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SOME INEQUALITIES OF HERMITE-HADAMARD TYPE FOR h-CONVEX FUNCTIONS

BO-YAN XI[1,*] AND FENG QI[2]

[1]College of Mathematics, Inner Mongolia University for Nationalities,
Tongliao City, Inner Mongolia Autonomous Region, 028043, China
[2]School of Mathematics and Informatics, Henan Polytechnic University,
Jiaozuo City, Henan Province, 454010, China

Abstract. In the paper, the authors establish some new integral inequalities of Hermite-Hadamard type for h-convex functions.

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1. Introduction

Throughout this paper, we use the following notations:

$$\mathbb{R} = (-\infty, \infty), \quad \mathbb{R}_0 = [0, \infty), \quad \text{and} \quad \mathbb{R}_+ = (0, \infty).$$

We first recite some definitions of various convex functions.

Definition 1.1. A function $f : I \subseteq \mathbb{R} \to \mathbb{R}$ is said to be convex if

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y) \tag{1.1}$$

holds for all $x, y \in I$ and $\lambda \in [0, 1]$.

*Corresponding author

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Definition 1.2 ([17])**.** Let $I, J \subseteq \mathbb{R}$ be intervals, $(0,1) \subseteq J$, and $h : J \to \mathbb{R}$ be a non-negative function. A function $f : I \to \mathbb{R}$ is called h-convex, or say, f belongs to the class $\mathrm{SX}(h, I)$, if f is non-negative and

$$f(tx + (1-t)y) \le h(t)f(x) + h(1-t)f(y) \tag{1.2}$$

for all $x, y \in I$ and $t \in [0,1]$.

The following theorems are some inequalities of Hermite-Hadamard type for the above mentioned convex functions.

Theorem 1.3 ([9, Theorem 2.2])**.** *Let $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping and $a, b \in I^\circ$ with $a < b$. If $|f'(x)|$ is convex on $[a,b]$, then*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{b-a}{8}\big(|f'(a)| + |f'(b)|\big). \tag{1.3}$$

Theorem 1.4 ([13, Theorem 4])**.** *Let $I \subseteq \mathbb{R}$ be an open interval, $a, b \in I$ with $a < b$, $f : I \to \mathbb{R}$ be a twice differentiable mapping such that $f''(x)$ is integrable. If $|f''(x)|$ is a convex function on $[a,b]$ and $0 \le \lambda \le 1$, then*

$$\begin{aligned} &\left|(\lambda-1)f\left(\frac{a+b}{2}\right) - \lambda\frac{f(a)+f(b)}{2} + \int_a^b f(x)\,\mathrm{d}x\right| \\ &\le \begin{cases} \dfrac{(b-a)^2}{24}\left\{\left[\lambda^4 + (1+\lambda)(1-\lambda)^3 + \dfrac{5\lambda-3}{4}\right]|f''(a)| \right. \\ \qquad \left. + \left[\lambda^4 + (2-\lambda)\lambda^3 + \dfrac{1-3\lambda}{4}\right]|f''(b)|\right\}, & 0 \le \lambda \le \dfrac{1}{2}; \\ \dfrac{(b-a)^2}{48}(3\lambda-1)(|f''(a)| + |f''(b)|), & \dfrac{1}{2} \le \lambda \le 1. \end{cases} \end{aligned} \tag{1.4}$$

For more information on this topic, please refer to [1, 2, 3, 6, 7, 10, 13, 14, 15, 17], recently published articles [4, 5, 8, 11, 12, 16, 18, 19, 20, 21, 22, 23, 24, 25] by the authors, and closely related references therein.

Theorem 1.5 ([2, Theorems 1 and 2])**.** *Let $I, J \subseteq \mathbb{R}$ be intervals, $(0,1) \subseteq J$, and $h : J \to \mathbb{R}_0$. Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L([a,b])$ for $a, b \in I$ with $a < b$. If $|f''|$ is an h-convex function on $[a,b]$ for some fixed $q > 1$, then*

$$\left|f\left(\frac{a+b}{2}\right)-\int_0^1 f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{1/p}\left[\int_a^b h(x)\,\mathrm{d}x\right]^{1/q} \times\left\{\left[|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right]^{1/q}\right\} \tag{1.5}$$

and

$$\left|f\left(\frac{a+b}{2}\right)-\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{48}\left[\int_0^1 h(x)\,\mathrm{d}x\right]^{1/q} \times\left\{\left[|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right]^{1/q}\right\}, \tag{1.6}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

In this paper, we will establish some new integral inequalities of Hermite-Hadamard type for h-convex functions.

2. Lemmas

For establishing our new integral inequalities of Hermite-Hadamard type for h-convex functions, we need the following integral identity.

Lemma 2.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a twice differentiable function I° and $\lambda \in \mathbb{R}$. If $f'' \in L([a,b])$ for $a,b \in I$ with $a<b$, then*

$$\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x = \frac{(b-a)^2}{16}\int_0^1 t(2\lambda-t)\left[f''\left((1-t)a+t\frac{a+b}{2}\right)+f''\left(t\frac{a+b}{2}+(1-t)b\right)\right]\mathrm{d}t. \tag{2.1}$$

Proof. Integrating by part yields

$$\begin{aligned}
&\frac{(b-a)^2}{16}\int_0^1 t(2\lambda-t)\left[f''\left((1-t)a+t\frac{a+b}{2}\right)+f''\left(t\frac{a+b}{2}+(1-t)b\right)\right]\mathrm{d}t \\
&=\frac{b-a}{8}\left[t(2\lambda-t)f'\left((1-t)a+t\frac{a+b}{2}\right)\Big|_0^1+\int_0^1(2t-2\lambda)f'\left((1-t)a+t\frac{a+b}{2}\right)\mathrm{d}t\right. \\
&\quad\left.-t(2\lambda-t)f'\left(t\frac{a+b}{2}+(1-t)b\right)\Big|_0^1-\int_0^1(2t-2\lambda)f'\left(t\frac{a+b}{2}+(1-t)b\right)\mathrm{d}t\right] \\
&=\frac{b-a}{4}\int_0^1(t-\lambda)\left[f'\left((1-t)a+t\frac{a+b}{2}\right)-f'\left(t\frac{a+b}{2}+(1-t)b\right)\right]\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\biggl[(t-\lambda)f\biggl((1-t)a+t\frac{a+b}{2}\biggr)\bigg|_0^1 - \int_0^1 f\biggl((1-t)a+t\frac{a+b}{2}\biggr)\,\mathrm{d}t \\
&\quad + (t-\lambda)f\biggl(t\frac{a+b}{2}+(1-t)b\biggr)\bigg|_0^1 - \int_0^1 f\biggl(t\frac{a+b}{2}+(1-t)b\biggr)\,\mathrm{d}t\biggr] \\
&= \lambda\frac{f(a)+f(b)}{2} + (1-\lambda)f\biggl(\frac{a+b}{2}\biggr) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x.
\end{aligned}
$$

The completes the proof of Lemma 2.1. □

Lemma 2.2. *Let* $0 \le \lambda \le 1$ *and* $r > -1$. *Then*

$$
\int_0^1 t|2\lambda - t|^r \,\mathrm{d}t = \begin{cases} \dfrac{(2\lambda)^{r+2} + (2\lambda + r + 1)(1-2\lambda)^{r+1}}{(r+1)(r+2)}, & 0 \le \lambda \le \dfrac{1}{2}, \\ \dfrac{(2\lambda)^{r+2} - (2\lambda + r + 1)(2\lambda-1)^{r+1}}{(r+1)(r+2)}, & \dfrac{1}{2} \le \lambda \le 1. \end{cases}
$$

Proof. The proof is straightforward. □

3. Some new integral inequalities of Hermite-Hadamard type

We are now in a position to establish some new integral inequalities of Hermite-Hadamard type for differentiable and h-convex functions.

Theorem 3.1. *Let* $I, J \subseteq \mathbb{R}$ *be intervals,* $(0,1) \subseteq J$, *and* $h : J \to \mathbb{R}_0$. *Let* $f : I \to \mathbb{R}$ *be a twice differentiable function on* I° *such that* $f'' \in L([a,b])$ *for* $a, b \in I$ *with* $a < b$. *If* $0 \le \lambda \le 1$ *and* $|f''|^q$ *is an* h*-convex function on* $[a,b]$ *for* $q \ge 1$, *then*

$$
\begin{aligned}
&\biggl|\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)f\biggl(\frac{a+b}{2}\biggr) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\biggr| \le \frac{(b-a)^2}{16}[H_1(\lambda)]^{1-1/q} \\
&\quad \times \Biggl\{\biggl[|f''(a)|^q \int_0^1 t|2\lambda - t|h(1-t)\,\mathrm{d}t + \biggl|f''\biggl(\frac{a+b}{2}\biggr)\biggr|^q \int_0^1 t|2\lambda - t|h(t)\,\mathrm{d}t\biggr]^{1/q} \\
&\qquad + \biggl[\biggl|f''\biggl(\frac{a+b}{2}\biggr)\biggr|^q \int_0^1 t|2\lambda - t|h(t)\,\mathrm{d}t + |f''(b)|^q \int_0^1 t|2\lambda - t|h(1-t)\,\mathrm{d}t\biggr]^{1/q}\Biggr\},
\end{aligned} \tag{3.1}
$$

where

$$
H_1(\lambda) = \begin{cases} \dfrac{8\lambda^3 - 3\lambda + 1}{3}, & 0 \le \lambda \le \dfrac{1}{2}, \\ \dfrac{3\lambda - 1}{3}, & \dfrac{1}{2} \le \lambda \le 1. \end{cases} \tag{3.2}
$$

Proof. From Lemma 2.1 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\
&\le\frac{(b-a)^2}{16}\left\{\int_0^1 t|2\lambda-t|\left[\left|f''\left((1-t)a+t\frac{a+b}{2}\right)\right|+\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}t\right\} \\
&\le\frac{(b-a)^2}{16}\left(\int_0^1 t|2\lambda-t|\,\mathrm{d}t\right)^{1-1/q}\left\{\left[\int_0^1 t|2\lambda-t|\left|f''\left((1-t)a+t\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right]^{1/q}\right. \\
&\quad\left.+\left[\int_0^1 t|2\lambda-t|\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q\mathrm{d}t\right]^{1/q}\right\}.
\end{aligned} \tag{3.3}$$

Using Lemma 2.2, we have

$$\int_0^1 t|2\lambda-t|\,\mathrm{d}t=\begin{cases}\dfrac{8\lambda^3-3\lambda+1}{3}, & 0\le\lambda\le\dfrac{1}{2}\\ \dfrac{3\lambda-1}{3}, & \dfrac{1}{2}\le\lambda\le 1.\end{cases} \tag{3.4}$$

Therefore, by the h-convexity of $|f''|^q$, we obtain

$$\begin{aligned}
&\int_0^1 t|2\lambda-t|\left|f''\left((1-t)a+t\frac{a+b}{2}\right)\right|^q\mathrm{d}t \\
&\qquad\le|f''(a)|^q\int_0^1 t|2\lambda-t|h(1-t)\,\mathrm{d}t+\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t|2\lambda-t|h(t)\,\mathrm{d}t
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
&\int_0^1 t|2\lambda-t|\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q\mathrm{d}t \\
&\qquad\le\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t|2\lambda-t|h(t)\,\mathrm{d}t+|f''(b)|^q\int_0^1 t|2\lambda-t|h(1-t)\,\mathrm{d}t.
\end{aligned} \tag{3.6}$$

Substituting the equation (3.4) and the inequalities (3.5) and (3.6) into the inequality (3.3) yields the inequality (3.1). Theorem 3.1 is thus proved. □

Corollary 3.2. *Under the conditions of Theorem 3.1, if $h: J\to\mathbb{R}_0$ is symmetric to $\frac{1}{2}$, then*

$$\begin{aligned}
&\left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{(b-a)^2}{16}\left[\int_0^1 t|2\lambda-t|h(t)\,\mathrm{d}t\right]^{1/q} \\
&\quad\times[H_1(\lambda)]^{1-1/q}\left\{\left[|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right]^{1/q}\right\};
\end{aligned}$$

Furthermore, if $\lambda=0$ also, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{3^{1/q}(b-a)^2}{48}\left[\int_0^1 t^2h(t)\,\mathrm{d}t\right]^{1/q}\times\left\{\left[|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right]^{1/q}\right\}. \tag{3.7}$$

Corollary 3.3. *Under the conditions of Theorem 3.1, when $q=1$, we have*

$$\left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{(b-a)^2}{16}\times\left\{\big[|f''(a)|+|f''(b)|\big]\int_0^1 t|2\lambda-t|h(1-t)\,\mathrm{d}t+2\left|f''\left(\frac{a+b}{2}\right)\right|\int_0^1 t|2\lambda-t|h(t)\,\mathrm{d}t\right\}. \tag{3.8}$$

Corollary 3.4. *Under the conditions of Theorem 3.1,*

(1) *when $\lambda=0$, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{3^{1/q}(b-a)^2}{48}\times\left\{\left[|f''(a)|^q\int_0^1 t^2h(1-t)\,\mathrm{d}t+\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t^2h(t)\,\mathrm{d}t\right]^{1/q}+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t^2h(t)\,\mathrm{d}t+|f''(b)|^q\int_0^1 t^2h(1-t)\,\mathrm{d}t\right]^{1/q}\right\}; \tag{3.9}$$

(2) *when $\lambda=\frac{1}{3}$, we have*

$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{(b-a)^2}{162}\left(\frac{81}{8}\right)^{1/q}\times\left\{\left[|f''(a)|^q\int_0^1 t\left|\frac{2}{3}-t\right|h(1-t)\,\mathrm{d}t+\left|f''\left(\frac{a+b}{2}\right)\right|\right]^{1/q}\int_0^1 t\left|\frac{2}{3}-t\right|h(t)\,\mathrm{d}t+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t\left|\frac{2}{3}-t\right|h(t)\,\mathrm{d}t+|f''(b)|^q\int_0^1 t\left|\frac{2}{3}-t\right|h(1-t)\,\mathrm{d}t\right]^{1/q}\right\}; \tag{3.10}$$

(3) *when $\lambda=\frac{1}{2}$, we have*

$$\left|\frac{1}{2}\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{6^{1/q}(b-a)^2}{96}\left(\int_0^1 t(1-t)h(t)\,\mathrm{d}t\right)\times\left\{\left[|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right]^{1/q}\right\}; \tag{3.11}$$

(4) *when* $\lambda = 1$, *we have*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{24}\left(\frac{3}{2}\right)^{1/q} \tag{3.12}$$
$$\times\left\{\left[|f''(a)|^q\int_0^1 t(2-t)h(1-t)\,\mathrm{d}t + \left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t(2-t)h(t)\,\mathrm{d}t\right]^{1/q}\right.$$
$$\left.+\left[\int_0^1 t(2-t)h(t)\,\mathrm{d}t\left|f''\left(\frac{a+b}{2}\right)\right|^q + |f''(b)|^q\int_0^1 t(2-t)h(1-t)\,\mathrm{d}t\right]^{1/q}\right\}.$$

Theorem 3.5. *Let* $I, J \subseteq \mathbb{R}$ *be intervals,* $(0,1) \subseteq J$, *and* $h : J \to \mathbb{R}_0$. *Let* $f : I \to \mathbb{R}$ *be a twice differentiable function on* I° *such that* $f'' \in L([a,b])$ *for* $a, b \in I$ *with* $a < b$. *If* $0 \le \lambda \le 1$ *and* $|f''|^q$ *is an* h*-convex function on* $[a,b]$ *for* $q > 1$, *then*

$$\left|\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{16}[H_2(\lambda)]^{1-1/q} \tag{3.13}$$
$$\times\left\{\left[|f''(a)|^q\int_0^1 th(1-t)\,\mathrm{d}t + \left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 th(t)\,\mathrm{d}t\right]^{1/q}\right.$$
$$\left.+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 th(t)\,\mathrm{d}t + |f''(b)|^q\int_0^1 th(1-t)\,\mathrm{d}t\right]^{1/q}\right\},$$

where

$$H_2(\lambda) = \begin{cases} \dfrac{(q-1)\big[(q-1)(2\lambda)^{(3q-2)/(q-1)} + (2q+2\lambda q-2\lambda-1)(1-2\lambda)^{(2q-1)/(q-1)}\big]}{(2q-1)(3q-2)}, & 0 \le \lambda \le \dfrac{1}{2}, \\ \dfrac{(q-1)\big\{(q-1)(2\lambda)^{(3q-2)/(q-1)} + \big[2q-4\lambda^2(q-1)-2\lambda q-1\big](2\lambda-1)^{q/(q-1)}\big\}}{(2q-1)(3q-2)}, & \dfrac{1}{2} \le \lambda \le 1. \end{cases} \tag{3.14}$$

Proof. Using Lemma 2.1, Hölder's inequality, and the h-convexity of $|f''|^q$, we have

$$\left|\lambda\frac{f(a)+f(b)}{2} + (1-\lambda)f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{16}$$
$$\times\left\{\int_0^1 t|2\lambda-t|\left[\left|f''\left((1-t)a+t\frac{a+b}{2}\right)\right| + \left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}t\right\}$$
$$\le \frac{(b-a)^2}{16}\left[\int_0^1 t|2\lambda-t|^{q/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\left\{\left[\int_0^1 t\left|f''\left((1-t)a+t\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right]^{1/q}\right.$$
$$\left.+\left[\int_0^1 t\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q\mathrm{d}t\right]^{1/q}\right\}$$

$$\le \frac{(b-a)^2}{16}[H_2(\lambda)]^{1-1/q}\left\{\left[|f''(a)|^q\int_0^1 th(1-t)\,\mathrm{d}t+\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 th(t)\,\mathrm{d}t\right]^{1/q}\right.$$
$$\left.+\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 th(t)\,\mathrm{d}t+|f''(b)|^q\int_0^1 th(1-t)\,\mathrm{d}t\right]^{1/q}\right\},$$

where we used Lemma 2.2 to deduce $H_2(\lambda)$. The proof of Theorem 3.5 is complete. □

Corollary 3.6. *Under the conditions of Theorem 3.5, if $h: J\to\mathbb{R}_0$ is symmetric to $\frac{1}{2}$, then*

$$\left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{(b-a)^2}{16}[H_2(\lambda)]^{1-1/q}$$
$$\times\left[\int_0^1 th(t)\,\mathrm{d}t\right]^{1/q}\left\{\left[|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right]^{1/q}\right\}; \tag{3.15}$$

Furthermore, if $\lambda=0$ also, then

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{(b-a)^2}{16}\left(\frac{q-1}{3q-2}\right)^{1-1/q}\left[\int_0^1 th(t)\,\mathrm{d}t\right]^{1/q}$$
$$\times\left\{\left[|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right]^{1/q}\right\}. \tag{3.16}$$

Remark 3.7. *Under the conditions of Theorem 3.5, if $\lambda=0,\frac{1}{3},\frac{1}{2},1$, we have*

$$H_2\left(\frac{1}{3}\right)=\frac{q-1}{(2q-1)(3q-2)}\left[(q-1)\left(\frac{2}{3}\right)^{(3q-2)/(q-1)}+(8q-5)\left(\frac{1}{3}\right)^{(3q-2)/(q-1)}\right],$$
$$H_2(0)=\frac{q-1}{3q-2},\quad H_2\left(\frac{1}{2}\right)=\frac{(q-1)^2}{(2q-1)(3q-2)},$$

and

$$H_2(1)=\frac{q-1}{(2q-1)(3q-2)}\left[2^{(3q-2)/(q-1)}(q-1)-4q+3\right].$$

Theorem 3.8. *Let $I,J\subseteq\mathbb{R}$ be intervals, $(0,1)\subseteq J$, and $h:J\to\mathbb{R}_0$. Let $f:I\to\mathbb{R}$ be a twice differentiable function on I° such that $f''\in L([a,b])$ for $a,b\in I$ with $a<b$. If $0\le\lambda\le1$ and $|f''|^q$ is an h-convex function on $[a,b]$ for $q>1$, then*

$$\left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{(b-a)^2}{16}[H_3(\lambda)]^{1-1/q}$$
$$\times\left\{\left[|f''(a)|^q\int_0^1 t^q h(1-t)\,\mathrm{d}t+\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t^q h(t)\,\mathrm{d}t\right]^{1/q}\right. \tag{3.17}$$

$$+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t^q h(t)\,\mathrm{d}t+|f''(b)|^q\int_0^1 t^q h(1-t)\,\mathrm{d}t\right]^{1/q}\right\},$$

where

$$H_3(\lambda)=\begin{cases}\dfrac{(q-1)\left[(2\lambda)^{(2q-1)/(q-1)}+(1-2\lambda)^{(2q-1)/(q-1)}\right]}{2q-1}, & 0\le\lambda\le\dfrac{1}{2},\\ \dfrac{(q-1)\left[(2\lambda)^{(2q-1)/(q-1)}-(2\lambda-1)^{(2q-1)/(q-1)}\right]}{2q-1}, & \dfrac{1}{2}\le\lambda\le 1.\end{cases} \tag{3.18}$$

Proof. By Lemma 2.1, Hölder's inequality, and the h-convexity of $|f''|^q$, we have

$$\begin{aligned}&\left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\\
&\le\frac{(b-a)^2}{16}\left\{\int_0^1 t|2\lambda-t|\left[\left|f''\left((1-t)a+t\frac{a+b}{2}\right)\right|+\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}t\right\}\\
&\le\frac{(b-a)^2}{16}\left[\int_0^1|2\lambda-t|^{q/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\left\{\left[\int_0^1 t^q\left|f''\left((1-t)a+t\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right]^{1/q}\right.\\
&\quad\left.+\left[\int_0^1 t^q\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q\mathrm{d}t\right]^{1/q}\right\}\\
&\le\frac{(b-a)^2}{16}[H_3(\lambda)]^{1-1/q}\left\{\left[|f''(a)|^q\int_0^1 t^q h(1-t)\,\mathrm{d}t+\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t^q h(t)\,\mathrm{d}t\right]^{1/q}\right.\\
&\quad\left.+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t^q h(t)\,\mathrm{d}t+|f''(b)|^q\int_0^1 t^q h(1-t)\,\mathrm{d}t\right]^{1/q}\right\}.\end{aligned}$$

The proof of Theorem 3.8 is complete. □

Corollary 3.9. *Under the conditions of Theorem 3.8, if $h:J\to\mathbb{R}_0$ is symmetric to $\frac{1}{2}$, then*

$$\begin{aligned}&\left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{(b-a)^2}{16}[H_3(\lambda)]^{1-1/q}\\
&\times\left[\int_0^1 t^q h(t)\,\mathrm{d}t\right]^{1/q}\left\{\left[|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right]^{1/q}\right\}.\end{aligned}$$

Remark 3.10. *Under the conditions of Theorem 3.8, if $\lambda=0,\frac{1}{3},\frac{1}{2},1$, we have*

$$H_3(0)=\frac{q-1}{2q-1},\quad H_3\left(\frac{1}{3}\right)=\frac{q-1}{2q-1}\left[\left(\frac{2}{3}\right)^{(2q-1)/(q-1)}+\left(\frac{1}{3}\right)^{(2q-1)/(q-1)}\right],$$

$$H_3\left(\frac{1}{2}\right)=\frac{q-1}{2q-1},\quad H_3(1)=\frac{q-1}{2q-1}\left[2^{(2q-1)/(q-1)}-1\right].$$

Theorem 3.11. *Let $I, J \subseteq \mathbb{R}$ be intervals, $(0,1) \subseteq J$, and $h : J \to \mathbb{R}_0$. Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L([a,b])$ for $a, b \in I$ with $a < b$. If $0 \le \lambda \le 1$ and $|f''|^q$ is an h-convex function on $[a,b]$ for $q > 1$, then*

$$\begin{aligned}(3.19)\quad & \left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{16}\left(\frac{q-1}{2q-1}\right)^{1-1/q} \\ & \times\left\{\left[|f''(a)|^q\int_0^1|2\lambda-t|^q h(1-t)\,\mathrm{d}t+\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1|2\lambda-t|^q h(t)\,\mathrm{d}t\right]^{1/q}\right. \\ & \left.+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1|2\lambda-t|^q h(t)\,\mathrm{d}t+|f''(b)|^q\int_0^1|2\lambda-t|^q h(1-t)\,\mathrm{d}t\right]^{1/q}\right\}.\end{aligned}$$

Proof. Using Lemma 2.1, Hölder's inequality, and the h-convexity of $|f''|^q$, we have

$$\begin{aligned} & \left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ & \le \frac{(b-a)^2}{16}\left\{\int_0^1 t|2\lambda-t|\left[\left|f''\left((1-t)a+t\frac{a+b}{2}\right)\right|+\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}t\right\} \\ & \le \frac{(b-a)^2}{16}\left[\int_0^1 t^{q/(q-1)}\,\mathrm{d}t\right]^{1-1/q}\left\{\left[\int_0^1|2\lambda-t|^q\left|f''\left((1-t)a+t\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right]^{1/q}\right. \\ & \left.+\left[\int_0^1|2\lambda-t|^q\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q\mathrm{d}t\right]^{1/q}\right\} \\ & \le \frac{(b-a)^2}{16}\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left\{\left[|f''(a)|^q\int_0^1|2\lambda-t|^q h(1-t)\,\mathrm{d}t\right.\right. \\ & \left.+\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1|2\lambda-t|^q h(t)\,\mathrm{d}t\right]^{1/q}+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1|2\lambda-t|^q h(t)\,\mathrm{d}t\right. \\ & \left.\left.+|f''(b)|^q\int_0^1|2\lambda-t|^q h(1-t)\,\mathrm{d}t\right]^{1/q}\right\}.\end{aligned}$$

The proof of Theorem 3.11 is complete. □

Corollary 3.12. *Under the conditions of Theorem 3.11, if $h : J \to \mathbb{R}_0$ is symmetric to $\frac{1}{2}$, then*

$$\begin{aligned} & \left|\lambda\frac{f(a)+f(b)}{2}+(1-\lambda)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{16}\left[\int_0^1|2\lambda-t|^q h(t)\,\mathrm{d}t\right]^{1/q} \\ & \times\left(\frac{q-1}{2q-1}\right)^{1-1/q}\left\{\left[|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right]^{1/q}\right\}.\end{aligned}$$

Theorem 3.13. *Let $I, J \subseteq \mathbb{R}$ be intervals, $(0,1) \subseteq J$, and $h : J \to \mathbb{R}_0$. Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L([a,b])$ for $a, b \in I$ with $a < b$. If $|f''|^q$ is an h-convex function on $[a,b]$ for $q > 1$ and $2q \ge r \ge 0$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{16}\left(\frac{q-1}{3q-r-1}\right)^{1-1/q} \times \left\{\left[|f''(a)|^q \int_0^1 t^r h(1-t)\,\mathrm{d}t + \left|f''\left(\frac{a+b}{2}\right)\right|^q \int_0^1 t^r h(t)\,\mathrm{d}t\right]^{1/q} + \left[\left|f''\left(\frac{a+b}{2}\right)\right|^q \int_0^1 t^r h(t)\,\mathrm{d}t + |f''(b)|^q \int_0^1 t^r h(1-t)\,\mathrm{d}t\right]^{1/q}\right\}. \tag{3.20}$$

Proof. Using Lemma 2.1, Hölder's inequality, and the h-convexity of $|f''|^q$, we have

$$\begin{aligned}&\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \\ &\le \frac{(b-a)^2}{16}\left(\int_0^1 t^{(2q-r)/(q-1)}\,\mathrm{d}t\right)^{1-1/q}\left\{\left[\int_0^1 t^r\left|f''\left((1-t)a + t\frac{a+b}{2}\right)\right|^q \mathrm{d}t\right]^{1/q}\right. \\ &\left.+ \left[\int_0^1 t^r\left|f''\left(t\frac{a+b}{2} + (1-t)b\right)\right|^q \mathrm{d}t\right]^{1/q}\right\} \le \frac{(b-a)^2}{16}\left(\frac{q-1}{3q-r-1}\right)^{1-1/q} \\ &\times \left\{\left[|f''(a)|^q \int_0^1 t^r h(1-t)\,\mathrm{d}t + \left|f''\left(\frac{a+b}{2}\right)\right|^q \int_0^1 t^r h(t)\,\mathrm{d}t\right]^{1/q}\right. \\ &\left.+ \left[\left|f''\left(\frac{a+b}{2}\right)\right|^q \int_0^1 t^r h(t)\,\mathrm{d}t + |f''(b)|^q \int_0^1 t^r h(1-t)\,\mathrm{d}t\right]^{1/q}\right\}.\end{aligned}$$

The proof of Theorem 3.13 is complete. □

Corollary 3.14. *Under the conditions of Theorem 3.13, if $h : J \to \mathbb{R}_0$ is symmetric to $\frac{1}{2}$, then*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{16}\left(\frac{q-1}{3q-r-1}\right)^{1-1/q}\left[\int_0^1 t^r h(t)\,\mathrm{d}t\right]^{1/q} \times \left\{\left[|f''(a)|^q + \left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q} + \left[\left|f''\left(\frac{a+b}{2}\right)\right|^q + |f''(b)|^q\right]^{1/q}\right\}; \tag{3.21}$$

Furthermore,

(1) *if $r = 0$ also, we have*

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right| \le \frac{(b-a)^2}{16}\left(\frac{q-1}{3q-1}\right)^{1-1/q}\left[\int_0^1 h(t)\,\mathrm{d}t\right]^{1/q}$$

$$\times\left\{\left[|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right]^{1/q}\right\};$$

(2) *if $r=2q$ also, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le\frac{(b-a)^2}{16}\left[\int_0^1 t^{2q}h(t)\,\mathrm{d}t\right]^{1/q}$$
$$\times\left\{\left[|f''(a)|^q+\left|f''\left(\frac{a+b}{2}\right)\right|^q\right]^{1/q}+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q+|f''(b)|^q\right]^{1/q}\right\}.$$

Theorem 3.15. *Let $I,J\subseteq\mathbb{R}$ be intervals, $(0,1)\subseteq J$, and $h:J\to\mathbb{R}_0$. Let $f:I\to\mathbb{R}$ be a twice differentiable function on I° such that $f''\in L([a,b])$ for $a,b\in I$ with $a<b$. If $|f''|^q$ is an h-convex function on $[a,b]$ for $q>1$ and $q\ge r,s\ge0$, then*

$$\left|\frac12\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|\le B^{1-1/q}\left(\frac{2q-r-1}{q-1},\frac{2q-s-1}{q-1}\right)$$
$$\times\frac{(b-a)^2}{16}\left\{\left[|f''(a)|^q\int_0^1 t^r(1-t)^sh(1-t)\,\mathrm{d}t+\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t^r(1-t)^sh(t)\,\mathrm{d}t\right]^{1/q}\right.$$
$$\left.+\left[\left|f''\left(\frac{a+b}{2}\right)\right|^q\int_0^1 t^r(1-t)^sh(t)\,\mathrm{d}t+|f''(b)|^q\int_0^1 t^r(1-t)^sh(1-t)\,\mathrm{d}t\right]^{1/q}\right\}.$$

where $B(\alpha,\beta)$ denotes the well known Beta function which may be defined by

$$B(\alpha,\beta)=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,\mathrm{d}t,\quad\alpha,\beta>0. \tag{3.22}$$

Proof. Using Lemma 2.1, Hölder's inequality, and the h-convexity of $|f''|^q$, we have

$$\left|\frac12\left[\frac{f(a)+f(b)}{2}+f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x\right|$$
$$\le\frac{(b-a)^2}{16}\left\{\int_0^1 t(1-t)\left[\left|f''\left((1-t)a+t\frac{a+b}{2}\right)\right|+\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|\right]\mathrm{d}t\right\}$$
$$\le\frac{(b-a)^2}{16}\left[\int_0^1 t^{(q-r)/(q-1)}(1-t)^{(q-s)/(q-1)}\,\mathrm{d}t\right]^{1-1/q}$$
$$\times\left\{\left[\int_0^1 t^r(1-t)^s\left|f''\left((1-t)a+t\frac{a+b}{2}\right)\right|^q\mathrm{d}t\right]^{1/q}\right.$$
$$\left.+\left[\int_0^1 t^r(1-t)^s\left|f''\left(t\frac{a+b}{2}+(1-t)b\right)\right|^q\mathrm{d}t\right]^{1/q}\right\}$$
$$\le\frac{(b-a)^2}{16}\left[B\left(\frac{2q-r-1}{q-1},\frac{2q-s-1}{q-1}\right)\right]^{1-1/q}$$

$$\times \left\{ \left[|f''(a)|^q \int_0^1 t^r(1-t)^s h(1-t)\,\mathrm{d}t + \left| f''\left(\frac{a+b}{2}\right)\right|^q \int_0^1 t^r(1-t)^s h(t)\,\mathrm{d}t \right]^{1/q} + \left[\left| f''\left(\frac{a+b}{2}\right)\right|^q \int_0^1 t^r(1-t)^s h(t)\,\mathrm{d}t + |f''(b)|^q \int_0^1 t^r(1-t)^s h(1-t)\,\mathrm{d}t \right]^{1/q} \right\}.$$

The proof of Theorem 3.15 is complete. □

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