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DIFFERENTIAL INEQUALITIES AND COMPARISON THEOREMS FOR FIRST ORDER HYBRID INTEGRO-DIFFERENTIAL EQUATIONS

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Abstract. In this paper, some basic results concerning the strict and nonstrict integro-differential inequalities and existence of the maximal and minimal solutions are proved for a first order hybrid integro-differential equation with a linear perturbations of second type.

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1. Introduction

Given a bounded interval $J = [t_0, t_0 + a)$ in $\mathbb{R}$ for some fixed $t_0, a \in \mathbb{R}$ with $a > 0$, consider the initial value problems of hybrid integro-differential equation (in short HIDE),

$$\left.\begin{aligned} \frac{d}{dt}\big[x(t) - f(t, x(t))\big] &= \int_{t_0}^{t} g(s, x(s))\, ds, \;\; t \in J \\ x(t_0) &= x_0 \in \mathbb{R}, \end{aligned}\right\} \tag{1.1}$$

where, $f, g : J \times \mathbb{R} \to \mathbb{R}$ are continuous.

By a *solution* of the HIDE (1.1) we mean a function $x \in C(J, \mathbb{R})$ such that

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(i) the function $t \mapsto x - f(t,x)$ is continuous for each $x \in \mathbb{R}$, and

(ii) x satisfies the equations in (1.1).

The importance of the investigations of hybrid integro-differential equations lies in the fact that they include several classes of differential and integral equations as special cases. The study of hybrid differential equations is implicit in the works of Krasnoselskii [9] and extensively treated in the several papers on hybrid differential equations with different perturbations. See Burton [1], Dhage [3] and the references therein. This class of hybrid integro-differential equations includes the perturbations of original integro-differential equations in different ways. A sharp classification of different types of perturbations of integro-differential equations appears in Dhage [5] which can be treated with hybrid fixed point theory (see Dhage [2, 4] and Dhage and Lakshmikantham [6]). In this paper, we initiate the basic theory of hybrid integro-differential equations of linear perturbations of second type involving two nonlinearities and prove some basic results such as integro-differential inequalities, existence theorem and maximal and minimal solutions etc. We claim that the results of this paper are basic and important contribution to the theory of nonlinear ordinary integro-differential equations.

2. Strict and Nonstrict Inequalities

We need frequently the following hypothesis in what follows.

(A_0) The function $x \mapsto x - f(t,x)$ is increasing in $\mathbb{R}$ for all $t \in J$.

We begin by proving the basic results dealing with hybrid integro-differential inequalities.

Theorem 2.1. Assume that the hypothesis (A_0) holds. Suppose that there exist $y, z \in C(J,\mathbb{R})$ such that

$$\frac{d}{dt}\big[y(t) - f(t,y(t))\big] \le \int_{t_0}^{t} g(s,y(s))\,ds, \quad t \in J \tag{2.1}$$

and

$$\frac{d}{dt}\big[z(t) - f(t,z(t))\big] \ge \int_{t_0}^{t} g(s,z(s))\,ds, \quad t \in J. \tag{2.2}$$

If one of the inequalities (2.1) and (2.2) is strict and

$$y(t_0) < z(t_0), \tag{2.3}$$

then

$$y(t) < z(t) \tag{2.4}$$

for all $t \in J$.

Proof. Suppose that the inequality (2.4) is false. Then the set Z defined by

$$Z = \{t \in J \mid y(t) \geq z(t)\} \tag{2.5}$$

is non-empty. Denote $t_1 = \inf Z$. Without loss of generality, we may assume that

$$y(t_1) = z(t_1) \text{ and } y(t) < z(t)$$

for all $t < t_1$.

Assume that

$$\frac{d}{dt}\big[z(t) - f(t, z(t))\big] > \int_0^t g(s, z(s))\, ds$$

for $t \in J$.

Denote

$$Y(t) = y(t) - f(t, y(t)) \quad \text{and} \quad Z(t) = z(t) - f(t, z(t)) \tag{2.6}$$

for $t \in J$.

As hypothesis (A_0) holds, it follows from (2.5) that

$$Y(t_1) = Z(t_1) \quad \text{and} \quad Y(t) < Z(t) \tag{2.7}$$

for all $t_0 \leq t < t_1$. The above relation (2.7) further yields

$$\frac{Y(t_1 + h) - Y(t_1)}{h} > \frac{Z(t_1 + h) - Z(t_1)}{h}$$

for small $h < 0$. Taking the limit as $h \to 0$, we obtain

$$Y'(t_1) \geq Z'(t_1). \tag{2.8}$$

Hence, from (2.7) and (2.8), we get

$$\int_{t_0}^{t_1} g(s, y(s))\, ds \geq Y'(t_1) \geq Z'(t_1) > \int_{t_0}^{t_1} g(s, z(s))\, ds.$$

This is a contradiction and the proof is complete.

The next result is about the nonstrict inequality for the HIDE (1.1) on J which requires a one-sided Lipschitz condition.

Theorem 2.2. Assume that the hypotheses of Theorem 2.1 hold. Suppose also that there exists a real number $L > 0$ such that

$$g(t, y(t)) - g(t, z(t)) \leq L \sup_{t_0 \leq s \leq t} \big[(y(s) - f(s, y(s))) - (z(s) - f(s, z(s)))\big] \tag{2.9}$$

whenever $y(s) \geq z(s)$, $t_0 \leq s \leq t$. Then,

$$y(t_0) \leq z(t_0) \tag{2.10}$$

implies

$$y(t) \leq z(t) \tag{2.11}$$

for all $t \in J$.

Proof. Let $\epsilon > 0$ and let a real number $L > 0$ be given. Set

$$z_\epsilon(t) - f(t, z_\epsilon(t)) = z(t) - f(t, x(t)) + \epsilon e^{2L(t-t_0)} \tag{2.12}$$

so that

$$z_\epsilon(t) - f(t, z_\epsilon(t)) > z(t) - f(t, x(t)).$$

Define

$$Z_\epsilon(t) = z_\epsilon(t) - f(t, z_\epsilon(t)) \quad \text{and} \quad Z(t) = z(t) - f(t, z(t))$$

for $t \in J$.

Now using the one-sided Lipschitz condition (2.9), we obtain

$$g(t, z_\epsilon(t)) - g(t, z(t)) \leq L \sup_{t_0 \leq s \leq t} [Z_\epsilon(s) - Z(s)] = L\epsilon e^{2L(t-t_0)}.$$

Now,

$$\begin{aligned}
Z'_\epsilon(t) &= Z'(t) + 2L\epsilon e^{2L(t-t_0)} \\
&\geq \int_{t_0}^{t} g(s, z(s))\, ds + 2L\epsilon e^{2L(t-t_0)} \\
&\geq \int_{t_0}^{t} g(s, z_\epsilon(s)) + 2L\epsilon e^{2L(t-t_0)} - L\epsilon e^{2L(t-t_0)} \\
&= \int_{t_0}^{t} g(s, z_\epsilon(s))\, ds + L\epsilon e^{2L(t-t_0)} \\
&> \int_{t_0}^{t} g(s, z_\epsilon(s))\, ds
\end{aligned}$$

for all $t \in J$. Also, we have

$$Z_\epsilon(t_0) > Z(t_0) \geq Y(t_0).$$

Now we apply Theorem 2.1 with $z = z_\epsilon$ to yield

$$Y(t) < Z_\epsilon(t)$$

for all $t \in J$. On taking $\epsilon \to 0$ in the above inequality, we get

$$Y(t) \leq Z(t)$$

which further in view of hypothesis (A_0) implies that (2.11) holds on J. This completes the proof.

Remark 2.1. The conclusion of Theorems 2.1 and 2.2 also remains true if we replace the derivative in the inequalities (2.1) and (2.2) by Dini-derivative D_- of the function $x(t) - f(t, x(t))$ on the bounded interval J.

3. Existence Result

In this section, we prove an existence result for the HIDE (1.1) on a closed and bounded interval $J = [t_0, t_0 + a]$ under mixed Lipschitz and compactness conditions on the nonlinearities involved in it. We place the HIDE (1.1) in the space $C(J, \mathbb{R})$ of continuous real-valued functions defined on J and use a hybrid fixed point of Dhage [2]. Define a supremum norm $\|\cdot\|$ in $C(J,\mathbb{R})$ defined by

$$\|x\| = \sup_{t\in J} |x(t)|.$$

Clearly $C(J,\mathbb{R})$ is a Banach space with respect to the above supremum norm. We prove the existence of solution for the HIDE (1.1) via a hybrid fixed point theorem in Banach space due to Dhage [2].

Theorem 3.1. Let S be a closed convex and bounded subset of the Banach space E and let $A : E \to E$ and $B : S \to E$ be two operators such that

(a) A is nonlinear contraction,

(b) B is compact and continuous, and

(c) $x = Ax + By$ for all $y \in S \Longrightarrow x \in S$.

Then the operator equation $Ax + Bx = x$ has a solution in S.

We consider the following hypotheses in what follows.

(A_1) There exists a constant $L > 0$ such that

$$|f(t,x) - f(t,y)| \le \frac{L|x-y|}{M + |x-y|}$$

for all $t \in J$ and $x, y \in \mathbb{R}$. Moreover, $L \le M$.

(A_2) There exists a continuous function $h : J \to \mathbb{R}$ such that

$$|g(t,x)| \le h(t),\ t \in J$$

for all $x \in \mathbb{R}$.

The following lemma is useful in the sequel.

Lemma 3.1. Assume that hypothesis (A_0) holds. Then for any continuous function $h : J \to \mathbb{R}$, the function $x \in C(J,\mathbb{R})$ is a solution of the HIDE

$$\left.\begin{aligned} \frac{d}{dt}\left[x(t) - f(t,x(t))\right] &= \int_{t_0}^{t} h(s)\,ds \ \ t \in J \\ x(0) &= x_0 \in \mathbb{R} \end{aligned}\right\} \tag{3.1}$$

if and only if x satisfies the hybrid integral equation (HIE)

$$x(t) = x_0 - f(t_0, x_0) + f(t, x(t)) + \int_{t_0}^{t} (t-s)h(s)\,ds, \quad t \in J. \tag{3.2}$$

Proof. Let $h \in C(J, \mathbb{R})$. Assume first that x is a solution of the HIDE (3.1). By definition, $x(t) - f(t, x(t))$ is continuous on J, and so, differentiable there, whence $\frac{d}{dt}\big[x(t) - f(t, x(t))\big]$ is integrable on J. Applying integration to (3.1) from t_0 to t, we obtain the HIE (3.2) on J.

Conversely, assume that x satisfies the HIE (3.2). Then by direct differentiation we obtain the first equation in (3.1). Again, substituting $t = t_0$ in (3.2) yields

$$x(t_0) - f(t_0, x(t_0)) = x_0 - f(t_0, x_0).$$

Since the mapping $x \mapsto x - f(t, x)$ is increasing in $\mathbb{R}$ for all $t \in J$, the mapping $x \mapsto x - f(t_0, x)$ is injective in $\mathbb{R}$, whence $x(t_0) = x_0$. Hence the proof of the lemma is complete.

Now we are in a position to prove the following existence theorem for the HIDE (1.1) on J.

Theorem 3.2. Assume that the hypotheses (A_0)-(A_2) hold. Then the HIDE (1.1) has a solution defined on J.

Proof. Set $E = C(J, \mathbb{R})$ and define a subset S of E defined by

$$S = \{x \in E \mid \|x\| \le N\} \tag{3.3}$$

where,

$$N = |x_0 - f(t_0, x_0)| + L + F_0 + \|h\|\, a(t_0 + a),$$

and $F_0 = \sup\{|f(t, 0)| \mid t \in J\}$.

Clearly S is a closed, convex and bounded subset of the Banach space E. Now, using the hypotheses (A_0) and (A_2) it can be shown by an application of Lemma 3.1 that the HIDE (1.1) is equivalent to the nonlinear HIE

$$x(t) = x_0 - f(t_0, x_0) + f(t, x(t)) + \int_{t_0}^{t} (t-s)g(s, x(s))\,ds \tag{3.4}$$

for $t \in J$.

Define two operators $A : E \to E$ and $B : S \to E$ by

$$Ax(t) = f(t, x(t)),\ t \in J, \tag{3.5}$$

and

$$Bx(t) = x_0 - f(t_0, x_0) + \int_{t_0}^{t} (t-s) g(s, x(s))\, ds,\ t \in J. \tag{3.6}$$

Then, the HIE (3.5) is transformed into an operator equation as

$$Ax(t) + Bx(t) = x(t),\ \ t \in J. \tag{3.7}$$

We shall show that the operators A and B satisfy all the conditions of Theorem 3.1.

First, we show that A is a Lipschitz operator on E with the Lipschitz constant L_1. Let $x, y \in E$. Then, by hypothesis (A_1),

$$|Ax(t) - Ay(t)| = |f(t, x(t)) - f(t, y(t))| \leq \frac{L|x(t) - y(t)|}{M + |x(t) - y(t)|} \leq \frac{L\|x - y\|}{M + \|x - y\|}$$

for all $t \in J$. Taking supremum over t, we obtain

$$\|Ax - Ay\| \leq \frac{L\|x - y\|}{M + \|x - y\|}$$

for all $x, y \in E$. This shows that A is a nonlinear contraction E with $\mathcal{D}$-function ψ defined by $\psi(r) = \dfrac{L\, r}{M + r}$.

Next, we show that B is a compact and continuous operator on S into E. First we show that B is continuous on S. Let $\{x_n\}$ be a sequence in S converging to a point $x \in S$. Then by dominated convergence theorem for integration, we obtain

$$\begin{aligned}
\lim_{n\to\infty} Bx_n(t) &= \lim_{n\to\infty} \left[x_0 - f(t_0, x_0) + \int_{t_0}^{t} (t-s) g(s, x_n(s))\, ds \right] \\
&= x_0 - f(t_0, x_0) + \lim_{n\to\infty} \int_{t_0}^{t} (t-s) g(s, x_n(s))\, ds \\
&= x_0 - f(t_0, x_0) + \int_{t_0}^{t} \left[\lim_{n\to\infty} (t-s) g(s, x_n(s)) \right]\, ds \\
&= x_0 - f(t_0, x_0) + \int_{t_0}^{t} (t-s) g(s, x(s))\, ds \\
&= Bx(t)
\end{aligned}$$

for all $t \in J$. Moreover, it can be shown as below that $\{Bx_n\}$ is an equicontinuous sequence of functions in X. Now, following the arguments similar to that given in Granas *et al.* [7], it is proved that B is a continuous operator on S.

Next, we show that B is compact operator on S. It is enough to show that $B(S)$ is a uniformly bounded and equi-continuous set in E. Let $x \in S$ be arbitrary. Then by hypothesis (A_2),

$$\begin{aligned}|Bx(t)| &\leq |x_0 - f(t_0, x_0)| + \int_{t_0}^{t} |t-s||g(s, x(s))|\,ds \\ &\leq |x_0 - f(t_0, x_0)| + \int_{t_0}^{t} (t_0 + a)h(s)\,ds \\ &\leq |x_0 - f(t_0, x_0)| + \|h\|\,a(t_0 + a)\end{aligned}$$

for all $t \in J$. Taking supremum over t,

$$\|Bx\| \leq \big|x_0 - f(t_0, x_0)\big| + \|h\|\,a(t_0 + a)$$

for all $x \in S$. This shows that B is uniformly bounded on S.

Again, let $t_1, t_2 \in J$. Then for any $x \in S$, one has

$$\begin{aligned}|Bx(t_1) - Bx(t_2)| &= \left|\int_{t_0}^{t_1} (t_1 - s)g(s, x(s))\,ds - \int_{t_0}^{t_2} (t_2 - s)g(s, x(s))\,ds\right| \\ &\leq \left|\int_{t_0}^{t_1} (t_1 - s)g(s, x(s))\,ds - \int_{t_0}^{t_1} (t_2 - s)g(s, x(s))\,ds\right| \\ &\quad + \left|\int_{t_0}^{t_1} (t_2 - s)g(s, x(s))\,ds - \int_{t_0}^{t_2} (t_2 - s)g(s, x(s))\,ds\right| \\ &\leq \left|\int_{t_0}^{t_0+a} |t_1 - t_2|h(s)\,ds\right| + \left|\int_{t_1}^{t_2} (t_0 + a)h(s)\,ds\right| \\ &\leq a\,|t_1 - t_2|\,\|h\| + (t_0 + a)|p(t_1) - p(t_2)|\end{aligned}$$

where, $p(t) = \int_{t_0}^{t} h(s)\,ds$. Since the function p is continuous on compact J, it is uniformly continuous. Hence, for $\epsilon > 0$, there exists a $\delta > 0$ such that

$$|t_1 - t_2| < \delta \Longrightarrow |Bx(t_1) - Bx(t_2)| < \epsilon$$

for all $t_1, t_2 \in J$ and for all $x \in S$. This shows that $B(S)$ is an equi-continuous set in E. Now the set $B(S)$ is uniformly bounded and equicontinuous set in E, so it is compact by Arzelá-Ascoli theorem. As a result, B is a continuous and compact operator on S.

Next, we show that hypothesis (c) of Theorem 3.1 is satisfied. Let $x \in E$ and $y \in S$ be arbitrary such that $x = Ax + By$. Then, by assumption (A_1), we have

$$\begin{aligned}
|x(t)| &\leq |Ax(t)| + |Bx(t)| \\
&\leq |x_0 - f(t_0, x_0)| + |f(t, x(t))| + \int_{t_0}^{t} |t-s||g(s, y(s))|\, ds \\
&\leq |x_0 - f(t_0, x_0)| + \big[|f(t, x(t)) - f(t, 0)| + |f(t, 0)|\big] + \int_{t_0}^{t} (t_0 + a)|g(s, y(s))|\, ds \\
&\leq |x_0 - f(t_0, x_0)| + L + F_0 + \int_{t_0}^{t} (t_0 + a)h(s)\, ds \\
&\leq |x_0 - f(t_0, x_0)| + L + F_0 + \|h\| a(t_0 + a).
\end{aligned}$$

Taking supremum over t,

$$\|x\| \leq |x_0 - f(t_0, x_0)| + L + F_0 + \|h\| a(t_0 + a).$$

Thus, all the conditions of Theorem 3.1 are satisfied and hence the operator equation $Ax + Bx = x$ has a solution in S. As a result, the HIDE (1.1) has a solution defined on J. This completes the proof.

4. Maximal and Minimal Solutions

In this section, we shall prove the existence of maximal and minimal solutions for the HIDE (1.1) on $J = [t_0, t_0 + a]$. We need the following definition in what follows.

Definition 4.1. A solution r of the HIDE (1.1) is said to be maximal if for any other solution x to the HIDE (1.1) one has $x(t) \leq r(t)$, for all $t \in J$. Again, a solution ρ of the HIDE (1.1) is said to be minimal if $\rho(t) \leq x(t)$, for all $t \in J$, where x is any solution of the HIDE (1.1) existing on J.

We discuss the case of maximal solution only, as the case of minimal solution is similar and can be obtained with the similar arguments with appropriate modifications. Given a arbitrary small real number $\epsilon > 0$, consider the the following initial value problem of HIDE,

$$\left.\begin{aligned} \tfrac{d}{dt}\left[x(t) - f(t, x(t))\right] &= \textstyle\int_0^t g(s, x(s))\, ds + \epsilon, \quad t \in J \\ x(t_0) &= x_0 + \epsilon \end{aligned}\right\} \tag{4.1}$$

where, $f, g \in C(J \times \mathbb{R}, \mathbb{R})$.

An existence theorem for the HIDE (4.1) can be stated as follows:

Theorem 4.1. Assume that the hypotheses (A_0)-(A_2) hold. Then for every small number $\epsilon > 0$, the HIDE (4.1) has a solution defined on J.

Proof. The proof is similar to Theorem 3.1 and we omit the details.

Our main existence theorem for maximal solution for the HIDE (1.1) is

Theorem 4.2. Assume that the hypotheses (A_0)-(A_2) hold. Further, if $L \leq M$, then the HIDE (1.1) has a maximal solution defined on J.

Proof. Let $\left\{\epsilon_n\right\}_0^\infty$ be a decreasing sequence of positive real numbers such that $\lim_{n\to\infty} \epsilon_n = 0$. Then for any solution u of the HIDE (1.1), by Theorem 2.1, one has

$$u(t) < r(t, \epsilon_n) \tag{4.2}$$

for all $t \in J$ and $n \in \mathbb{N} \cup \{0\}$, where $r(t, \epsilon_n)$ is a solution of the HIDE,

$$\left.\begin{aligned} \tfrac{d}{dt}\left[x(t) - f(t, x(t))\right] &= \textstyle\int_0^t g(s, x(s))\, ds + \epsilon_n, \quad t \in J \\ x(t_0) &= x_0 + \epsilon_n \end{aligned}\right\} \tag{4.3}$$

defined on J.

Since, by Theorems 3.1 and 3.2, $\{r(t, \epsilon_n)\}$ is a decreasing sequence of positive real numbers, the limit

$$r(t) = \lim_{n\to\infty} r(t, \epsilon_n) \tag{4.4}$$

exists. We show that the convergence in (4.4) is uniform on J. To finish, it is enough to prove that the sequence $\{r(t, \epsilon_n)\}$ is equi-continuous in $C(J, \mathbb{R})$. Let $t_1, t_2 \in J$ be arbitrary. Then,

$$
\begin{aligned}
&|r(t_1,\epsilon_n) - r(t_2,\epsilon_n)| \\
&\quad \le \big|f(t_1, r(t_1,\epsilon_n)) - f(t_2, r(t_2,\epsilon_n))\big| \\
&\qquad + \left|\int_{t_0}^{t_1} (t_1 - s) g(s, r_{\epsilon_n}(s))\, ds - \int_{t_0}^{t_2} (t_2 - s) g(s, r_{\epsilon_n}(s))\, ds\right| \\
&\qquad + \left|\int_{t_0}^{t_1} (t_1 - s)\epsilon_n\, ds - \int_{t_0}^{t_2} (t_2 - s)\epsilon_n\, ds\right| \\
&\quad \le \big|f(t_1, r(t_1,\epsilon_n)) - f(t_2, r(t_2,\epsilon_n))\big| \\
&\qquad + \left|\int_{t_0}^{t_1} (t_1 - s) g(s, r_{\epsilon_n}(s))\, ds - \int_{t_0}^{t_1} (t_2 - s) g(s, r_{\epsilon_n}(s))\, ds\right| \\
&\qquad + \left|\int_{t_0}^{t_1} (t_2 - s) g(s, r_{\epsilon_n}(s))\, ds - \int_{t_0}^{t_2} (t_2 - s) g(s, r_{\epsilon_n}(s))\, ds\right| \\
&\qquad + \left|\int_{t_0}^{t_1} (t_1 - s)\epsilon_n\, ds - \int_{t_0}^{t_1} (t_2 - s)\epsilon_n\, ds\right| \\
&\qquad + \left|\int_{t_0}^{t_1} (t_2 - s)\epsilon_n\, ds - \int_{t_0}^{t_2} (t_2 - s)\epsilon_n\, ds\right| \\
&\quad \le \big|f(t_1, r(t_1,\epsilon_n)) - f(t_2, r(t_2,\epsilon_n))\big| \\
&\qquad + \left|\int_{t_0}^{t_1} (t_1 - t_2) g(s, r_{\epsilon_n}(s))\, ds\right| + \left|\int_{t_1}^{t_2} (t_2 - s) g(s, r_{\epsilon_n}(s))\, ds\right| \\
&\qquad + \left|\int_{t_0}^{t_1} (t_1 - t_2)\epsilon_n\, ds\right| + \left|\int_{t_1}^{t_2} (t_2 - s)\epsilon_n\, ds\right| \\
&\quad \le \big|f(t_1, r(t_1,\epsilon_n)) - f(t_2, r(t_2,\epsilon_n))\big| \\
&\qquad + \int_{t_0}^{t_0+a} |t_1 - t_2||g(s, r_{\epsilon_n}(s))|\, ds + \left|\int_{t_1}^{t_2} |g(s, r_{\epsilon_n}(s))|\, ds\right| \\
&\qquad + \int_{t_0}^{t_0+a} |t_1 - t_2|\epsilon_n\, ds + \left|\int_{t_1}^{t_2} (t_0 + a)\epsilon_n\, ds\right| \\
&\quad \le \big|f(t_1, r(t_1,\epsilon_n)) - f(t_2, r(t_2,\epsilon_n))\big| \\
&\qquad + a\|h\||t_1 - t_2| + \left|\int_{t_1}^{t_2} h(s)\, ds\right| + a|t_1 - t_2|\epsilon_n + |t_1 - t_2|(t_0 + a)\epsilon_n \\
&\quad \le \big|f(t_1, r(t_1,\epsilon_n)) - f(t_2, r(t_2,\epsilon_n))\big| + c|t_1 - t_2| + |p(t_1) - p(t_2)|
\end{aligned}
\tag{4.5}
$$

where, $p(t) = \int_{t_0}^{t} h(s)\,ds$ and $c = [(t_0 + 2a)\epsilon_n + a\|h\|]$.

Since f is continuous on compact set $J \times [-N, N]$, they are uniformly continuous there. Hence,

$$\big|f(t_1, r(t_1, \epsilon_n)) - f(t_2, r(t_2, \epsilon_n))\big| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $n \in \mathbb{N}$. Similarly, since the function p is continuous on compact set J, it is uniformly continuous and hence

$$\big|p(t_1) - p(t_2)\big| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $t_1, t_2 \in J$.

Therefore, from the above inequality (4.5), it follows that

$$|r(t_1, \epsilon_n) - r(t_1, \epsilon_n)| \to 0 \quad \text{as} \quad t_1 \to t_2$$

uniformly for all $n \in \mathbb{N}$. Therefore,

$$r(t, \epsilon_n) \to r(t) \quad \text{as} \quad n \to \infty$$

for all $t \in J$. Next, we show that the function $r(t)$ is a solution of the HIDE (3.1) defined on J. Now, since $r(t, \epsilon_n)$ is a solution of the HIDE (4.3), we have

$$r(t, \epsilon_n) = x_0 + \epsilon_n + f(t, r(t, \epsilon_n)) + \int_{t_0}^{t} (t - s) g(s, r_{\epsilon_n}(s))\,ds \tag{4.6}$$

for all $t \in J$. Taking the limit as $n \to \infty$ in the above equation (4.6) yields

$$r(t) = x_0 - f(t_0, x_0) + f(t, r(t)) + \int_{t_0}^{t} (t - s) g(s, r(s))\,ds$$

for $t \in J$. Thus, the function r is a solution of the HIDE (1.1) on J. Finally, form the inequality (4.2) it follows that

$$u(t) \leq r(t)$$

for all $t \in J$. Hence the HIDE (1.1) has a maximal solution on J. This completes the proof.

In the following section we prove the comparison principle for the hybrid integro-differential equation ((1.1).

5. Comparison Theorems

The main problem of the integro-differential inequalities is to estimate a bound for the solution set for the integro-differential inequality related to the HIDE (1.1). In this section we prove that the maximal and minimal solutions serve the bounds for the solutions of the related integro-differential inequality to HIDE (1.1) on $J = [t_0, t_0 + a]$.

Theorem 5.1. Assume that the hypotheses (A_0)-(A_2) hold. Further, if there exists a function $u \in C(J, \mathbb{R})$ such that

$$\left.\begin{aligned} \tfrac{d}{dt}\left[u(t) - f(t, u(t))\right] &\leq \int_{t_0}^{t} g(s, u(s))\, ds \;\; t \in J \\ u(t_0) &\leq x_0. \end{aligned}\right\} \tag{5.1}$$

Then,

$$u(t) \leq r(t) \tag{5.2}$$

for all $t \in J$, where r is a maximal solution of the HIDE (1.1) on J.

Proof. Let $\epsilon > 0$ be arbitrary small. Then, by Theorem 4.2, $r(t, \epsilon)$ is a maximal solution of the HIDE (4.1)) and that the limit

$$r(t) = \lim_{\epsilon \to 0} r(t, \epsilon) \tag{5.3}$$

is uniform on J and the function r is a maximal solution of the HIDE (1.1) on J. Hence, we obtain

$$\left.\begin{aligned} \tfrac{d}{dt}\left[r(t, \epsilon) - f(t, r(t, \epsilon))\right] &= \int_{t_0}^{t} g(s, r(s, \epsilon)) + \epsilon, \;\; t \in J \\ r(t_0, \epsilon) &= x_0 + \epsilon. \end{aligned}\right\} \tag{5.4}$$

From above inequality it follows that

$$\left.\begin{aligned} \tfrac{d}{dt}\left[r(t, \epsilon) - f(t, r(t, \epsilon))\right] &> \int_{t_0}^{t} g(s, r(s, \epsilon)), \;\; t \in J \\ r(t_0, \epsilon) &> x_0. \end{aligned}\right\} \tag{5.5}$$

Now we apply Theorem 2.1 to the inequalities (5.1) and (5.5) and conclude that

$$u(t) < r(t, \epsilon) \tag{5.6}$$

for all $t \in J$. This further in view of limit (5.3) implies that inequality (5.2) holds on J. This completes the proof.

Theorem 5.2. Assume that the hypotheses (A_0)-(A_2) hold. Further, if there exists a function $v \in C(J, \mathbb{R})$ such that

$$\left.\begin{array}{rcl} \frac{d}{dt}\left[v(t) - f(t, v(t))\right] & \geq & \int_{t_0}^{t} g(s, v(s)), \ \ t \in J \\ v(t_0) & \geq & x_0. \end{array}\right\} \tag{5.7}$$

Then,

$$\rho(t) \leq v(t) \tag{5.8}$$

for all $t \in J$, where ρ is a minimal solution of the HIDE (1.1) on J.

Note that Theorem 5.1 is useful to prove the boundedness and uniqueness of the solutions for the HIDE (1.1) on J. A result in this direction is

Theorem 5.3. Assume that the hypotheses (A_0)-(A_2) hold. Suppose that there exists a function $G : J \times \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned} &\big|g(t, x_1(t)) - g(t, x_2(t))\big| \\ &\qquad \leq G\left(t, \sup_{t_0 \leq s \leq t} \big|\big(x_1(s) - f(s, x_1(s))\big) - \big(x_2(s) - f(s, x_2(s))\big)\big|\right) \end{aligned} \tag{5.9}$$

for all $t \in J$ and $x_1, x_2 \in E$. If identically zero function is the only solution of the integro-differential equation

$$m'(t) = \int_{t_0}^{t} G(s, m(s))\, ds, \ \ t \in J, \ \ m(t_0) = 0, \tag{5.10}$$

then the HIDE (1.1) has a unique solution defined on J.

Proof. By Theorem 3.2, the HIDE (1.1) has a solution defined on J. Suppose that there are two solutions u_1 and u_2 of the HIDE (1.1) existing on J. Define a function $m : J \to \mathbb{R}_+$ by

$$m(t) = \big|\big(u_1(t) - f(t, u_1(t))\big) - \big(u_2(t) - f(t, u_2(t))\big)\big|. \tag{5.11}$$

As $(|x(t)|)' \leq |x'(t)|$ for $t \in J$, we have that

$$
\begin{aligned}
m'(t) &\leq \left| \frac{d}{dt}\left[u_1(t) - f(t, u_1(t))\right] - \frac{d}{dt}\left[u_2(t) - f(t, u_2(t))\right] \right| \\
&\leq \int_{t_0}^{t} |g(s, u_1(s)) - g(s, u_2(s))| \, ds \\
&\leq \int_{t_0}^{t} G\Big(s, \sup_{t_0 \leq s \leq t} \Big| \big(u_1(s) - f(s, u_1(s))\big) - \big(u_2(s) - f(s, u_2(s))\big) \Big| \Big) \, ds \\
&= \int_{t_0}^{t} G(s, m(s)) \, ds
\end{aligned}
$$

for all $t \in J$; and that $m(t_0) = 0$.

Now, we apply Theorem 5.1 with $f \equiv 0$ to get that $m(t) = 0$ for all $t \in J$. This gives

$$
u_1(t) - f(t, u_1(t)) = u_2(t) - f(t, u_2(t))
$$

for all $t \in J$. Finally, in view of hypothesis (A$_0$) we conclude that $u_1(t) = u_2(t)$ on J. This completes the proof.

6. Extremal Solutions in Vector Segments

Sometimes it is desirable to have knowledge of existence of extremal solutions for the HIDE (1.1) in a vector segment defined on J. Therefore, in this section we shall prove the existence of maximal and minimal solutions for HIDE (1.1) between the given upper and lower solutions on $J = [t_0, t_0 + a]$. We use a hybrid fixed point theorem of Dhage [3] in ordered Banach space for establishing our results. We need the following preliminaries in the sequel.

A non-empty closed set K in a Banach space E is called a **cone** with vertex 0, if (i) $K + K \subseteq K$, (ii) $\lambda K \subseteq K$ for $\lambda \in \mathbb{R}, \lambda \geq 0$ and (iii) $\{-K\} \cap K = 0$, where 0 is the zero element of E. We introduce an order relation $\leq$ in E as follows. Let $x, y \in E$. Then $x \leq y$ if and only if $y - x \in K$. A cone K is called to be **normal** if the norm $\| \cdot \|$ is semi-monotone increasing on K, that is, there is a constant $N > 0$ such that $\|x\| \leq N\|y\|$ for all $x, y \in K$ with $x \leq y$. It is known that if the cone K is normal in E, then every order-bounded set in E is norm-bounded. The details of cones and their properties appear in Heikkilä and Lakshmikantham [8].

For any $a, b \in E, a \leq b$, the order interval $[a, b]$ is a set in E given by

$$[a, b] = \{x \in E : a \leq x \leq b\}.$$

Definition A mapping $T : [a, b] \to E$ is said to be **nondecreasing** or **monotone increasing** if $x \leq y$ implies $Tx \leq Ty$ for all $x, y \in [a, b]$.

We use the following fixed point theorems of Dhage [4] for proving the existence of extremal solutions for the IVP (1.1) under certain monotonicity conditions.

Theorem 6.1. Let K be a cone in a Banach space E and let $a, b \in E$. be such that $a \leq b$. Suppose that $A, B : [a, b] \to E$ are two nondecreasing operators such that

(a) A is nonlinear contraction,

(b) B is completely continuous, and

(c) $Ax + Cx \in [a, b]$ for each $x \in [a, b]$.

Further, if the cone K is normal, then the operator equation $Ax + Cx = x$ has a least and a greatest solution in $[a, b]$.

We equip the space $C(J, \mathbb{R})$ with the order relation $\leq$ with the help of the cone K in it defined by

$$K = \big\{x \in C(J, \mathbb{R}) : x(t) \geq 0 \text{ for all } t \in J\big\}. \tag{6.1}$$

It is well known that the cone K is a normal in $C(J, \mathbb{R})$. We need the following definitions in the sequel.

Definition A function $a \in C(J, \mathbb{R})$ is called a lower solution of the HIDE (1.1) defined on J if the map $t \mapsto x - f(t, x)$ is continuous for every $x \in \mathbb{R}$ and satisfies

$$\left.\begin{array}{rcl} \frac{d}{dt}\left[a(t) - f(t, a(t))\right] & \leq & \int_{t_0}^{t} g(s, a(s))\, ds, \;\; t \in J \\ a(t_0) & \leq & x_0. \end{array}\right\} \tag{6.2}$$

Similarly, a function $b \in C(J, \mathbb{R})$ is called an upper solution of the HIDE (1.1) defined on J if the map $t \mapsto x - f(t, x)$ is continuous for every $x \in \mathbb{R}$ and satisfies

$$\left.\begin{array}{rcl} \frac{d}{dt}\left[b(t) - f(t, b(t))\right] & \geq & \int_{t_0}^{t} g(s, b(s))\, ds, \;\; t \in J \\ b(t_0) & \geq & x_0. \end{array}\right\} \tag{6.3}$$

A solution to the HIDE (1.1) is a lower as well as an upper solution for the HIDE (1.1) defined on J and vice versa.

We consider the following set of assumptions:

(B_1) The HIDE (1.1) has a lower solution a and an upper solution b defined on J with $a \leq b$.

(B_2) The function $x \mapsto x - f(t, x)$ is increasing in the interval $\left[\min_{t \in J} a(t), \max_{t \in J} b(t)\right]$ for $t \in J$.

(B_3) The functions $f(t, x)$ and $g(t, x)$ are nondecreasing in x for all $t \in J$.

(B_4) There exists a continuous function $h : J \to \mathbb{R}$ such that

$$g(s, b(s)) \leq h(t)$$

for all $t \in J$.

Theorem 6.2. Suppose that the assumptions (A_1) and (B_1) through (B_4) hold. Then the HIDE (1.1) has a minimal and a maximal solution in $[a, b]$ defined on J.

Proof. Now, the HIDE (1.1) is equivalent to hybrid integral equation (3.4) defined on J. Let $E = C(J, \mathbb{R})$. Define three operators A and B on $[a, b]$ by (3.5) and (3.6) respectively. Then the integral equation (3.4) is transformed into an operator equation as $Ax(t) + Bx(t) = x(t)$ in the ordered Banach space E. Notice that hypothesis (B_1) implies $A, B : [a, b] \to E$. Since the cone K in E is normal, $[a, b]$ is a norm-bounded set in E. Now it is shown, as in the proof of Theorem 3.2, that the operators A is nonlinear contraction. Similarly, B is completely continuous operator on $[a, b]$ into E. Again, the hypothesis (B_3) implies that A and B are nondecreasing on $[a, b]$. To see this, let $x, y \in [a, b]$ be such that $x \le y$. Then, by hypothesis (B_3),

$$Ax(t) = f(t, x(t)) \le f(t, y(t)) = Ay(t)$$

for all $t \in J$. Similarly, we have

$$\begin{aligned} Bx(t) &= x_0 - f(t_0, x_0) + \int_{t_0}^{t} (t-s) g(s, x(s))\, ds \\ &\le x_0 - f(t_0, x_0) + \int_{t_0}^{t} (t-s) g(s, y(s))\, ds \\ &= By(t) \end{aligned}$$

for all $t \in J$. So A and B are nondecreasing operators on $[a, b]$. Hence, for any $x \in [a, b]$ we obtain

$$\begin{aligned} a(t) &\le x_0 - f(t_0, x_0) + f(t, a(t)) + \int_{t_0}^{t} (t-s) g(s, a(s))\, ds \\ &\le x_0 - f(t_0, x_0) + f(t, x(t)) + \int_{t_0}^{t} (t-s) g(s, x(s))\, ds \\ &\le x_0 - f(t_0, x_0) + f(t, b(t)) + \int_{t_0}^{t} (t-s) g(s, b(s))\, ds \\ &\le b(t), \end{aligned}$$

for all $t \in J$. As a result $a(t) \le Ax(t) + Bx(t) \le b(t)$ for all $t \in J$ and $x \in [a, b]$. Hence, $Ax + Bx \in [a, b]$ for all $x \in [a, b]$.

Now, we apply Theorem 6.1 to the operator equation $Ax + Bx = x$ to yield that the HIDE (1.1) has a minimal and a maximal solution in $[a, b]$ defined on J. This completes the proof.

Remark The hybrid integro-differential equations is a rich area for variety of nonlinear ordinary as well as partial integro-differential equations. Here, in this paper, we have considered a very simple hybrid integro-differential equation involving two nonlinearities, however, a more complex hybrid integro-differential equation can also be studied on similar lines with appropriate modifications. Again, the results proved in this paper are very fundamental in nature and therefore, all other problems for the hybrid integro-differential equation in question are still open. In a forthcoming paper we plan to prove some approximation results for the hybrid integro-differential equation considered in this paper.

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