |\alpha| \right\} - \alpha. \tag{18}$$

The result is best possible and the equality holds for the polynomial $P(z) = e^{i\gamma}F(z)$ where $\gamma \in \mathbb{R}$.

Proof of Theorem 2.1. Since polynomial $F(z)$ of degree n has all its zeros in $|z| \leq k$ and $P(z)$ is a polynomial of degree at most n such that

$$|P(z)| \leq |F(z)| \quad \text{for} \quad |z| = k, \tag{19}$$

therefore, if $F(z)$ has a zero of multiplicity s at $z = ke^{i\theta_0}$, $0 \leq \theta_0 < 2\pi$, then $P(z)$ has a zero of multiplicity at least s at $z = ke^{i\theta_0}$. If $P(z)/F(z)$ is a constant, then inequality (17) is obvious. We now assume that $P(z)/F(z)$ is not a constant, so that by the maximum modulus principle, it follows that

$$|P(z)| < |F(z)| \text{ for } |z| > k \ .$$

Suppose $F(z)$ has m zeros on $|z| = k$ where $0 \leq m < n$, so that we can write

$$F(z) = F_1(z)F_2(z)$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z| = k$ and $F_2(z)$ is a polynomial of degree exactly $n - m$ having all its zeros in $|z| < k$. This implies with the help of inequality (19) that

$$P(z) = P_1(z)F_1(z)$$

where $P_1(z)$ is a polynomial of degree at most $n - m$. Again, from inequality (19), we have

$$|P_1(z)| \leq |F_2(z)| \ \ for \ |z| = k$$

where $F_2(z) \neq 0 \ for \ |z| = k$. Therefore for every real or complex number λ with $|\lambda| > 1$, a direct application of Rouche's theorem shows that the zeros of the polynomial $P_1(z) - \lambda F_2(z)$ of degree $n - m \geq 1$ lie in $|z| < k$ hence the polynomial

$$G(z) = F_1(z)\left(P_1(z) - \lambda F_2(z)\right) = P(z) - \lambda F(z)$$

has all its zeros in $|z| \leq k$ with at least one zero in $|z| < k$, so that we can write

$$G(z) = (z - te^{i\delta})H(z)$$

where $t < k$ and $H(z)$ is a polynomial of degree $n - 1$ having all its zeros in $|z| \leq k$. Applying Lemma 1.1 to the polynomial $H(z)$, we obtain for every $R > r \geq k$ and

$0 \leq \theta < 2\pi$,

$$\begin{aligned}|G(Re^{i\theta})| =&|Re^{i\theta} - te^{i\delta}||H(Re^{i\theta})| \\ \geq&|Re^{i\theta} - te^{i\delta}| \left(\frac{R+k}{k+r}\right)^{n-1} |H(re^{i\theta})|, \\ =&\left(\frac{R+k}{k+r}\right)^{n-1} \frac{|Re^{i\theta} - te^{i\delta}|}{|re^{i\theta} - te^{i\delta}|}|(re^{i\theta} - te^{i\delta})H(re^{i\theta})|, \\ \geq&\left(\frac{R+k}{k+r}\right)^{n-1} \left(\frac{R+t}{r+t}\right)|G(re^{i\theta})|.\end{aligned}$$

This implies for $R > r \geq k$ and $0 \leq \theta < 2\pi$,

$$\left(\frac{r+t}{R+t}\right)|G(Re^{i\theta})| \geq \left(\frac{R+k}{k+r}\right)^{n-1} |G(re^{i\theta})|. \tag{20}$$

Since $R > r \geq k$ so that $G(Re^{i\theta}) \neq 0$ for $0 \leq \theta < 2\pi$ and $\frac{r+k}{k+R} > \frac{r+t}{R+t}$, from inequality (20), we obtain

$$|G(Re^{i\theta})| > \left(\frac{R+k}{k+r}\right)^{n} |G(re^{i\theta})|, \quad R > r \geq k \text{ and } 0 \leq \theta < 2\pi. \tag{21}$$

Equivalently,

$$|G(Rz)| > \left(\frac{R+k}{k+r}\right)^{n} |G(rz)|$$

for $|z| = 1$ and $R > r \geq k$. Hence for every real or complex number α with $|\alpha| \leq 1$ and $R > r \geq k$, we have

$$\begin{aligned}|G(Rz) - \alpha G(rz)| &\geq |G(Rz)| - |\alpha|\,|G(rz)| \\ &> \left\{\left(\frac{R+k}{k+r}\right)^{n} - |\alpha|\right\}|G(rz)|, \text{ for } |z| = 1.\end{aligned} \tag{22}$$

Also, inequality (21) can be written in the form

$$|G(re^{i\theta})| < \left(\frac{k+r}{R+k}\right)^{n} |G(Re^{i\theta})| \tag{23}$$

for every $R > r \geq k$ and $0 \leq \theta < 2\pi$. Since $G(Re^{i\theta}) \neq 0$ and $\left(\frac{k+r}{R+k}\right)^n < 1$, from inequality (23), we obtain for $0 \leq \theta < 2\pi$ and $R > r \geq k$,

$$|G(re^{i\theta})| < |G(Re^{i\theta})|.$$

That is,

$$|G(rz)| < |G(Rz)| \text{ for } |z| = 1.$$

Since all the zeros of $G(Rz)$ lie in $|z| \leq (k/R) < 1$, a direct application of Rouche's theorem shows that the polynomial $G(Rz) - \alpha G(rz)$ has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| \leq 1$. Applying Rouche's theorem again, it follows from (22) that for arbitrary real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq k$, all the zeros of the polynomial

$$\begin{aligned} T(z) =& G(Rz) - \alpha G(rz) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} G(rz) \\ =& \left[P(Rz) - \alpha P(rz) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} P(rz) \right] \\ & - \lambda \left[F(Rz) - \alpha F(rz) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} F(rz) \right] \end{aligned}$$

lie in $|z| < 1$.

Applying Lemma 1.3 to the polynomial $T(z)$ and noting that B is a linear operator, it follows that all the zeros of polynomial

$$\begin{aligned} B[T(z)] = & \left[B[P(Rz)] - \alpha B[P(rz)] + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} B[P(rz)] \right] \\ & - \lambda \left[B[F(Rz)] - \alpha B[F(rz)] + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} [F(rz)] \right] \end{aligned}$$

lie in $|z| < 1$. This implies

$$|B[P(Rz)] + \Phi_k(R, r, \alpha, \beta) B[P(rz)]| \leq |B[P(Rz)] + \Phi_k(R, r, \alpha, \beta) B[P(rz)]|, \tag{24}$$

for $|z| \geq 1$ and $R > r \geq k$. If inequality (24) is not true, then there a point $z = z_0$ with $|z_0| \geq 1$ such that

$$\left| \{B[P(Rz)] + \Phi_k(R, r, \alpha, \beta) B[P(rz)]\}_{z=z_0} \right| \geq \left| \{B[F(Rz)] + \Phi_k(R, r, \alpha, \beta) B[F(rz)]\}_{z=z_0} \right|,$$

But all the zeros of $F(Rz)$ lie in $|z| < (k/R) < 1$, therefore, it follows (as in case of $G(z)$) that all the zeros of $F(Rz) - \alpha F(rz) + \beta \left\{ \left(\frac{R+k}{k+r} \right)^n - |\alpha| \right\} F(rz)$ lie in $|z| < 1$. Hence, by Lemma 1.3,

$$\{B[F(Rz)] + \Phi_k(R, r, \alpha, \beta) B[F(rz)]\}_{z=z_0} \neq 0$$

with $|z_0| \geq 1$.We take

$$\lambda = \frac{\{B[P(Rz)] + \Phi_k(R, r, \alpha, \beta) B[P(rz)]\}_{z=z_0}}{\{B[P(Rz)] + \Phi_k(R, r, \alpha, \beta) B[P(rz)]\}_{z=z_0}},$$

then λ is a well defined real or complex number with $|\lambda| > 1$ and with this choice of λ, we obtain $\{B[T(z)]\}_{z=z_0} = 0$ where $|z_0| \geq 1$. This contradicts the fact that all the zeros of $B[T(z)]$ lie in $|z| < 1$. Thus (24) holds for $|\alpha| \leq 1$, $|\beta| \leq 1$, $|z| \geq 1$, and $R > r \geq k$.

□

For $\alpha = 0$ in Theorem 2.1, we obtain the following result.

Corollary 2.2. *If $F \in \mathscr{P}_n$ and $F(z)$ has all its zeros in the disk $|z| \leq k$, where $k > 0$ and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \quad for \quad |z| = k,$$

then for $|\beta| \leq 1$, $R > r \geq k$ and $|z| \geq 1$,

$$\left|B[P(Rz)] + \beta\left(\frac{R+k}{k+r}\right)^n B[P(rz)]\right| \leq \left|B[F(Rz)] + \beta\left(\frac{R+k}{k+r}\right)^n B[F(rz)]\right|. \tag{25}$$

The result is sharp, and the equality holds for the polynomial $P(z) = e^{i\gamma}F(z)$ where $\gamma \in \mathbb{R}$.

If we choose $F(z) = z^n M/k^n$, where $M = Max_{|z|=k}|P(z)|$ in Theorem 2.1, we get the following result.

Corollary 2.3. *If $P \in \mathscr{P}_n$ then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k > 0$ and $|z| = 1$,*

$$\begin{aligned}\big|B[P(Rz)]+&\Phi_k(R,r,\alpha,\beta)B[P(rz)]\big| \\ &\leq \frac{1}{k^n}\left|R^n + r^n\Phi_k(R,r,\alpha,\beta)\right|\left|B[z^n]\right|\underset{|z|=k}{Max}\left|P(z)\right|,\end{aligned} \tag{26}$$

where $\Phi_k(R,r,\alpha,\beta)$ is given by (18)*. The result is best possible and equality in* (26) *holds for $P(z) = az^n$, $a \neq 0$.*

Next, we take $P(z) = z^n m/k^n$, where $m = Min_{|z|=k}|P(z)|$ in Theorem 2.1, we get the following result.

Corollary 2.4. *If $F \in \mathscr{P}_n$ and $F(z)$ have all its zeros in the disk $|z| \leq k$, where $k > 0$ then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k > 0$*

$$\begin{aligned} &\underset{|z|=1}{Min}\left|B[F(Rz)]+\Phi_k(R,r,\alpha,\beta)B[F(rz)]\right| \\ &\qquad \geq \frac{|B[z^n]|}{k^n}\left|R^n + r^n\Phi_k(R,r,\alpha,\beta)\right| \underset{|z|=k}{Min}|P(z)|, \end{aligned} \tag{27}$$

where $\Phi_k(R,r,\alpha,\beta)$ is given by (18). The result is Sharp.

If we take $\beta = 0$ in (26), we get the following result.

Corollary 2.5. *If $P \in \mathscr{P}_n$ then for $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $R > r \geq k > 0$ and $|z| \geq 1$,*

$$|B[P(Rz)] - \alpha B[P(rz)]| \leq \frac{1}{k^n}|R^n - \alpha r^n|\,|B[z^n]|\underset{|z|=k}{Max}|P(z)|, \tag{28}$$

The result is best possible as shown by $P(z) = az^n, a \neq 0$.

For polynomials $P \in \mathscr{P}_n$ having no zero in $|z| < k$, we establish the following result which leads to the compact generalization of inequalities (3),(4),(8) and (12).

Theorem 2.6. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in the disk $|z| < k$, where $k \leq 1$, then for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq k > 0$ and $|z| \geq 1$,*

$$\begin{aligned} |B[P(Rz)] + \Phi_k(R,r,\alpha,\beta)B[P(rz)]| \leq \frac{1}{2}\Bigg[&\frac{|B[z^n]|}{k^n}\left|R^n + r^n\Phi_k(R,r,\alpha,\beta)\right| \\ &+ |1 + \Phi_k(R,r,\alpha,\beta)|\,|\lambda_0|\Bigg]\underset{|z|=k}{Max}|P(z)| \end{aligned} \tag{29}$$

where $\Phi_k(R,r,\alpha,\beta)$ is given by (18).

Proof of Theorem 2.6. Since $P(z)$ does not vanish in $|z| < k$, $k \leq 1$, by Lemma 1.3, we have for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > 1$ and $|z| \geq 1$,

$$\begin{aligned} &\left|B[P(Rz)]+\Phi_k(R,r,\alpha,\beta)B[P(rz)]\right| \\ &\qquad \leq k^n\left|B[Q(Rz/k^2)] + \Phi_k(R,r,\alpha,\beta)B[Q(rz/k^2)]\right|, \end{aligned} \tag{30}$$

where $Q(z) = z^n\overline{P(1/\overline{z})}$. Inequality (30) in conjunction with Lemma 1.4 gives for all $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\leq 1,\ |\beta|\leq 1,\ R>r\geq k$ and $|z|\geq 1$,

$$\begin{aligned}2\big|B[P(Rz)]&+\Phi_k(R,r,\alpha,\beta)B[P(rz)]\big|\\ &\leq \big|B[P(Rz)]+\Phi_k(R,r,\alpha,\beta)B[P(rz)]\big|+k^n\left|B[Q(Rz/k^2)]+\Phi_k(R,r,\alpha,\beta)B[Q(rz/k^2)]\right|\\ &\leq \left\{|\lambda_0|\big|1+\Phi_k(R,r,\alpha,\beta)\big|+\frac{|B[z^n]|}{k^n}\left|R^n+r^n\Phi_k(R,r,\alpha,\beta)\right|\right\}|\underset{|z|=k}{Max}\left|P(z)\right|.\end{aligned}$$

This completes the proof of Theorem 2.6.

□

We finally prove the following result, which is the refinement of Theorem 2.6.

Theorem 2.7. *If $P\in\mathscr{P}_n$ and $P(z)$ does not vanish in the disk $|z|<k$, where $k\leq 1$, then for all $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\leq 1$, $|\beta|\leq 1$, $R>r\geq k>0$ and $|z|=1$,*

$$\begin{aligned}\Big|B[P(Rz)]+&\Phi_k(R,r,\alpha,\beta)B[P(rz)]\Big|\\ &\leq\frac{1}{2}\Bigg[\left\{\frac{|B[z^n]|}{k^n}\left|R^n+r^n\Phi_k(R,r,\alpha,\beta)\right|+\left|1+\Phi_k(R,r,\alpha,\beta)\right||\lambda_0|\right\}\underset{|z|=k}{Max}\left|P(z)\right|\\ &\quad-\left\{\frac{|B[z^n]|}{k^n}\left|R^n+r^n\Phi_k(R,r,\alpha,\beta)\right|-\left|1+\Phi_k(R,r,\alpha,\beta)\right||\lambda_0|\right\}\underset{|z|=k}{Min}\left|P(z)\right|\Bigg],\end{aligned}\tag{31}$$

where $\Phi_k(R,r,\alpha,\beta)$ is given by (18).

Proof of Theorem 2.7. Let $m=Min_{|z|=k}\left|P(z)\right|$. If $P(z)$ has a zero on $|z|=k$, then the result follows from Theorem 2.6. We assume that $P(z)$ has all its zeros in $|z|>k$ where $k\leq 1$ so that $m>0$. Now for every δ with $|\delta|<1$, it follows by Rouche's theorem $h(z)=P(z)-\delta m$ does not vanish in $|z|<k$. Applying Lemma 1.3 to the polynomial $h(z)$, we get for all $\alpha,\beta\in\mathbb{C}$ with $|\alpha|\leq 1, |\beta|\leq 1$, $R>r\geq k$ and $|z|\geq 1$

$$|B[h(Rz)]+\Phi_k(R,r,\alpha,\beta)B[h(rz)]|\leq k^n\left|B[q(Rz/k^2)]+\Phi_k(R,r,\alpha,\beta)B[q(rz/k^2)]\right|,$$

where $q(z)=z^n\overline{h(1/\overline{z})}=z^n\overline{P(1/\overline{z})}-\overline{\delta}mz^n$. Equivalently,

$$\begin{aligned}\big|B[P(Rz)]+&\Phi_k(R,r,\alpha,\beta)B[P(rz)]-\delta\lambda_0\left(1+\Phi_k(R,r,\alpha,\beta)\right)m\big|\\ &\leq k^n\bigg|B[Q(Rz/k^2)]+\Phi_k(R,r,\alpha,\beta)B[Q(rz/k^2)]\\ &\qquad-\frac{\overline{\delta}}{k^{2n}}\left(R^n+r^n\Phi_k(R,r,\alpha,\beta)\right)mB[z^n]\bigg| \end{aligned} \tag{32}$$

where $Q(z)=z^n\overline{P(1/\overline{z})}$. Since all the zeros of $Q(z/k^2)$ lie in $|z|\leq k$, $k\leq 1$ by Corollary 2.4 applied to $Q(z/k^2)$, we have for $R>1$ and $|z|=1$,

$$\begin{aligned}\big|B[Q(Rz/k^2)]&+\Phi_k(R,r,\alpha,\beta)B[Q(rz/k^2)]\big|\\ &\geq \frac{1}{k^n}\big|R^n+r^n\Phi_k(R,r,\alpha,\beta)\big||B[z^n]|\underset{|z|=k}{Min}Q(z/k^2)\\ &=\frac{1}{k^{2n}}\big|R^n+r^n\Phi_k(R,r,\alpha,\beta)\big||B[z^n]|m. \end{aligned} \tag{33}$$

Now, choosing the argument of δ on the right hand side of inequality (32) such that

$$\begin{aligned}k^n&\bigg|B[Q(Rz/k^2)]+\Phi_k(R,r,\alpha,\beta)B[Q(rz/k^2)]-\frac{\overline{\delta}}{k^{2n}}\left(R^n+r^n\Phi_k(R,r,\alpha,\beta)\right)mB[z^n]\bigg|\\ &=k^n\big|B[Q(Rz/k^2)]+\Phi_k(R,r,\alpha,\beta)B[Q(rz/k^2)]\big|-\frac{1}{k^n}\big|R^n+r^n\Phi_k(R,r,\alpha,\beta)\big||B[z^n]|m. \end{aligned}$$

for $|z|=1$, which is possible by inequality (33). We get for $|z|=1$,

$$\begin{aligned}\big|B[P(Rz)]+\Phi_k(R,r,\alpha,\beta)B[P(rz)]\big|&-|\delta||\lambda_0||1+\Phi_k(R,r,\alpha,\beta)\big|m\\ &\leq k^n\big|B[Q(Rz/k^2)]+\Phi_k(R,r,\alpha,\beta)B[Q(rz/k^2)]\big|\\ &\qquad-\frac{|\delta|}{k^n}\big|R^n+r^n\Phi_k(R,r,\alpha,\beta)\big||B[z^n]|m. \end{aligned} \tag{34}$$

Equivalently for $|z|=1, R>r\geq k$, we have

$$\begin{aligned}\big|B[P(Rz)]+\Phi_k(R,r,\alpha,\beta)B[P(rz)]\big|-k^n\big|B[Q(Rz/k^2)]+\Phi_k(R,r,\alpha,\beta)B[Q(rz/k^2)]\big|\\ \leq|\delta|\left\{|\lambda_0||1+\Phi_k(R,r,\alpha,\beta)\big|-\frac{1}{k^n}\big|R^n+r^n\Phi_k(R,r,\alpha,\beta)\big||B[z^n]|\right\}m. \end{aligned} \tag{35}$$

Letting $|\delta| \to 1$ in inequality (35), we obtain for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1, |\beta| \leq 1, R > r \geq k$ and $|z| = 1$,

$$\begin{aligned} &\Big|B[P(Rz)] + \Phi_k(R,r,\alpha,\beta)B[P(rz)]\Big| - k^n\Big|B[Q(Rz/k^2)] + \Phi_k(R,r,\alpha,\beta)B[Q(rz/k^2)]\Big| \\ &\qquad \leq \left\{|\lambda_0||1+\Phi_k(R,r,\alpha,\beta)| - \frac{1}{k^n}\Big|R^n + r^n\Phi_k(R,r,\alpha,\beta)\Big||B[z^n]|\right\} m. \end{aligned} \tag{36}$$

Inequality (36) in conjunction with Lemma 1.4 gives for all $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1, R > 1$ and $|z| = 1$,

$$\begin{aligned} &2\Big|B[P(Rz)] + \Phi_k(R,r,\alpha,\beta)B[P(rz)]\Big| \\ &\qquad \leq \left\{|\lambda_0|\Big|1+\Phi_k(R,r,\alpha,\beta)\Big| + \frac{1}{k^n}\left|R^n + r^n\Phi_k(R,r,\alpha,\beta)\right||B[z^n]|\right\} |\underset{|z|=k}{Max}|P(z)| \\ &\qquad + \left\{|\lambda_0||1+\Phi_k(R,r,\alpha,\beta)| - \frac{1}{k^n}\Big|R^n + r^n\Phi_k(R,r,\alpha,\beta)\Big||B[z^n]|\right\} \underset{|z|=k}{Min}|P(z)|. \end{aligned}$$

which is equivalent to inequality (31) and thus completes the proof of theorem 2.7.

□

If we take $\alpha = 0$, we get the following.

Corollary 2.8. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \leq 1$, then for all $\beta \in \mathbb{C}$ with $|\beta| \leq 1$, $R > r \geq k$ and $|z| = 1$,*

$$\begin{aligned} &\left|B[P(Rz)] + \beta\left(\frac{R+k}{k+r}\right)^n B[P(rz)]\right| \\ &\qquad \leq \frac{1}{2}\left[\left\{\frac{|B[z^n]|}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{k+1}\right)^n\right| + \left|1+\beta\left(\frac{R+k}{k+1}\right)^n\right||\lambda_0|\right\}\underset{|z|=k}{Max}|B[P(z)]|\right. \\ &\qquad \left. - \left\{\frac{|B[z^n]|}{k^n}\left|R^n + r^n\beta\left(\frac{R+k}{k+1}\right)^n\right| - \left|1+\beta\left(\frac{R+k}{k+1}\right)^n\right||\lambda_0|\right\}\underset{|z|=k}{Min}|B[P(z)]|\right]. \end{aligned} \tag{37}$$

For $\beta = 0$, Theorem 2.6 reduces to the following result.

Corollary 2.9. *If $P \in \mathscr{P}_n$ and $P(z)$ does not vanish in $|z| < k$ where $k \le 1$, then for all $\alpha \in \mathbb{C}$ with $|\alpha| \le 1$, $R > r \ge k$ and $|z| = 1$,*

$$\begin{aligned} |B[P(Rz)] - \alpha B[P(z)]| \le \frac{1}{2}\Bigg[&\left\{ \frac{|B[z^n]|}{k^n} |R^n - \alpha r^n| + |1-\alpha|\,|\lambda_0| \right\} \underset{|z|=k}{Max} |P(z)| \\ &- \left\{ \frac{|B[z^n]|}{k^n} |R^n - \alpha r^n| - |1-\alpha|\,|\lambda_0| \right\} \underset{|z|=k}{Min} |P(z)| \Bigg]. \end{aligned} \tag{38}$$

The result is sharp and extremal polynomial is $P(z) = az^n + b, |a| = |b| \neq 0$.

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