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NEW OPERATOR PRESERVING L_p INEQUALITIES BETWEEN POLYNOMIALS

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Abstract. For a polynomial $P(z)$ of degree n having no zero in $|z| < 1$, it was recently asserted by Shah and Liman [17] that for every $R \geq 1$, $p \geq 1$,

$$\|B[P \circ \sigma](z)\|_p \leq \frac{R^n|\Lambda_n| + |\lambda_0|}{\|1+z\|_p} \|P(z)\|_p,$$

where B is a B_n-operator with parameters $\lambda_0, \lambda_1, \lambda_2$ in the sense of Rahman and Schmeisser [15], $\Lambda = \lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}$ and $\sigma(z) = Rz,\ R \geq 1$. The proof of this result is incorrect. In this paper, we present certain new L_p inequlities for B_n-operators which not only provide a correct proof of the above inequality and other related results but also extend these inequalities for $0 \leq p < 1$ as well.

Keywords: L^p inequalities, B_n-operators, polynomials.

2000 AMS Subject Classification: 26D10, 41A17

1. Introduction

Let $\mathcal{P}_n$ denote the space of all complex polynomials $P(z) = \sum_{j=0}^{n} a_j z^j$ of degree n. For $P \in \mathcal{P}_n$, define

$$\|P(z)\|_0 := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \log\left|P(e^{i\theta})\right| d\theta\right\},$$

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$$\|P(z)\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left|P(e^{i\theta})\right|^p \right\}^{1/p}, \ 1 \leq p < \infty,$$

$$\|P(z)\|_\infty := \max_{|z|=1} |P(z)|$$

and denote for any complex function $\psi : \mathbb{C} \to \mathbb{C}$ the composite function of P and ψ, defined by $(P \circ \psi)(z) := P(\psi(z)) \quad (z \in \mathbb{C})$, as $P \circ \psi$.

If $P \in \mathcal{P}_n$, then

$$\|P'(z)\|_p \leq n \|P(z)\|_p, \quad p \geq 1 \tag{1}$$

and

$$\|P(Rz)\|_p \leq R^n \|P(z)\|_p, \ \ R > 1, \ \ p > 0, \tag{2}$$

Inequality (1) was found out by Zygmund [18] whereas inequality (2) is a simple consequence of a result of Hardy [8]. Arestov [2] proved that (3) remains true for $0 < p < 1$ as well. For $p = \infty$, the inequality (1) is due to Bernstein (for reference, see [11,15,16]) whereas the case $p = \infty$ of inequality (2) is a simple consequence of the maximum modulus principle (see [11,12,15]). Both the inequalities (1) and (2) can be sharpened if we restrict ourselves to the class of polynomials having no zero in $|z| < 1$. In fact, if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then inequalities (1) and (2) can be respectively replaced by

$$\|P'(z)\|_p \leq n \frac{\|P(z)\|_p}{\|1+z\|_p}, \ \ p \geq 0 \tag{3}$$

and

$$\|P(Rz)\|_p \leq \frac{\|R^n z + 1\|_p}{\|1+z\|_p} \|P(z)\|_p, \ \ R > 1, \ \ p > 0. \tag{4}$$

Inequality (3) is due to De-Bruijn [6](see also [3]) for $p \geq 1$. Rahman and Schmeisser [14] extended it for $0 < p < 1$ whereas the inequality (4) was proved by Boas and Rahman [5] for $p \geq 1$ and later it was extended for $0 < p < 1$ by Rahman and Schmeisser [14]. For $p = \infty$, the inequality (3) was conjectured by Erdös and later verified by Lax [9] whereas inequality (4) was proved by Ankeny and Rivlin [1].

As a compact generalization of inequalities (1) and (2), Aziz and Rather [4] proved that if $P \in \mathcal{P}_n$, then for every real or complex number α with $|\alpha| \leq 1$, $R \geq 1$, and $p > 0$,

$$\|P(Rz) - \alpha P(z)\|_p \leq |R^n - \alpha| \, \|P(z)\|_p . \tag{5}$$

and if $P \in \mathcal{P}_n$ and $P(z) \neq 0$ in $|z| < 1$, then for every real or complex number α with $|\alpha| \leq 1$, $R \geq 1$, and $p > 0$,

$$\|P(Rz) - \alpha P(z)\|_p \leq \frac{\|(R^n - \alpha)z + (1 - \alpha)\|_p}{\|1 + z\|_p} \|P(z)\|_p . \tag{6}$$

Inequality (6) is the corresponding compact generalization of inequalities (3) and (4).

Rahman [13] (see also Rahman and Schmeisser [15, p. 538]) introduced a class B_n of operators B that maps $P \in \mathcal{P}_n$ into itself. That is, the operator B carries $P \in \mathcal{P}_n$ into

$$B[P](z) := \lambda_0 P(z) + \lambda_1 \left(\frac{nz}{2}\right) \frac{P'(z)}{1!} + \lambda_2 \left(\frac{nz}{2}\right)^2 \frac{P''(z)}{2!} \tag{7}$$

where λ_0, λ_1 and λ_2 are such that all the zeros of

$$u(z) := \lambda_0 + C(n,1)\lambda_1 z + C(n,2)\lambda_2 z^2, \;\; C(n,r) = n!/r!(n-r)!, \tag{8}$$

lie in the half plane

$$|z| \leq |z - n/2| \tag{9}$$

and proved that if $P \in \mathcal{P}_n$, then

$$|B[P \circ \sigma](z)| \leq R^n \, |\Lambda_n| \, \|P(z)\|_\infty \;\; for \;\; |z| = 1. \tag{10}$$

and if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then

$$|B[P \circ \sigma](z)| \leq \frac{1}{2} \left\{R^n \, |\Lambda_n| + |\lambda_0|\right\} \|P(z)\|_\infty \;\; for \;\; |z| = 1, \tag{11}$$

(see [13, Inequality (5.2) and (5.3)]) where $\sigma(z) = Rz$, $R \geq 1$ and

$$\Lambda_n := \lambda_0 + \lambda_1 \frac{n^2}{2} + \lambda_2 \frac{n^3(n-1)}{8}. \tag{12}$$

As an extension of inequality (10) to L_p-norm, recently W.M.Shah and A.Liman [17, Theorem 1] proved that if $P \in \mathcal{P}_n$, then for every $R \geq 1$ and $p \geq 1$,

$$\|B[P \circ \sigma](z)\|_p \leq R^n |\Lambda_n| \, \|P(z)\|_p \tag{13}$$

where $B \in B_n$ and $\sigma(z) = Rz$ and Λ_n is defined by (12).

While seeking the desired extension of inequality (11) to L_p-norm, they [17, Theorem 2] have made an incomplete attempt by claiming to have proved that if $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for each $R \geq 1$ and $p \geq 1$,

$$\|B[P \circ \sigma](z)\|_p \leq \frac{R^n|\Lambda_n| + |\lambda_0|}{\|1+z\|_p} \|P(z)\|_p \, . \tag{14}$$

where $B \in B_n$ and $\sigma(z) = Rz$ and Λ_n is defined by (12).

Further, it has been claimed to have proved the inequality (14) for self-inversive polynomials as well.

The proof of inequality (14) and other related results including the Lemma 4 in [17] given by Shah and Liman is not correct. The reason being that the authors in [17] deduce line 10 from line 7 on page 84, line 19 on page 85 from Lemma 3 [17] and line 16 from line 14 on page 86 by using the fact that if $P^*(z) := z^n \overline{P(1/\overline{z})}$, then for $\sigma(z) = Rz$, $R \geq 1$ and $|z| = 1$,

$$|B[P^* \circ \sigma](z)| = |B[(P^* \circ \sigma)^*](z)|,$$

which is not true, in general, for every $R \geq 1$ and $|z| = 1$. To see this, let

$$P(z) = a_n z^n + \cdots + a_k z^k + \cdots + a_1 z + a_0$$

be an arbitrary polynomial of degree n, then

$$P^\star(z) =: z^n \overline{P(1/\overline{z})} = \bar{a_0} z^n + \bar{a_1} z^{n-1} + \cdots + \bar{a_k} z^{n-k} + \cdots + \bar{a_n}.$$

Now with $\mu_1 := \lambda_1 n/2$ and $\mu_2 := \lambda_2 n^2/8$, we have

$$B[P^\star \circ \rho](z) = \sum_{k=0}^{n} \left(\lambda_0 + \mu_1(n-k) + \mu_2(n-k)(n-k-1)\right) \bar{a_k} z^{n-k} R^{n-k},$$

and in particular for $|z| = 1$, we get

$$B[P^\star \circ \rho](z) = R^n z^n \sum_{k=0}^{n} \overline{\left(\lambda_0 + \mu_1(n-k) + \mu_2(n-k)(n-k-1)\right) a_k \left(\frac{z}{R}\right)^k},$$

whence

$$|B[P^\star \circ \rho](z)| = R^n \left| \sum_{k=0}^{n} \overline{(\lambda_0 + \mu_1(n-k) + \mu_2(n-k)(n-k-1))} a_k \left(\frac{z}{R}\right)^k \right|.$$

But

$$|B[(P^\star \circ \rho)^\star](z)| = R^n \left| \sum_{k=0}^{n} \left(\lambda_0 + \mu_1 k + \mu_2 k(k-1)\right) a_k \left(\frac{z}{R}\right)^k \right|,$$

so the asserted identity does not hold in general for every $R \geq 1$ and $|z| = 1$ as e.g. the immediate counterexample of $P(z) := z^n$ demonstrates in view of $P^\star(z) = 1$, $|B[P^\star \circ \rho](z)| = |\lambda_0|$ and

$$|B[(P^\star \circ \rho)^\star](z)| = |\lambda_0 + \lambda_1(n^2/2) + \lambda_2 n^3(n-1)/8|, \quad |z| = 1.$$

The main aim of this paper is to present correct proofs of the results mentioned in [17] by investigating the dependence of

$$\left\| B[P \circ \sigma](z) + \phi_n\left(R, r, \alpha, \beta\right) B[P \circ \rho](z) \right\|_p$$

on $\|P(z)\|_p$ for arbitrary real or complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$, $0 \leq p < \infty$, $\sigma(z) := Rz$, $\rho(z) := rz$ and

$$\phi_n\left(R, r, \alpha, \beta\right) := \beta \left\{ \left(\frac{R+1}{r+1}\right)^n - |\alpha| \right\} - \alpha, \tag{15}$$

and establish certain generalized L_p-mean extensions of the inequalities (10) and (11) for $0 \leq p < \infty$.

2. Lemmas

For the proofs of our results, we need the following lemmas. The first Lemma is easy to prove.

Lemma 2.1. *If $P \in P_n$ and $P(z)$ has all its zeros in $|z| \leq 1$, then for every $R \geq r \geq 1$ and $|z| = 1$,*

$$|P(Rz)| \geq \left(\frac{R+1}{r+1}\right)^n |P(rz)| .$$

The following Lemma follows from Corollary 18.3 of [7, p.65].

Lemma 2.2. *If all the zeros of polynomial $P \in \mathcal{P}_n$ lie in $|z| \leq 1$, then all the zeros of the polynomial $B[P](z)$ also lie in $|z| \leq 1$.*

Lemma 2.3. If $F \in P_n$ *has all its zeros in $|z| \leq 1$ and $P(z)$ is a polynomial of degree at most n such that*

$$|P(z)| \leq |F(z)| \text{ for } |z| = 1,$$

then for arbitrary real or complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$, $R \geq r \geq 1$, and $|z| \geq 1$,

$$|B[P \circ \sigma](z) + \phi(R, r, \alpha, \beta)B[P \circ \rho](z)| \leq |B[P^* \circ \sigma](z) + \phi(R, r, \alpha, \beta)B[P^* \circ \rho](z)| \tag{16}$$

where $P^*(z) := z^n\overline{P(1/\overline{z})}$, $B \in B_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, Λ_n *and* $\phi_n(R, r, \alpha, \beta)$ *are defined by* (12) *and* (15) *respectively.*

Proof. Since the polynomial $F(z)$ of degree n has all its zeros in $|z| \leq 1$ and $P(z)$ is a polynomial of degree at most n such that

$$|P(z)| \leq |F(z)| \text{ for } |z| = 1, \tag{17}$$

therefore, if $F(z)$ has a zero of multiplicity s at $z = e^{i\theta_0}$, then $P(z)$ has a zero of multiplicity at least s at $z = e^{i\theta_0}$. If $P(z)/F(z)$ is a constant, then the inequality (16) is obvious.

We now assume that $P(z)/F(z)$ is not a constant, so that by the maximum modulus principle, it follows that

$$|P(z)| < |F(z)| \text{ for } |z| > 1 .$$

Suppose $F(z)$ has m zeros on $|z| = 1$ where $0 \leq m \leq n$, so that we can write

$$F(z) = F_1(z)F_2(z)$$

where $F_1(z)$ is a polynomial of degree m whose all zeros lie on $|z| = 1$ and $F_2(z)$ is a polynomial of degree exactly $n - m$ having all its zeros in $|z| < 1$. This implies with the help of inequality (17) that

$$P(z) = P_1(z)F_1(z)$$

where $P_1(z)$ is a polynomial of degree at most $n - m$. Now, from inequality (17), we get

$$|P_1(z)| \leq |F_2(z)| \text{ for } |z| = 1$$

where $F_2(z) \neq 0\ for\ |z| = 1$. Therefore for every real or complex number λ with $|\lambda| > 1$, a direct application of Rouche's theorem shows that the zeros of the polynomial $P_1(z) - \lambda F_2(z)$ of degree $n - m \geq 1$ lie in $|z| < 1$. Hence the polynomial

$$f(z) = F_1(z)\left(P_1(z) - \lambda F_2(z)\right) = P(z) - \lambda F(z)$$

has all its zeros in $|z| \leq 1$ with at least one zero in $|z| < 1$, so that we can write

$$f(z) = (z - te^{i\delta})H(z)$$

where $t < 1$ and $H(z)$ is a polynomial of degree $n - 1$ having all its zeros in $|z| \leq 1$. Applying Lemma 1.1 to the polynomial $f(z)$ with $k = 1$, we obtain for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned}
|f(Re^{i\theta})| &= |Re^{i\theta} - te^{i\delta}||H(Re^{i\theta})| \\
&\geq |Re^{i\theta} - te^{i\delta}| \left(\frac{R+1}{r+1}\right)^{n-1} |H(re^{i\theta})| \\
&= \left(\frac{R+1}{r+1}\right)^{n-1} \frac{|Re^{i\theta} - te^{i\delta}|}{|re^{i\theta} - te^{i\delta}|} |(re^{i\theta} - te^{i\delta})H(re^{i\theta})| \\
&\geq \left(\frac{R+1}{r+1}\right)^{n-1} \left(\frac{R+t}{r+t}\right) |f(re^{i\theta})|.
\end{aligned}$$

This implies for $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\left(\frac{r+t}{R+t}\right)|f(Re^{i\theta})| \geq \left(\frac{R+1}{r+1}\right)^{n-1}|f(re^{i\theta})|. \tag{18}$$

Since $R > r \geq 1 > t$ so that $f(Re^{i\theta}) \neq 0$ for $0 \leq \theta < 2\pi$ and $\frac{1+r}{1+R} > \frac{r+t}{R+t}$, from inequality (18), we obtain $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$|f(Re^{i\theta}| > \left(\frac{R+1}{r+1}\right)^{n}|f(re^{i\theta})|. \tag{19}$$

Equivalently,

$$|f(Rz)| > \left(\frac{R+1}{r+1}\right)^{n}|f(rz)|$$

for $|z| = 1$ and $R > r \geq 1$. Hence for every real or complex number α with $|\alpha| \leq 1$ and $R > r \geq 1$, we have

$$\begin{aligned} |f(Rz) - \alpha f(rz)| &\geq |f(Rz)| - |\alpha||f(rz)| \\ &> \left\{\left(\frac{R+1}{r+1}\right)^{n} - |\alpha|\right\}|f(rz)|, \quad |z| = 1. \end{aligned} \tag{20}$$

Also, inequality (19) can be written in the form

$$|f(re^{i\theta})| < \left(\frac{r+1}{R+1}\right)^{n}|f(Re^{i\theta})| \tag{21}$$

for every $R > r \geq 1$ and $0 \leq \theta < 2\pi$. Since $f(Re^{i\theta}) \neq 0$ and $\left(\frac{r+1}{R+1}\right)^{n} < 1$, from inequality (21), we obtain for $0 \leq \theta < 2\pi$ and $R > r \geq 1$,

$$|f(re^{i\theta}| < |f(Re^{i\theta}).$$

Equivalently,

$$|f(rz)| < |f(Rz)| \text{ for } |z| = 1.$$

Since all the zeros of $f(Rz)$ lie in $|z| \leq (1/R) < 1$, a direct application of Rouche's theorem shows that the polynomial $f(Rz) - \alpha f(rz)$ has all its zeros in $|z| < 1$ for every real or complex number α with $|\alpha| \leq 1$. Applying Rouche's theorem again, it follows from (20) that for arbitrary real or complex numbers α, β with $|\alpha| \leq 1, |\beta| \leq 1$ and $R > r \geq 1$, all the zeros of the polynomial

$$
\begin{aligned}
T(z) =& f(Rz) - \alpha f(rz) + \beta \left\{ \left(\frac{R+1}{r+1} \right)^n - |\alpha| \right\} f(rz) \\
&= f(Rz) + \phi(R, r, \alpha, \beta) f(rz) \\
&= \big(P(Rz) - \lambda F(Rz)\big) + \phi(R, r, \alpha, \beta)\big(P(rz) - \lambda F(rz)\big) \\
&= \big(P(Rz) + \phi(R, r, \alpha, \beta) P(rz)\big) - \lambda\big(F(Rz) + \phi(R, r, \alpha, \beta) F(rz)\big)
\end{aligned}
$$

lie in $|z| < 1$ for every λ with $|\lambda| > 1$. Using Lemma 2.2 and the fact that B is a linear operator, we conclude that all the zeros of polynomial

$$
\begin{aligned}
W(z) &= B[T](z) \\
&= (B[P \circ \sigma](z) + \phi(R, r, \alpha, \beta) B[P \circ \rho](z)) \\
&\quad -\lambda(B[F \circ \sigma](z) + \phi(R, r, \alpha, \beta) B[F \circ \rho](z))
\end{aligned}
$$

also lie in $|z| < 1$ for every λ with $|\lambda| > 1$. This implies

$$
|B[P \circ \sigma](z) + \phi(R, r, \alpha, \beta) B[P \circ \rho](z)| \leq |B[F \circ \sigma](z) + \phi(R, r, \alpha, \beta) B[F \circ \rho](z)| \tag{22}
$$

for $|z| \geq 1$ and $R > r \geq 1$. If inequality (22) is not true, then exist a point $z = z_0$ with $|z_0| \geq 1$ such that

$$
|B[P \circ \sigma](z_0) + \phi(R, r, \alpha, \beta) B[P \circ \rho](z_0)| > |B[F \circ \sigma](z_0) + \phi(R, r, \alpha, \beta) B[F \circ \rho](z_0)|.
$$

But all the zeros of $F(Rz)$ lie in $|z| < 1$, therefore, it follows (as in case of $f(z)$) that all the zeros of $F(Rz) + \phi(R, r, \alpha, \beta) F(rz)$ lie in $|z| < 1$. Hence by Lemma 2.2, all the zeros of $B[F \circ \sigma](z) + \phi(R, r, \alpha, \beta) B[F \circ \rho](z)$ also lie in $|z| < 1$, which shows that

$$
B[F \circ \sigma](z_0) + \phi(R, r, \alpha, \beta) B[F \circ \rho](z_0) \neq 0.
$$

We take

$$
\lambda = \frac{B[P \circ \sigma](z_0) + \phi(R, r, \alpha, \beta) B[P \circ \rho](z_0)}{B[F \circ \sigma](z_0) + \phi(R, r, \alpha, \beta) B[F \circ \rho](z_0)},
$$

then λ is a well defined real or complex number with $|\lambda| > 1$ and with this choice of λ, we obtain $W(z_0) = 0$. This contradicts the fact that all the zeros of $W(z)$ lie in $|z| < 1$.

Thus (22) holds and this completes the proof of Lemma 2.3.

Lemma 2.4. If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for arbitrary real or complex numbers α, β with $|\alpha| \le 1, |\beta| \le 1, R > r \ge 1$ and $|z| \ge 1$,

$$\begin{aligned} &|B[P\circ\sigma](z)+\phi(R,r,\alpha,\beta)\,B[P\circ\rho](z)| \\ &\qquad \le |B[P^*\circ\sigma](z)+\phi(R,r,\alpha,\beta)\,B[P^*\circ\rho](z)| \end{aligned} \tag{23}$$

where $P^*(z) := z^n\overline{P(1/\overline{z})}$, $B \in B_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, and $\phi(R,r,\alpha,\beta)$ is defined by (15).

Proof. By hpyothesis the polynomial $P(z)$ of degree n does not vanish in $|z| < 1$, therefore, all the zeros of the polynomial $P^*(z) = z^n\overline{P(1/\overline{z})}$ of degree n lie in $|z| \le 1$. Applying Lemma 2.3 with $F(z)$ replaced by $P^*(z)$, it follows that

$$\begin{aligned} &|B[P\circ\sigma](z)+\phi(R,r,\alpha,\beta)\,B[P\circ\rho](z)| \\ &\qquad \le |B[P^*\circ\sigma](z)+\phi(R,r,\alpha,\beta)\,B[P^*\circ\rho](z)| \end{aligned}$$

for $|z| \ge 1, |\alpha| \le 1, |\beta| \le 1$ and $R > r \ge 1$. This proves the Lemma 2.4.

Next we describe a result of Arestov[2].

For $\gamma = (\gamma_0, \gamma_1, \cdots, \gamma_n) \in \mathbb{C}^{n+1}$ *and* $P(z) = \sum_{j=0}^{n} a_j z^j$, we define

$$C_\gamma P(z) = \sum_{j=0}^{n} \gamma_j a_j z^j.$$

The operator C_γ is said to be admissible if it preserves one of the following properties:

(i) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \le 1\}$,

(ii) $P(z)$ has all its zeros in $\{z \in \mathbb{C} : |z| \ge 1\}$.

The result of Arestov may now be stated as follows.

Lemma 2.5. [2,Th.2] *Let $\phi(x) = \psi(\log x)$ where ψ is a convex nondecreasing function on $\mathbb{R}$. Then for all $P \in \mathcal{P}_n$ and each admissible operator Λ_γ,*

$$\int_0^{2\pi} \phi\left(|C_\gamma P(e^{i\theta})|\right) d\theta \le \int_0^{2\pi} \phi\left(c(\gamma, n)|P(e^{i\theta})|\right) d\theta$$

where $c(\gamma, n) = max\left(|\gamma_0|, |\gamma_n|\right)$.

In particular Lemma 2.5 applies with $\phi : x \to x^p$ for every $p \in (0, \infty)$ and $\phi : x \to \log x$ as well. Therefore, we have for $0 \le p < \infty$,

$$\left\{\int_0^{2\pi} \phi\left(|C_\gamma P(e^{i\theta})|^p\right) d\theta\right\}^{1/p} \le c(\gamma, n)\left\{\int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\right\}^{1/p}. \tag{24}$$

From Lemma 2.5, we deduce the following result.

Lemma 2.6. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for each $p > 0$,* $R > 1$ and η real, $0 \le \eta < 2\pi$,

$$\begin{aligned}\int_0^{2\pi} &|\big(B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta)B[P \circ \rho](e^{i\theta})\big)e^{i\eta} \\ &\qquad + \big(B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta})B[P^* \circ \rho]^*(e^{i\theta})\big)|^p d\theta \\ &\le |(R^n + \phi_n(R, r,\alpha, \beta)r^n)\Lambda_n e^{i\eta} + (1 + \phi_n(R, r, \bar{\alpha}, \bar{\beta}))\bar{\lambda_0}|^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta\end{aligned}$$

where $B \in B_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, $B[P^* \circ \sigma]^*(z) := (B[P^* \circ \sigma](z))^*$, Λ_n *and* $\phi_n(R, r, \alpha, \beta)$ *are defined by* (12) *and* (15) *respectively.*

Proof. Since $P(z)$ does not vanish in $|z| < 1$ and $P^*(z) = z^n\overline{P(1/\bar{z})}$, by Lemma 3, we have for $R > r \ge 1$,

$$\begin{aligned}&|B[P \circ \sigma](z) + \phi\left(R, r, \alpha, \beta\right)B[P \circ \rho](z) \\ &\qquad\qquad \le |B[P^* \circ \sigma](z) + \phi\left(R, r, \alpha, \beta\right)B[P^* \circ \rho](z)|\end{aligned} \tag{25}$$

Also, since

$P^*(Rz) + \phi(R,r,\alpha,\beta)\, P^*(rz) = R^n z^n \overline{P(1/R\bar{z})} + \phi(R,r,\alpha,\beta)\, r^n z^n \overline{P(1/r\bar{z})}$, therefore,

$$\begin{aligned}
&B[P^* \circ \sigma](z) + \phi_n(R,r,\alpha,\beta) B[P^* \circ \rho](z)\\
&= \lambda_0\big(R^n z^n \overline{P(1/R\bar{z})} + \phi(R,r,\alpha,\beta)\, r^n z^n \overline{P(1/r\bar{z})}\big) + \lambda_1 \left(\frac{nz}{2}\right) \Big(nR^n z^{n-1} \overline{P(1/R\bar{z})}\\
&\quad - R^{n-1} z^{n-2} \overline{P'(1/R\bar{z})} + \phi(R,r,\alpha,\beta) \big(nr^n z^{n-1} \overline{P(1/r\bar{z})} - r^{n-1} z^{n-2} \overline{P'(1/r\bar{z})}\big)\Big)\\
&\quad + \frac{\lambda_2}{2!}\left(\frac{nz}{2}\right)^2 \Big(n(n-1)R^n z^{n-2} \overline{P(1/R\bar{z})} - 2(n-1)R^{n-1} z^{n-3} \overline{P'(1/R\bar{z})}\\
&\quad + R^{n-2} z^{n-4} \overline{P''(1/R\bar{z})} + \phi(R,r,\alpha,\beta) \big(n(n-1) r^n z^{n-2} \overline{P(1/r\bar{z})}\\
&\quad - 2(n-1) r^{n-1} z^{n-3} \overline{P'(1/r\bar{z})} + r^{n-2} z^{n-4} \overline{P''(1/r\bar{z})}\big)\Big)
\end{aligned}$$

and hence,

$$\begin{aligned}
&B[P^* \circ \sigma]^*(z) + \phi\big(R,r,\bar{\alpha},\bar{\beta}\big)\, B[P^* \circ \rho]^*(z)\\
&= \big(B[P^* \circ \sigma](z) + \phi(R,r,\alpha,\beta)\, B[P^* \circ \rho](z)\big)^*\\
&= \left(\bar{\lambda}_0 + \bar{\lambda}_1 \frac{n^2}{2} + \bar{\lambda}_2 \frac{n^3(n-1)}{8}\right) \big(R^n P(z/R) + \phi\big(R,r,\bar{\alpha},\bar{\beta}\big)\, r^n P(z/r)\big)\\
&\quad - \left(\bar{\lambda}_1 \frac{n}{2} + \bar{\lambda}_2 \frac{n^2(n-1)}{4}\right) \Big(R^{n-1} z P'(z/R) + \phi\big(R,r,\bar{\alpha},\bar{\beta}\big)\, r^{n-1} z P'(z/r)\Big)\\
&\quad + \bar{\lambda}_2 \frac{n^2}{8} \Big(R^{n-2} z^2 P''(z/R) + \phi\big(R,r,\bar{\alpha},\bar{\beta}\big)\, r^{n-2} z^2 P''(z/r)\Big). \qquad (26)
\end{aligned}$$

Also, for $|z| = 1$

$$\begin{aligned}
|B[P^* \circ \sigma](z) &+ \phi(R,r,\alpha,\beta)\, B[P^* \circ \rho](z)|\\
&= |B[P^* \circ \sigma]^*(z) + \phi\big(R,r,\bar{\alpha},\bar{\beta}\big)\, B[P^* \circ \rho]^*(z)|.
\end{aligned}$$

Using this in (25), we get for $|z| = 1$ and $R > r \geq 1$,

$$\begin{aligned}
|B[P \circ \sigma](z) &+ \phi(R,r,\alpha,\beta)\, B[P \circ \rho](z)|\\
&\leq |B[P^* \circ \sigma]^*(z) + \phi\big(R,r,\bar{\alpha},\bar{\beta}\big)\, B[P^* \circ \rho]^*(z)|.
\end{aligned}$$

Since all the zeros of $P^*(z)$ lie in $|z| \leq 1$, as before, all the zeros of $P^*(Rz) + \phi_n(R,r,\alpha,\beta)P^*(rz)$ lie in $|z| < 1$ for all real or complex numbers α, β with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R > r \geq 1$. Hence by Lemma 2.2, all the zeros of $B[P^* \circ \sigma](z) + \phi_n(R,r,\alpha,\beta)B[P^* \circ$

$\rho](z)$ lie in $|z| < 1$, therefore, all the zeros of $B[P^* \circ \sigma]^*(z) + \phi_n(R, r, \bar{\alpha}, \bar{\beta})B[P^* \circ \rho]^*(z)$ lie in $|z| > 1$. Hence by the maximum modulus principle,

$$\begin{aligned} &|B[P \circ \sigma](z) + \phi\,(R, r, \alpha, \beta)\, B[P^* \circ \rho](z)| \\ &\qquad < |B[P^* \circ \sigma]^*(z) + \phi\left(R, r, \bar{\alpha}, \bar{\beta}\right) B[P^* \circ \rho]^*(z)| \quad \text{for} \quad |z| < 1. \end{aligned} \tag{27}$$

A direct application of Rouche's theorem shows that

$$\begin{aligned} C_\gamma P(z) =& \big(B[P \circ \sigma](z) + \phi_n(R, r, \alpha, \beta)B[P \circ \rho](z)\big)e^{i\eta} \\ &+ \big(B[P^* \circ \sigma]^*(z) + \phi_n(R, r, \bar{\alpha}, \bar{\beta})B[P^* \circ \rho]^*(z)\big) \\ =& \left\{(R^n + \phi_n(R, r, \alpha, \beta)r^n)\Lambda_n e^{i\eta} + (1 + \phi_n(R, r, \bar{\alpha}, \bar{\beta}))\bar{\lambda_0}\right\} a_n z^n \\ &+ \cdots + \left\{(R^n + \phi_n(R, r, \bar{\alpha}, \bar{\beta})r^n)\bar{\Lambda_n} + e^{i\eta}(1 + \phi_n(R, r, \alpha, \beta))\lambda_0\right\} a_0 \end{aligned}$$

does not vanish in $|z| < 1$. Therefore, C_γ is an admissible operator. Applying (24) of Lemma 2.5, the desired result follows immediately for each $p > 0$.

From Lemma 2.6, we deduce the following more general result.

Lemma 2.7. *If* $P \in P_n$, *then for every* $p > 0$, $R > 1$ and η real, $0 \le \eta < 2\pi$,

$$\begin{aligned} &\int_0^{2\pi} |\big(B[P \circ \sigma](e^{i\theta}) + \phi_n(R, r, \alpha, \beta)B[P \circ \rho](e^{i\theta})\big)e^{i\eta} \\ &\qquad\qquad + \big(B[P^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta})B[P^* \circ \rho]^*(e^{i\theta})\big)|^p d\theta \\ &\le |(R^n + \phi_n(R, r, \alpha, \beta)r^n)\Lambda_n e^{i\eta} + (1 + \phi_n(R, r, \bar{\alpha}, \bar{\beta}))\bar{\lambda_0}|^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta \end{aligned}$$

where $B \in B_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, $B[P^* \circ \sigma]^*(z) := (B[P^* \circ \sigma](z))^*$, Λ_n *and* $\phi_n\,(R, r, \alpha, \beta)$ *are defined by* (12) *and* (15) *respectively.*

Proof. If all the zeros of $P(z)$ lie in $|z| \ge 1$, then the result follows by Lemma 2.6. Henceforth, we assume that $P(z)$ has at least one zero in $|z| < 1$ so that we can write

$$P(z) = P_1(z)P_2(z) = a\prod_{j=1}^{k}(z - z_j) \prod_{j=k+1}^{n}(z - z_j), \;\; 0 \le k \le n-1, \;\; a \ne 0$$

where all the zeros of $P_1(z)$ lie in $|z| \geq 1$ and all the zeros of $P_2(z)$ lie in $|z| < 1$. First we assume that $P_1(z)$ has no zero on $|z| = 1$ so that all the zeros of $P_1(z)$ lie in $|z| > 1$. Let $P_2^*(z) = z^{n-k}\overline{P_2(1/\bar{z})}$, then all the zeros of $P_2^*(z)$ lie in $|z| > 1$ and $|P_2^*(z)| = |P_2(z)|$ for $|z| = 1$. Now consider the polynomial

$$f(z) = P_1(z)P_2^*(z) = a\prod_{j=1}^{k}(z - z_j)\prod_{j=k+1}^{n}(1 - z\bar{z}_j),$$

then all the zeros of $f(z)$ lie in $|z| > 1$ and for $|z| = 1$,

$$|f(z)| = |P_1(z)|\,|P_2^*(z)| = |P_1(z)|\,|P_2(z)| = |P(z)|\,. \tag{28}$$

Therefore, it follows by Rouche's theorem that the polynomial $g(z) = P(z) + \mu f(z)$ does not vanish in $|z| \leq 1$ for every μ with $|\mu| > 1$, so that all the zeros of $g(z)$ lie in $|z| \geq \delta$ for some $\delta > 1$ and hence all the zeros of $T(z) = g(\delta z)$ lie in $|z| \geq 1$. Applying (27) and (26) to the polynomial $T(z)$, we get for $R > 1$ and $|z| < 1$,

$$\begin{aligned}
&|B[T\circ\sigma](z) + \phi\left(R,r,\alpha,\beta\right)B[T\circ\rho](z)| \\
&\quad < |B[T^*\circ\sigma]^*(z) + \phi\left(R,r,\bar{\alpha},\bar{\beta}\right)B[T^*\circ\rho]^*(z)| \\
&\quad = |\left(\bar{\lambda}_0 + \bar{\lambda}_1\frac{n^2}{2} + \bar{\lambda}_2\frac{n^3(n-1)}{8}\right)\left(R^nT(z/R) + \phi\left(R,r,\bar{\alpha},\bar{\beta}\right)r^nT(z/r)\right) \\
&\quad - \left(\bar{\lambda}_1\frac{n}{2} + \bar{\lambda}_2\frac{n^2(n-1)}{4}\right)\left(R^{n-1}zT'(z/R) + \phi\left(R,r,\bar{\alpha},\bar{\beta}\right)r^{n-1}zT'(z/r)\right) \\
&\quad\quad + \bar{\lambda}_2\frac{n^2}{8}\left(R^{n-2}z^2T''(z/R) + \phi\left(R,r,\bar{\alpha},\bar{\beta}\right)r^{n-2}z^2T''(z/r)\right)|,
\end{aligned}$$

that is,

$$\begin{aligned}
&|B[T\circ\sigma](z) + \phi\left(R,r,\alpha,\beta\right)B[T\circ\rho](z)| \\
&\quad = |\left(\bar{\lambda}_0 + \bar{\lambda}_1\frac{n^2}{2} + \bar{\lambda}_2\frac{n^3(n-1)}{8}\right)\left(R^ng(\delta z/R) + \phi\left(R,r,\bar{\alpha},\bar{\beta}\right)r^ng(\delta z/r)\right) \\
&\quad - \left(\bar{\lambda}_1\frac{n}{2} + \bar{\lambda}_2\frac{n^2(n-1)}{4}\right)\left(R^{n-1}\delta zg'(\delta z/R) + \phi\left(R,r,\bar{\alpha},\bar{\beta}\right)r^{n-1}\delta zg'(\delta z/r)\right) \\
&\quad\quad + \bar{\lambda}_2\frac{n^2}{8}\left(R^{n-2}\delta^2z^2g''(\delta z/R) + \phi\left(R,r,\bar{\alpha},\bar{\beta}\right)r^{n-2}\delta^2z^2g''(\delta z/r)\right)|
\end{aligned}$$

for $|z| < 1$. If $z = e^{i\theta}/\delta, 0 \leq \theta < 2\pi$, then $|z| = (1/\delta) < 1$ as $\delta > 1$ and we get

$$\begin{aligned}
&|B[T \circ \sigma](e^{i\theta}/\delta) + \phi_n(R, r, \alpha, \beta) B[T \circ \rho](e^{i\theta}/\delta)| \\
&= |\left(\bar{\lambda_0} + \bar{\lambda_1}\frac{n^2}{2} + \bar{\lambda_2}\frac{n^3(n-1)}{8}\right)\left(R^n g(e^{i\theta}/R) + \phi\left(R, r, \bar{\alpha}, \bar{\beta}\right) r^n g(e^{i\theta}/r)\right) \\
&- \left(\bar{\lambda_1}\frac{n}{2} + \bar{\lambda_2}\frac{n^2(n-1)}{4}\right)\left(R^{n-1}e^{i\theta}g'(e^{i\theta}/R) + \phi\left(R, r, \bar{\alpha}, \bar{\beta}\right) r^{n-1}e^{i\theta}g'(e^{i\theta}/r)\right) \\
&\quad + \bar{\lambda_2}\frac{n^2}{8}\left(R^{n-2}e^{2i\theta}g''(e^{i\theta}/R) + \phi\left(R, r, \bar{\alpha}, \bar{\beta}\right) r^{n-2}e^{2i\theta}g''(e^{i\theta}/r)\right)| \\
&= |B[g^* \circ \sigma]^*(e^{i\theta}) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[g^* \circ \rho]^*(e^{i\theta})|.
\end{aligned}$$

Equivalently for $|z| = 1$,

$$\begin{aligned}
|B[g \circ \sigma](z)] + &\phi\left(R, r, \alpha, \beta\right) B[g \circ \rho](z)| \\
&< |B[g^* \circ \sigma]^*(z) + \phi\left(R, r, \bar{\alpha}, \bar{\beta}\right) B[g^* \circ \rho]^*(z)|.
\end{aligned}$$

Since all the zeros of $g(z)$ lie in $|z| \geq 1$, all the zeros of $g^*(z) = z^n\overline{g(1/\bar{z})}$ lie in $|z| \leq 1$ and hence as before, all the zeros of $g^*(Rz) + \phi(R, r, \alpha, \beta)g^*(rz)$ lie in $|z| < 1$. By Lemma 2.2, all the zeros of $B[g^* \circ \sigma](z) + \phi(R, r, \alpha, \beta)B[g^* \circ \rho](z)$ lie in $|z| < 1$ and therefore, all the zeros of $B[g^* \circ \sigma]^*(z) + \phi(R, r, \bar{\alpha}, \bar{\beta})B[g^* \circ \rho]^*(z)$ lie in $|z| > 1$. Thus

$$B[g^* \circ \sigma]^*(z) + \phi(R, r, \bar{\alpha}, \bar{\beta})B[g^* \circ \rho]^*(z) \neq 0 \quad \text{for} \quad |z| \leq 1.$$

An application of Rouche's theorem shows that the polynomial

$$\begin{aligned}
M(z) = &\big(B[g \circ \sigma](z) + \phi_n(R, r, \alpha, \beta) B[g \circ \rho](z)\big)e^{i\eta} \\
&+ B[g^* \circ \sigma]^*(z) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[g^* \circ \rho]^*(z)
\end{aligned} \tag{29}$$

does not vanish in $|z| \leq 1$. Replacing $g(z)$ by $P(z) + \mu f(z)$ and noting that B is a linear operator, it follows that the polynomial

$$\begin{aligned}
M(z) = \big(B[P \circ \sigma](z) + &\phi_n(R, r, \alpha, \beta) B[P \circ \rho](z)\big)e^{i\eta} \\
&+ \big(B[P^* \circ \sigma]^*(z) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[P^* \circ \rho]^*(z)\big) \\
+\mu\big(\big(B[f \circ \sigma](z) &+ \phi_n(R, r, \alpha, \beta) B[f \circ \rho](z)\big)e^{i\eta} \\
&+ (B[f^* \circ \sigma]^*(z) + \phi_n(R, r, \bar{\alpha}, \bar{\beta}) B[f^* \circ \rho]^*(z))\big)
\end{aligned} \tag{30}$$

does not vanish in $|z| \leq 1$ for every μ with $|\mu| > 1$.

We claim

$$\begin{aligned}
&|\big(B[P\circ\sigma](z)+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](z)\big)e^{i\eta}\\
&\qquad\qquad + B[P^*\circ\sigma]^*(z)+\phi_n(R,r,\bar{\alpha},\bar{\beta})B[P^*\circ\rho]^*(z)|\\
&\leq |\big(B[f\circ\sigma](z)+\phi_n(R,r,\alpha,\beta)B[f\circ\rho](z)\big)e^{i\eta}\\
&\qquad\qquad + B[f^*\circ\sigma]^*(z)+\phi_n(R,r,\bar{\alpha},\bar{\beta})B[f^*\circ\rho]^*(z)| \qquad (31)
\end{aligned}$$

for $|z| \leq 1$. If inequality (31) is not true, then there a point $z = z_0$ with $|z_0| \leq 1$ such that

$$\begin{aligned}
&|\big(B[P\circ\sigma](z_0)+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](z_0)\big)e^{i\eta}\\
&\qquad\qquad + B[P^*\circ\sigma]^*(z_0)+\phi_n(R,r,\bar{\alpha},\bar{\beta})B[P^*\circ\rho]^*(z_0)|\\
&> |\big(B[f\circ\sigma](z_0)+\phi_n(R,r,\alpha,\beta)B[f\circ\rho](z_0)\big)e^{i\eta}\\
&\qquad\qquad + B[f^*\circ\sigma]^*(z_0)+\phi_n(R,r,\bar{\alpha},\bar{\beta})B[f^*\circ\rho]^*(z_0)|
\end{aligned}$$

Since $f(z)$ does not vanish in $|z| \leq 1$, proceeding similarly as in the proof of (29), it follows that the polynomial

$$\begin{aligned}
&\big(B[f\circ\sigma](z)+\phi_n(R,r,\alpha,\beta)B[f\circ\rho](z)\big)e^{i\eta}\\
&\qquad\qquad + B[f^*\circ\sigma]^*(z)+\phi_n(R,r,\bar{\alpha},\bar{\beta})B[f^*\circ\rho]^*(z)
\end{aligned}$$

does not vanish in $|z| \leq 1$. Hence

$$\begin{aligned}
&\big(B[f\circ\sigma](z_0)+\phi_n(R,r,\alpha,\beta)B[f\circ\rho](z_0)\big)e^{i\eta}\\
&\qquad\qquad + B[f^*\circ\sigma]^*(z_0)+\phi_n(R,r,\bar{\alpha},\bar{\beta})B[f^*\circ\rho]^*(z_0) \neq 0.
\end{aligned}$$

We take

$$\mu = -\frac{(B[P\circ\sigma](z_0)+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](z_0))e^{i\eta}+B[P^*\circ\sigma]^*(z_0)+\phi_n(R,r,\bar{\alpha},\bar{\beta})B[P^*\circ\rho]^*(z_0)}{(B[f\circ\sigma](z_0)+\phi_n(R,r,\alpha,\beta)B[f\circ\rho](z_0))e^{i\eta}+B[f^*\circ\sigma]^*(z_0)+\phi_n(R,r,\bar{\alpha},\bar{\beta})B[f^*\circ\rho]^*(z_0)},$$

so that μ is well-defined real or complex number with $|\mu| > 1$ and with this choice of μ, from (30), we get $M(z_0) = 0$. This clearly is a contradiction to the fact that $M(z)$ does not vanish in $|z| \leq 1$. Thus (31) holds, which in particular gives for each $p > 0$ and η real,

$$\begin{aligned}
&\int_0^{2\pi} |\big(B[P\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\big)e^{i\theta} \\
&\qquad\qquad + B[P^*\circ\sigma]^*(e^{i\theta}) + \phi_n(R,r,\bar{\alpha},\bar{\beta})B[P^*\circ\rho]^*(e^{i\theta})|^p d\theta \\
&\leq \int_0^{2\pi} |\big(B[f\circ\sigma](z) + \phi_n(R,r,\alpha,\beta)B[f\circ\rho](z)\big)e^{i\theta} \\
&\qquad\qquad + B[f^*\circ\sigma]^*(e^{i\theta}) + \phi_n(R,r,\bar{\alpha},\bar{\beta})B[f^*\circ\rho]^*(e^{i\theta})|d\theta
\end{aligned} \tag{32}$$

Using Lemma 2.7 and (28), we get for each $p > 0$,

$$\begin{aligned}
&\int_0^{2\pi} |\big(B[P\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\big)e^{i\eta} \\
&\qquad\qquad + B[P^*\circ\sigma]^*(e^{i\theta}) + \phi_n(R,r,\bar{\alpha},\bar{\beta})B[P^*\circ\rho]^*(e^{i\theta})|^p d\theta \\
&\leq |(R^n + \phi_n(R,r,\alpha,\beta)r^n)\Lambda_n e^{i\eta} + (1+\phi_n(R,r,\bar{\alpha},\bar{\beta}))\bar{\lambda_0}|^p \int_0^{2\pi} \left|f(e^{i\theta})\right|^p d\theta \\
&= |(R^n + \phi_n(R,r,\alpha,\beta)r^n)\Lambda_n e^{i\eta} + (1+\phi_n(R,r,\bar{\alpha},\bar{\beta}))\bar{\lambda_0}|^p \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta
\end{aligned} \tag{33}$$

Now if $P_1(z)$ has a zero on $|z| = 1$, then applying (33) to the polynomial $Q(z) = P_1(tz)P_2(z)$ where $t < 1$, we get for each $p > 0$, $R > r \geq 1$ and η real,

$$\begin{aligned}
&\int_0^{2\pi} |\big(B[Q\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[Q\circ\rho](e^{i\theta})\big)e^{i\eta} \\
&\qquad\qquad +\big(B[Q^*\circ\sigma]^*(e^{i\theta}) + \phi_n(R,r,\bar{\alpha},\bar{\beta})B[Q^*\circ\rho]^*(e^{i\theta})\big)|^p d\theta \\
&\leq |(R^n + \phi_n(R,r,\alpha,\beta)r^n)\Lambda_n e^{i\eta} + (1+\phi_n(R,r,\bar{\alpha},\bar{\beta}))\bar{\lambda_0}|^p \int_0^{2\pi} \left|Q(e^{i\theta})\right|^p d\theta.
\end{aligned} \tag{34}$$

Letting $t \to 1$ in (34) and using continuity, the desired result follows immediately and this proves Lemma 2.7.

Lemma 2.8. *If* $P \in P_n$ and $P^*(z) = z^n\overline{P(1/\bar{z})}$, *then for every* $p > 0$, $\alpha, \beta \in \mathbb{C}$ with $|\alpha| \leq 1$, $|\beta| \leq 1$ and $R > r \geq 1$,

$$\begin{aligned}\int_0^{2\pi}\int_0^{2\pi} &\big|\big(B[P\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\big)e^{i\eta} \\ &+\big(B[P^*\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](e^{i\theta})\big)\big|^p d\theta \\ &\qquad \leq \int_0^{2\pi} |(R^n + \phi_n(R,r,\alpha,\beta)r^n)\Lambda_n e^{i\eta} + (1+\phi_n(R,r,\alpha,\beta))\lambda_0|^p d\eta \\ &\qquad\qquad \times \int_0^{2\pi} \left|P(e^{i\theta})\right|^p d\theta \end{aligned} \tag{35}$$

where $B \in B_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, Λ_n *and* $\phi_n(R,r,\alpha,\beta)$ *are defined by* (12) *and* (15) *respectively. The result is best possible and the extremal polynomial is* $P(z) = \beta z^n, \ \beta \neq 0$.

Proof. Since $B[P^*\circ\sigma]^*(z) + \phi_n(R,r,\bar{\alpha},\bar{\beta})B[P^*\circ\rho]^*(z)$ is the conjugate polynomial of $B[P^*\circ\sigma](z) + \phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](z)$,

$$\begin{aligned}&|B[P^*\circ\sigma]^*(e^{i\theta}) + \phi_n(R,r,\bar{\alpha},\bar{\beta})B[P^*\circ\rho]^*(e^{i\theta})| \\ &\qquad = |B[P^*\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](e^{i\theta})|, \ 0 \leq \theta < 2\pi\end{aligned}$$

and therefore for each $p > 0$, $R > r \geq 1$ and $0 \leq \theta < 2\pi$, we have

$$\begin{aligned}\int_0^{2\pi} &\big|\big(B[P\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\big)e^{i\eta} \\ &\qquad +\big(B[P^*\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](e^{i\theta})\big)\big|^p d\eta \\ = \int_0^{2\pi} &||B[P\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})|e^{i\eta} \\ &\qquad +|B[P^*\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](e^{i\theta})||^p d\eta \\ = \int_0^{2\pi} &||B[P\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})|e^{i\eta} \\ &\qquad +|B[P^*\circ\sigma]^*(e^{i\theta}) + \phi_n(R,r,\bar{\alpha},\bar{\beta})B[P^*\circ\rho]^*(e^{i\theta})||^p d\eta.\end{aligned} \tag{36}$$

Integrating both sides of (36) with respect to θ from 0 to 2π and using Lemma 2.7, we get

$$\begin{aligned}
&\int_0^{2\pi}\int_0^{2\pi} |\big(B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\big)e^{i\eta} \\
&\qquad\qquad +\big(B[P^*\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](e^{i\theta})\big)|^p d\eta d\theta \\
&=\int_0^{2\pi}\int_0^{2\pi} ||B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})|e^{i\eta} \\
&\qquad\qquad +|B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n(R,r,\bar{\alpha},\bar{\beta})B[P^*\circ\rho]^*(e^{i\theta})||^p d\eta d\theta \\
&=\int_0^{2\pi}\bigg(\int_0^{2\pi} \big(B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\big)e^{i\eta} \\
&\qquad\qquad +\big(B[P^*\circ\sigma]^*(e^{i\theta})+\phi_n(R,r,\bar{\alpha},\bar{\beta})B[P^*\circ\rho]^*(e^{i\theta})\big)|^p d\theta\bigg)d\eta \\
&\le\int_0^{2\pi} |(R^n+\phi_n(R,r,\alpha,\beta)r^n)\Lambda_n e^{i\eta}+(1+\phi_n(R,r,\bar{\alpha},\bar{\beta}))\bar{\lambda_0}|^p d\eta \\
&\qquad\qquad\qquad\qquad \times\int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta \\
&\le\int_0^{2\pi} |(R^n+\phi_n(R,r,\alpha,\beta)r^n)\Lambda_n e^{i\eta}+(1+\phi_n(R,r,\alpha,\beta))\lambda_0|^p d\eta \\
&\qquad\qquad\qquad\qquad \times\int_0^{2\pi}|P(e^{i\theta})|^p d\theta.
\end{aligned}$$

This completes the proof of Lemma 2.8.

3. Main results

We first present the following result which is a compact generalization of the inequalities (1),(2), (5) and (10) and extends inequality (13) for $0 \le p < 1$ as well.

Theorem 3.1. *If $P \in \mathcal{P}_n$, then for arbitrary real or complex numbers α, β* with *$|\alpha| \le 1$, $|\beta| \le 1$, $R > r \ge 1$ and $0 \le p < \infty$,*

$$\|B[P\circ\sigma](z)+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](z)\|_p$$

$$\leq |R^n+\phi_n(R,r,\alpha,\beta)r^n|\,|\Lambda_n|\,\|P(z)\|_p \tag{37}$$

where $B\in B_n$, $\sigma(z):=Rz$, $\rho(z):=rz$, Λ_n and $\phi_n(R,r,\alpha,\beta)$ are defined by (12) *and* (15) *respectively. The result is best possible and equality in* (37) *holds for $P(z)=az^n, a\neq 0$.*

Proof. By hypothesis $P\in P_n$, we can write

$$P(z)=P_1(z)P_2(z)=c\prod_{j=1}^{k}(z-z_j)\prod_{j=k+1}^{n}(z-z_j),\ \ k\geq 1,\ c\neq 0$$

where all the zeros of $P_1(z)$ lie in $|z|\leq 1$ and all the zeros of $P_2(z)$ lie in $|z|>1$. First we suppose that all the zeros of $P_1(z)$ lie in $|z|<1$. Let $P_2^*(z)=z^{n-k}\overline{P_2(1/\bar{z})}$, then all the zeros of $P_2^*(z)$ lie in $|z|<1$ and $|P_2^*(z)|=|P_2(z)|$ for $|z|=1$. Now consider the polynomial

$$F(z)=P_1(z)P_2^*(z)=c\prod_{j=1}^{k}(z-z_j)\prod_{j=k+1}^{n}(1-z\bar{z}_j),$$

then all the zeros of $F(z)$ lie in $|z|<1$ and for $|z|=1$,

$$|F(z)|=|P_1(z)|\,|P_2^*(z)|=|P_1(z)|\,|P_2(z)|=|P(z)|\,. \tag{38}$$

Observe that $P(z)/F(z)\to 1/\prod_{j=k+1}^{n}(-\bar{z}_j)$ when $z\to\infty$, so it is regular even at ∞ and thus from (38) and by the maximum modulus principle, it follows that

$$|P(z)|\leq|F(z)|\ \text{ for }\ |z|\geq 1.$$

Since $F(z)\neq 0$ for $|z|\geq 1$, a direct application of Rouche's theorem shows that the polynomial $H(z)=P(z)+\lambda F(z)$ has all its zeros in $|z|<1$ for every λ with $|\lambda|>1$. Therefore, for all real or complex numbers α,β with $|\alpha|\leq 1,|\beta|\leq 1$ and $R>r\geq 1$, it follows that all the zeros of $h(z)=H(Rz)+\phi_n(R,r,\alpha,\beta)H(rz)$ lie in $|z|<1$. Applying Lemma 2.2 to the polynomial $h(z)$ and noting that B is a linear operator, it follows that all the zeros of

$$\begin{aligned} B[h](z) &= B[H \circ \sigma](z) + \phi_n(R,r,\alpha,\beta)B[H \circ \rho](z) \\ &= B[P \circ \sigma](z) + \phi_n(R,r,\alpha,\beta)B[P \circ \rho](z) \\ &\qquad\qquad + \lambda(B[F \circ \sigma](z) + \phi_n(R,r,\alpha,\beta)B[F \circ \rho](z)) \end{aligned}$$

lie in $|z| < 1$ for every λ with $|\lambda| > 1$. This implies

$$|B[P \circ \sigma](z) + \phi_n(R,r,\alpha,\beta)B[P \circ \rho](z)| \leq |B[F \circ \sigma](z) + \phi_n(R,r,\alpha,\beta)B[F \circ \rho](z)|$$

for $|z| \geq 1$, which, in particular, gives for each $p > 0$, $R > r \geq 1$ and $0 \leq \theta < 2\pi$,

$$\begin{aligned} &\int_0^{2\pi} |B[P \circ \sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P \circ \rho](e^{i\theta})|^p d\theta \\ &\leq \int_0^{2\pi} |B[F \circ \sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[F \circ \rho](e^{i\theta})|^p d\theta \end{aligned} \tag{39}$$

Again, since all the zeros of $F(z)$ lie in $|z| < 1$, it follows, as before, that all the zeros of $B[F(Rz)] + \phi_n(R,r,\alpha,\beta)F(rz)$ also lie in $|z| < 1$. Therefore, if $F(z) = b_n z^n + b_{n-1}z^{n-1} + \cdots + b_0$, then the operator C_γ defined by

$$\begin{aligned} C_\gamma F(z) &= B[F \circ \sigma](z) + \phi_n(R,r,\alpha,\beta)B[F \circ \rho](z) \\ &= (R^n + \phi_n(R,r,\alpha,\beta)r^n)(\lambda_0 + \lambda_1\frac{n^2}{2} + \lambda_2\frac{n^3(n-1)}{8})b_n z^n + \cdots + \lambda_0 b_0 \end{aligned}$$

is admissible. Hence by (24) of Lemma 2.5, for each $p > 0$, we have

$$\begin{aligned} &\int_0^{2\pi} |B[F \circ \sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[F \circ \rho](e^{i\theta})|^p d\theta \\ &\leq |R^n + \phi_n(R,r,\alpha,\beta)r^n||\lambda_0 + \lambda_1\frac{n^2}{2} + \lambda_2\frac{n^3(n-1)}{8}|^p \int_0^{2\pi} |F(e^{i\theta})|^p d\theta. \end{aligned} \tag{40}$$

Combining inequalities (39) and (40) and noting that $|F(e^{i\theta})| = |P(e^{i\theta})|$, we obtain for each $p > 0$ and $R > r \geq 1$,

$$\begin{aligned} &\int_0^{2\pi} |B[P \circ \sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P \circ \rho](e^{i\theta})|^p d\theta \\ &\qquad\qquad \leq |R^n + \phi_n(R,r,\alpha,\beta)r^n||\Lambda_n| \int_0^{2\pi} |P(e^{i\theta})|^p d\theta. \end{aligned} \tag{41}$$

In case $P_1(z)$ has a zero on $|z| = 1$, then the inequality (41) follows by continuity. To obtain this result for $p = 0$, we simply make $p \to 0+$.

A variety of interesting results can be deduced from Theorem 3.1 as special cases. Here we mention a few of these.

The following result follows from Theorem 3.1 by taking $\beta = 0$.

Corollary 3.2. *If $P \in \mathcal{P}_n$, then for every real or complex number α* with $|\alpha| \leq 1$, $R > r \geq 1$ *and* $0 \leq p < \infty$,

$$\left\|B[P\circ\sigma](z) - \alpha B[P\circ\rho](z)\right\|_p \leq |R^n - \alpha r^n|\,|\Lambda_n|\,\|P(z)\|_p \tag{42}$$

where $B \in B_n$, $\sigma(z) := Rz$, $\rho(z) := rz$ and Λ_n *is defined by* (12). *The result is best possible and equality in* (42) *holds for* $P(z) = az^n, a \neq 0$.

Setting $\alpha = 0$ in Corollary 3.2, we get the following sharp result.

Corollary 3.3. *If $P \in \mathcal{P}_n$, then for $R > 1$ and $0 \leq p < \infty$,*

$$\left\|B[P\circ\sigma](z)\right\|_p \leq |R^n|\,|\Lambda_n|\,\|P(z)\|_p \tag{43}$$

where $B \in B_n$, $\sigma(z) := Rz$ and Λ_n *is defined by* (12). *The result is best possible and equality in* (43) *holds for* $P(z) = az^n, a \neq 0$.

Remark 3.4. Corollary 3.3 not only includes inequality (13) as a special case but also extends it for $0 \leq p < 1$ as well. Further inequality (10) follows from Corollary 3.3 by letting $p \to \infty$ in (43).

The case $B[P](z) = P(z)$ of Theorem 3.1 yields the following interesting result which is a compact generalization of inequalities (1), (2) and (5).

Corollary 3.5. *If $P \in \mathcal{P}_n$, then for every real or complex number α with $|\alpha| \leq 1$, $R > r \geq 1$, and $p > 0$,*

$$\|P(Rz) + \phi_n(R, r, \alpha, \beta) P(rz)\|_p \leq |R^n + \phi_n(R, r, \alpha, \beta) r^n| \, \|P(z)\|_p \tag{44}$$

where $\phi_n(R, r, \alpha, \beta)$ is defined by (15). The result is best possible and equality in (44) holds for $P(z) = az^n, a \neq 0$.

Remark 3.6. If we divide the two sides of (44) by $R - r$ with $\alpha = 1$ and then let $R \to r$, we get for $P \in \mathcal{P}_n$, $r \geq 1$, $|\beta| \leq 1$ and $0 \leq p < \infty$,

$$\left\| zP'(rz) + \beta \frac{n}{1+r} P(rz) \right\|_p \leq n \left| r^{n-1} + \beta \frac{r^n}{1+r} \right| \|P(z)\|_p . \tag{45}$$

The result is best possible and equality in (45) holds for $P(z) = az^n, a \neq 0$.

Taking $\alpha = 0$ in (41), we obtain:

Corollary 3.7. *If $P \in \mathcal{P}_n$, then for every real or complex number β with $|\beta| \leq 1$, $R > r \geq 1$ and $0 \leq p < \infty$,*

$$\left\| B[P(Rz)] + \beta \left(\frac{R+1}{r+1} \right)^n B[P(rz)] \right\|_p \leq \left| R^n + \beta \left(\frac{R+1}{r+1} \right)^n r^n \right| |\Lambda_n| \, \|P(z)\|_p . \tag{46}$$

where $B \in B_n$ and $\phi(R, r, \alpha, \beta)$ is defined by (15).The result is best possible and equality in (46) holds for $P(z) = \lambda z^n, \lambda \neq 0$.

Theorem 3.1 can be sharpened if we restrict ourselves to the class of polynomials $P \in P_n$ having no zero in $|z| < 1$. In this direction, we next present the following result which in particular includes a generalized L_p mean extension of the inequality (11) for $0 \leq p < \infty$ and among other things yields a correct proof of inequality (14) for each $p \geq 0$ as a special case.

Theorem 3.8. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for then for arbitrary real or complex numbers α, β* with $|\alpha| \le 1$, $|\beta| \le 1$, $R > r \ge 1$ *and* $0 \le p < \infty$,

$$\begin{aligned} &\left\|B[P\circ\sigma](z) + \phi_n(R,r,\alpha,\beta)B[P\circ\rho](z)\right\|_p \\ &\qquad \le \frac{\left\|\left(R^n + \phi_n(R,r,\alpha,\beta)r^n\right)\Lambda_n z + \left(1+\phi_n(R,r,\alpha,\beta)\right)\lambda_0\right\|_p}{\left\|1+z\right\|_p}\left\|P(z)\right\|_p . \end{aligned} \tag{47}$$

where $B \in B_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, Λ_n *and* $\phi_n(R,r,\alpha,\beta)$ *are defined by* (12) *and* (15) *respectively. The result is best possible and equality in* (47) *holds for* $P(z) = az^n + b, |a| = |b| \neq 0$.

Proof. By hypothesis $P \in P_n$ does not vanish in $|z| < 1$, $\sigma(z) = Rz$, $\rho(z) = rz$ therefore, if $P^*(z) = z^n\overline{P(1/\bar{z})}$, then by Lemma 2.3, we have for $0 \le \theta < 2\pi$,

$$\begin{aligned} &|B[P\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})| \\ &\qquad \le |B[P^*\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](e^{i\theta})| \end{aligned} \tag{48}$$

Also, by Lemma 2.8, for each $p > 0$ and η real and $R > r \ge 1$,

$$\begin{aligned} &\int_0^{2\pi}\int_0^{2\pi} |\big(B[P\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\big)e^{i\eta} \\ &\quad +\big(B[P^*\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](e^{i\theta})\big)|^p d\theta d\eta \\ &\qquad \le \int_0^{2\pi} |(R^n + \phi_n(R,r,\alpha,\beta)r^n)\Lambda_n e^{i\eta} \\ &\qquad\quad +(1+\phi_n(R,r,\alpha,\beta))\lambda_0|^p d\eta \int_0^{2\pi}\left|P(e^{i\theta})\right|^p d\theta. \end{aligned} \tag{49}$$

Now it can be easily verified that for every real number α and $s \ge 1$,

$$\left|s + e^{i\alpha}\right| \ge \left|1 + e^{i\alpha}\right|.$$

This implies for each $p > 0$,

$$\int_0^{2\pi}\left|s + e^{i\alpha}\right|^p d\alpha \ge \int_0^{2\pi}\left|1 + e^{i\alpha}\right|^p d\alpha. \tag{50}$$

If $B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\bar{\alpha},\bar{\beta})B[P\circ\rho](e^{i\theta})\neq 0$, we take

$$s=\frac{|B[P^*\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](e^{i\theta})|}{|B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})|},$$

then by (48), $s\geq 1$ and from (50), we get

$$\begin{aligned}
&\int_0^{2\pi}|\big(B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\big)e^{i\eta}\\
&\qquad+\big(B[P^*\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](e^{i\theta})\big)|^p d\eta\\
&=|B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})|^p\\
&\qquad\times\int_0^{2\pi}\left|e^{i\eta}+\frac{B[P^*\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](e^{i\theta})}{B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})}\right|^p d\eta\\
&=|B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})|^p\\
&\qquad\times\int_0^{2\pi}\left|e^{i\eta}+\left|\frac{B[P^*\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](e^{i\theta})}{B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})}\right|\right|^p d\eta\\
&\geq|B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})|^p\int_0^{2\pi}|1+e^{i\eta}|^p d\eta.
\end{aligned}$$

For $B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})=0$, this inequality is trivially true. Using this in (49), we conclude that for each $p>0$,

$$\begin{aligned}
&\int_0^{2\pi}\left|B[P\circ\sigma](e^{i\theta})+\phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})\right|^p d\theta\int_0^{2\pi}\left|1+e^{i\eta}\right|^p d\eta\\
&\qquad\leq\int_0^{2\pi}|(R^n+\phi_n(R,r,\alpha,\beta)r^n)\Lambda_n e^{i\eta}\\
&\qquad\qquad+(1+\phi_n(R,r,\alpha,\beta))\lambda_0|^p d\eta\int_0^{2\pi}|P(e^{i\theta})|^p d\theta,
\end{aligned}$$

from which Theorem 3.8 follows for $p>0$. To establish this result for $p=0$, we simply let $p\to 0+$. This completes the proof of Theorem 3.8.

For $\beta=0$, inequality (47) reduces to the following result.

Corollary 3.9. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for every real or complex number α* with $|\alpha| \leq 1$, $R > r \geq 1$ *and* $0 \leq p < \infty$,

$$\left\| B[P \circ \sigma](z) - \alpha B[P \circ \rho](z) \right\|_p \leq \frac{\left\| (R^n - \alpha r^n)\Lambda_n z + (1-\alpha)\lambda_0 \right\|_p}{\left\| 1+z \right\|_p} \left\| P(z) \right\|_p \tag{51}$$

where $B \in B_n$, $\sigma(z) := Rz$, $\rho(z) := rz$ and Λ_n *is defined by* (12). *The result is best possible and equality in* (51) *holds for* $P(z) = az^n + b$, $|a| = |b| \neq 0$.

For $\alpha = 0$, Corollary 3.9 yields the following interesting result.

Corollary 3.10. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $R > r \geq 1$ and $0 \leq p < \infty$,*

$$\left\| B[P \circ \sigma](z) \right\|_p \leq \frac{\left\| R^n \Lambda_n z + \lambda_0 \right\|_p}{\left\| 1+z \right\|_p} \left\| P(z) \right\|_p \tag{52}$$

where $B \in B_n$, $\sigma(z) := Rz$ and Λ_n *is defined by* (12). *The result is best possible and equality in* (52) *holds for* $P(z) = az^n + b$, $|a| = |b| \neq 0$.

Remark 3.11. If we choose $\alpha = \lambda_0 = \lambda_2 = 0$ in (49), we get for $R > 1$ and $0 \leq p < \infty$

$$\left\| P'(Rz) \right\|_p \leq \frac{nR^{n-1}}{\left\| 1+z \right\|_p} \left\| P(z) \right\|_p \tag{53}$$

which in particular yields inequality (3).

By the triangle inequality, the following result immediately follows from Corollary 3.10.

Corollary 3.12. *If $P \in \mathcal{P}_n$ and $P(z)$ does not vanish in $|z| < 1$, then for $0 \leq p < \infty$ and $R > 1$,*

$$\left\| B[P \circ \sigma](z) \right\|_p \leq \frac{R^n |\Lambda_n| + |\lambda_0|}{\left\| 1+z \right\|_p} \left\| P(z) \right\|_p . \tag{54}$$

where $B \in B_n$, $\sigma(z) := Rz$, Λ_n *is defined by* (12).

Remark 3.13. Corollary 3.12 not only validates the inequality (13) for $p \geq 1$ but also extends it for $0 \leq p < 1$ as well.

A polynomial $P \in \mathcal{P}_n$ is said be self-inversive if $P(z) = uP^*(z)$ where $|u| = 1$ and $P^*(z)$ is the conjugate polynomial of $P(z)$, that is, $P^*(z) = z^n\overline{P(1/\overline{z})}$. Finally in this paper, we establish the following result for self-inversive polynomials which includes a correct proof of another result of Shah and Liman [17, Theorem 3] as a special case.

Theorem 3.14. *If $P \in \mathcal{P}_n$ is a self-inversive polynomial, then for arbitrary real or complex numbers α, β* with *$|\alpha| \leq 1$, $|\beta| \leq 1$, $R > r \geq 1$ and $0 \leq p < \infty$,*

$$\|B[P\circ\sigma](z) + \phi_n(R,r,\alpha,\beta)B[P\circ\rho](z)\|_p$$

$$\leq \frac{\|(R^n + \phi_n(R,r,\alpha,\beta)r^n)\,\Lambda_n z + (1+\phi_n(R,r,\alpha,\beta))\,\lambda_0\|_p}{\|1+z\|_p}\,\|P(z)\|_p\,. \tag{55}$$

where $B \in B_n$, $\sigma(z) := Rz$, $\rho(z) := rz$, Λ_n and $\phi_n\,(R,r,\alpha,\beta)$ are defined by (12) *and* (15) *respectively. The result is best possible and equality in* (55) *holds for $P(z) = z^n + 1$.*

Proof. Since $P \in P_n$ is self-inversive polynomial, we have for some u with $|u|$=1, $P^*(z) = uP(z)$ for all $z \in \mathbb{C}$ where $P^*(z) = z^n\overline{P(1/\overline{z})}$. This gives for $0 \leq \theta < 2\pi$,

$$|B[P\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P\circ\rho](e^{i\theta})|$$
$$= |B[P^*\circ\sigma](e^{i\theta}) + \phi_n(R,r,\alpha,\beta)B[P^*\circ\rho](e^{i\theta})|.$$

Using this in (35) and proceeding similarly as in the proof of Theorem 3.8, we get the desired result for each $p > 0$. To extension to $p = 0$ is obtains by letting $p \to 0+$.

The following result is an immediate consequence of Theorem 3.14.

Corollary 3.15. *If $P \in \mathcal{P}_n$ is a self-inversive polynomial, then for $|\alpha| \leq 1$, $0 \leq p < \infty$ and $R > r \geq 1$,*

$$\|B[P \circ \sigma](z) - \alpha B[P \circ \rho)](z)\|_p \leq \frac{\|(R^n - \alpha r^n)\Lambda_n z + (1-\alpha)\lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p . \tag{56}$$

where $B \in B_n$ *and* $\sigma(z) := Rz$, $\rho(z) := rz$ *and* Λ_n *is defined by* (12). *The result is sharp and equality in* (56) *holds for* $P(z) = z^n + 1$.

For $\alpha = 0$, Corollary 3.15 reduces to the following interesting result.

Corollary 3.16. *If $P \in \mathcal{P}_n$ is a self-inversive polynomial, then for $0 \leq p < \infty$ and $R > 1$,*

$$\|B[P \circ \sigma](z)\|_p \leq \frac{\|R^n \Lambda_n z + \lambda_0\|_p}{\|1+z\|_p} \|P(z)\|_p . \tag{57}$$

where $B \in B_n$, $\sigma(z) := Rz$ *and* Λ_n *is defined by* (12). *The result is best possible and equality in* (57) *holds for* $P(z) = z^n + 1$.

By the triangle inequality, the following result follows immediately from Corollary 3.16.

Corollary 3.17. *If $P \in \mathcal{P}_n$ is a self-inversive polynomial, then for $0 \leq p < \infty$ and $R > 1$,*

$$\|B[P \circ \sigma](z)\|_p \leq \frac{R^n |\Lambda_n| + |\lambda_0|}{\|1+z\|_p} \|P(z)\|_p . \tag{58}$$

where $B \in B_n$, $\sigma(z) := Rz$ *and* Λ_n *is defined by* (12).

Remark 3.18. Corollary 3.16 establishes a correct proof of a result due to Shah and Liman [17, Theorem 3] for $p \geq 1$ and also extends it for $0 \leq p < 1$ as well.

Lastly letting $p \to \infty$ and setting $\alpha = \beta = 0$ in (57), we obtain the following result.

Corollary 3.19. *If $P \in \mathcal{P}_n$ is a self-inversive polynomial, then for $|z| = 1$ and $R > 1$,*

$$|B[P \circ \sigma](z)| \leq \frac{1}{2} \{R^n |\Lambda_n| + |\lambda_0|\} \|P(z)\|_\infty .$$

where $B \in B_n$, $\sigma(z) := Rz$ *and* Λ_n *is defined by* (12). The result is sharp.

References

[1] N.C. Ankeny and T.J.Rivlin, On a theorm of S.Bernstein, Pacific J. Math., 5(1955),849 - 852.

[2] V.V.Arestov, On integral inequalities for trigonometric polynimials and their derivatives, Izv. Akad. Nauk SSSR Ser. Mat. 45 (1981),3-22[in Russian]. English translation; Math.USSR-Izv.,18 (1982),1-17.

[3] A.Aziz, A new proof and a generalization of a theorem of De Bruijn, proc. Amer Math. Soc., 106(1989),345-350.

[4] A. Aziz and N.A. Rather, Some compact generalizations of Zygmund-type inequalities for polynomials, Nonlinear Studies, 6(1999), 241 - 255.

[5] R.P.Boas,Jr. and Q.I.Rahman , L^p inequalities for polynomials and entire functions, Arch. Rational Mech. Anal., 11(1962),34-39.

[6] N.G.Bruijn, Inequalities concerning polynomials in the complex domain,Nederal. Akad.Wetensch. Proc.,50(1947),1265-1272.

[7] K.K.Dewan and N.K.Govil, An inequality for self- inversive polynomials, J.Math. Anal. Appl.,45(1983), 490.

[8] G.H.Hardy, The mean value of the modulus of an analytic functions , Proc. London Math. Soc., 14(1915), 269-277.

[9] P.D.Lax, Proof of a conjecture of P.Erdös on the derivative of a polynomial, Bull. Amer. Math.Soc.,50(1944),509-513.

[10] M.Marden, Geometry of polynomials, Math. Surveys, No.3, Amer. Math.Soc. Providence, RI, 1949.

[11] G.V.Milovanovic,D.S.Mitrinovic and Th.M.Rassias, Topics in Polynomials: Extremal Properties, Inequalities, Zeros, World scientific Publishing Co., Singapore,(1994).

[12] G.Polya and G.Szegö, Aufgaben und lehrsätze aus der analysis, Springer-Verlag, Berlin(1925).

[13] Q.I.Rahman, Functions of exponential type, Trans. Amer. Math. Soc., 135(1969),295-309.

[14] Q.I.Rahman and G.Schmeisser, L^p inequalities for polynomials, J. Approx. Theory,53(1988),26-32.

[15] Q.I.Rahman and G.Schmisser, Analytic Theory of Polynomials, Oxford University Press, New York, 2002.

[16] A.C.Schaffer, Inequalities of A.Markov and S.Bernstein for polynomials and related functions, Bull.Amer. Math. Soc., 47(1941), 565-579.

[17] W.M.Shah and A.Liman, Integral estimstes for the family of B-operators, Operators and Matrices, 5(2011), 79-87.

[18] A. Zygmund, A remark on conjugate series, Proc. London Math. Soc., 34(1932),292-400.