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SOME GENERALIZATIONS OF THE ENESTRÖM-KAKEYA THEOREM

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Abstract. In this paper, we obtain some generalizations of a well-known result of Eneström-Kakeya concerning the bounds for the moduli of the zeros of polynomials which extend certain known results in this direction.

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1 Introduction

The following result due to Eneström-Kakeya (see[13, 14]) is well known in theory of the distribution of zeros of polynomials.

Theorem 1.1 (Eneström-Kakeya). *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with real coefficients satisfying*

$$a_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$$

then all the zeros of $P(z)$ lie in $|z| \leq 1$.

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In literature [1, 2, 5 - 7, 9 - 14] there exits several extensions of Eneström-Kakeya theorem. Govil and Rahman [8] generalized this theorem to polynomials with complex coefficients by proving the following.

Theorem 1.2. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients such that*

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2 \ \ , j = 0, 1, \cdots, n,$$

for some real β *and*

$$|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of $P(z)$ *lie in*

$$|z| \leq (\cos\alpha + \sin\alpha) + 2\sin\alpha \sum_{j=0}^{n} \frac{|a_j|}{|a_n|}.$$

A. Aziz and Q. G. Muhammad [3] used matrix method and proved among other things the following generalization of Theorem 1.2.

Theorem 1.3. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients such that*

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2, \ \ j = 0, 1, \cdots, n,$$

for some real β *and for some* $t > 0$,

$$t^n|a_n| \leq t^{n-1}|a_{n-1}| \leq \cdots \leq t^k|a_k| \geq t^{k-1}|a_{k-1}| \geq \cdots \geq t|a_1| \geq |a_0|,$$

where $0 \leq k \leq n$, *then all the zeros of* $P(z)$ *lie in*

$$|z| \leq \left\{ \left(\frac{2t^k|a_k|}{t^n|a_n|} - 1 \right) \cos\alpha + \sin\alpha \right\} + 2\sin\alpha \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n| t^{n-j-1}}.$$

For $k = n$, $\alpha = \beta = 0$ and $t = 1$, this reduces to Theorem 1.1 and for $k = n$ and $t = 1$, Theorem 1.3 reduces to Theorem 1.2. The following generalization of Theorem 1.1 and a result due to Govil and Rahman [8, Theorem 4] is also due to A. Aziz and Q. G. Mohammad [3].

Theorem 1.4. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with* $\mathrm{Re}\ a_j = \alpha_j$ *and* $\mathrm{Im}\ a_j = \beta_j$, $j = 1, 2, \cdots, n$ *such that for some* $t > 0$,

$$0 < t^n \alpha_n \leq t^{n-1}\alpha_{n-1} \leq \cdots \leq t^k \alpha_k \geq \cdots \geq t\alpha_1 \geq \alpha_0 \geq 0,$$

where $0 \leq k \leq n$, *then all the zeros of* $P(z)$ *lie in*

$$|z| \leq t\left(\frac{2t^k\alpha_k}{t^n\alpha_n} - 1\right) + 2\sum_{j=0}^{n} \frac{|\beta_j|}{\alpha_n t^{n-j-1}}.$$

A. Aziz and B. A. Zargar [4] generalized Theorem 1.1 by establishing.

Theorem 1.5. *If* $P(z) = \sum\limits_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *such that for some* $k \geq 1$,

$$ka_n \geq a_{n-1} \geq \cdots \geq a_1 \geq a_0 > 0$$

then all the zeros of $P(z)$ *lie in* $|z + k - 1| \leq k$.

2 Preliminaries

For the proof of our main results, we need the following lemma.

Lemma 2.1. *If for some real numbers* α, β *and* $a_j \in \mathbb{C}$, *where* $j = 0, 1, 2, \cdots n$

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2$$

then for $t > 0$ *and* $j = 1, 2, 3, \cdots, n$,

$$|ta_j - a_{j-1}| \leq |t|a_j| - |a_{j-1}||\cos\alpha + (t|a_j| + |a_{j-1}|)\sin\alpha.$$

This Lemma can be easily verified (for reference see [8]).

3 Main results

In this paper, we prove the following results which is a generalization of theorems 1.2 and 1.5.

Theorem 3.1. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n with complex coefficients such that for some real β,*

$$|\arg a_j-\beta|\leq\alpha\leq\pi/2,\ \ j=0,1,\cdots,n,$$

and for some $k\geq 1$,

$$k|a_n|\geq|a_{n-1}|\geq\cdots\geq|a_r|\leq|a_{r-1}|\leq\cdots\leq|a_1|\leq|a_0|,$$

where $0\leq r\leq n-1$, then all the zeros of $P(z)$ lie in

$$|z+k-1|\leq\left\{k+\frac{|a_0|-2|a_r|}{|a_n|}\right\}\cos\alpha+\left\{k+2\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|}-\frac{|a_0|}{|a_n|}\right\}\sin\alpha+\frac{|a_0|}{|a_n|}.$$

Remark 3.2. . For $\alpha=\beta=0$ and $r=0$, Theorem 3.1 reduces to Theorem 1.5.

For $k=1$ and $r=0$ Theorem 3.1 reduces to the following result which is the improvement of Theorem 1.2 due to Govil and Rahman [8].

Corollary 3.3. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n with complex coefficients such that*

$$|\arg a_j-\beta|\leq\alpha\leq\pi/2\ \ ,j=0,1,\cdots,n,$$

for some real β and

$$|a_n|\geq|a_{n-1}|\geq\cdots\geq|a_1|\geq|a_0|,$$

then all the zeros of $P(z)$ lie in

$$|z|\leq\left\{1-\frac{|a_0|}{|a_n|}\right\}(\cos\alpha+\sin\alpha)+\frac{|a_0|}{|a_n|}+2\sin\alpha\sum_{j=0}^{n}\frac{|a_j|}{|a_n|}.$$

The following result which is a generalization of Theorem 1.2 follows by taking $r=0$.

Corollary 3.4. *If $P(z)=\sum_{j=0}^{n}a_jz^j$ is a polynomial of degree n with complex coefficients such that for some real β,*

$$|\arg a_j-\beta|\leq\alpha\leq\pi/2,\ \ j=0,1,\cdots,n,$$

and for some $k \geq 1$,

$$k|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_1| \geq |a_0|,$$

then all the zeros of $P(z)$ *lie in*

$$|z+k-1| \leq \left\{k - \frac{|a_0|}{|a_n|}\right\}(\cos\alpha + \sin\alpha) + \frac{|a_0|}{|a_n|} + 2\sin\alpha \sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|}.$$

Applying Theorem 3.1 to the polynomial $P(tz)$, we immediately get the following result.

Corollary 3.5. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients such that for some real* β,

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2, \;\; j = 0, 1, \cdots, n,$$

and for some $k \geq 1$, *and* $t > 0$

$$kt^n|a_n| \geq t^{n-1}|a_{n-1}| \geq \cdots \geq t^r|a_r| \leq t^{r-1}|a_{r-1}| \leq \cdots \leq t|a_1| \leq |a_0|,$$

where $0 \leq r \leq n-1$, *then all the zeros of* $P(z)$ *lie in*

$$|z+kt-t| \leq t\left\{k + \frac{|a_0| - 2t^r|a_r|}{t^n|a_n|}\right\}\cos\alpha + t\left\{k + 2\sum_{j=0}^{n-1} \frac{|a_j|}{t^{n-j}|a_n|} - \frac{|a_0|}{t^n|a_n|}\right\}\sin\alpha + \frac{|a_0|}{t^{n-1}|a_n|}.$$

Instead of proving Theorem 3.1, we prove the following more general result.

Theorem 3.6. *If* $P(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *with complex coefficients such that for some real* β,

$$|\arg a_j - \beta| \leq \alpha \leq \pi/2 \;\;, j = 0, 1, \cdots, n,$$

for $k \geq 1$ *and* $\tau \geq 1$,

$$k|a_n| \geq |a_{n-1}| \geq \cdots \geq |a_r| \leq |a_{r-1}| \leq \cdots \leq |a_1| \leq \tau|a_0|,$$

where $0 \leq r \leq n-1$, *then all the zeros of* $P(z)$ *lie in*

$$|z+k-1| \leq \left\{k + \frac{\tau|a_0| - 2|a_r|}{|a_n|}\right\}\cos\alpha + \left\{k + 2\sum_{j=0}^{n-1} \frac{|a_j|}{|a_n|} + (\tau-2)\frac{|a_0|}{|a_n|}\right\}\sin\alpha + \frac{\tau|a_0|}{|a_n|}.$$

Proof of Theorem 3.6. Consider the polynomial

$$\begin{aligned}F(z) &= (1-z)P(z)\\ &= -a_n z^{n+1} + (a_n - a_{n-1})z^n + \sum_{j=2}^{n-1}(a_j - a_{j-1})z^j + (a_1 - a_0)z + a_0.\end{aligned}$$

Let $|z| > 1$, so that $1/|z|^{n-j} < 1$, where $j = 0, 1, 2, \cdots, n-1$, then for $k \geq 1$ and $\tau \geq 1$, we have

$$\begin{aligned}|F(z)| &= \left| -a_n z^{n+1} + (a_n - a_{n-1})z^n + \sum_{j=2}^{n-1}(a_j - a_{j-1})z^j + (a_1 - a_0)z + a_0 \right|,\\ &\geq |z|^n \Bigg[|a_n||z+k-1| - \Bigg\{ |ka_n - a_{n-1}| + \sum_{j=2}^{n-1} \frac{|a_j - a_{j-1}|}{|z|^{n-j}}\\ &\qquad + \frac{|a_1 - \tau a_0|}{|z|^{n-1}} + \frac{|\tau a_0 - a_0|}{|z|^{n-1}} + \frac{|a_0|}{|z|^n} \Bigg\} \Bigg]\\ &> |z|^n \Bigg[|a_n||z+k-1| - \Bigg\{ |ka_n - a_{n-1}| + \sum_{j=2}^{n-1} |a_j - a_{j-1}| + |a_1 - \tau a_0| + \tau|a_0| \Bigg\} \Bigg]. \qquad (1)\end{aligned}$$

Now by Lemma 2.1 and for $1 \leq r \leq n-1$, we have

$$\begin{aligned}&|ka_n - a_{n-1}| + \sum_{j=2}^{n-1} |a_j - a_{j-1}| + |a_1 - \tau a_0| + \tau|a_0|\\ &\leq \Bigg[\Bigg\{ ||ka_n| - |a_{n-1}|| + \sum_{j=2}^{n-1} ||a_j| - |a_{j-1}|| + ||a_1| - \tau|a_0|| \Bigg\} \cos\alpha\\ &\qquad + \Bigg\{ (|ka_n| + |a_{n-1}|) + \sum_{j=2}^{n-1}(|a_j| + |a_{j-1}|) + (|a_1| + \tau|a_0|) \Bigg\} \sin\alpha \Bigg] + \tau|a_0|\\ &= \Bigg[\Bigg\{ (k|a_n| - |a_{n-1}|) + \sum_{j=2}^{r}(|a_{j-1}| - |a_j|) + \sum_{j=r+1}^{n-1}(|a_j| - |a_{j-1}|)\\ &\qquad + (\tau|a_0| - |a_1|) \Bigg\} \cos\alpha + \Bigg(k|a_n| + 2\sum_{j=1}^{n-1}|a_j| + \tau|a_0| \Bigg) \sin\alpha + \tau|a_0|\\ &= \Big(k|a_n| - 2|a_r| + \tau|a_0| \Big) \cos\alpha + \Bigg(k|a_n| + 2\sum_{j=1}^{n-1}|a_j| + \tau|a_0| \Bigg) \sin\alpha + \tau|a_0|. \qquad (2)\end{aligned}$$

Inequality (2) is also valid if $r = 0$. Therefore, it follows from (1) that for $|z| > 1$ and $0 \le r \le n-1$,

$$|F(z)| > |a_n||z|^n \Bigg[|z+k-1| - \Bigg\{\left(k + \frac{\tau|a_0| - 2|a_r|}{|a_n|}\right)\cos\alpha + \left(k + 2\sum_{j=0}^{n-1}\frac{|a_j|}{|a_n|} + (\tau-2)\frac{|a_0|}{|a_n|}\right)\sin\alpha + \frac{\tau|a_0|}{|a_n|}\Bigg\}\Bigg] > 0$$

if

$$|z+k-1| > \left\{k + \frac{\tau|a_0| - 2|a_r|}{|a_n|}\right\}\cos\alpha + \left\{k + 2\sum_{j=1}^{n-1}\frac{|a_j|}{|a_n|} + (\tau-2)\frac{|a_0|}{|a_n|}\right\}\sin\alpha + \frac{\tau|a_0|}{|a_n|}.$$

Thus all the zeros of $F(z)$ whose modulus is greater than one lie in the circle

$$|z+k-1| \le \left\{k + \frac{\tau|a_0| - 2|a_r|}{|a_n|}\right\}\cos\alpha + \left\{k + 2\sum_{j=1}^{n-1}\frac{|a_j|}{|a_n|} + (\tau-2)\frac{|a_0|}{|a_n|}\right\}\sin\alpha + \frac{\tau|a_0|}{|a_n|}. \tag{3}$$

But all those zeros of $F(z)$ whose modulus is less than or equal to one already satisfy the inequality (3), we conclude that all the zeros of $P(z)$ lie in the circle defined by (3) and this proves the desired result. □

Remark 3.7. If $\tau = 1$, then Theorem 3.6 reduces to Theorem 3.1.

Applying Theorem 3.6 to the polynomial $P(tz)$ we immediately get the following result.

Corollary 3.8. *If $P(z) = \sum_{j=0}^{n} a_j z^j$ is a polynomial of degree n with complex coefficients such that for some real β,*

$$|\arg a_j - \beta| \le \alpha \le \pi/2 \ \ , j = 0, 1, \cdots, n,$$

for $k \ge 1$ and $\tau \ge 1$,

$$kt^n|a_n| \ge t^{n-1}|a_{n-1}| \ge \cdots \ge t^r|a_r| \le t^{r-1}|a_{r-1}| \le \cdots \le t|a_1| \le \tau|a_0|,$$

where $0 \le r \le n-1$, *then all the zeros of* $P(z)$ *lie in*

$$|z+kt-t| \le t\left\{k+\frac{\tau|a_0|-2t^r|a_r|}{t^n|a_n|}\right\}\cos\alpha + t\left\{k+2\sum_{j=1}^{n-1}\frac{|a_j|}{t^{n-j}|a_n|}+(\tau-2)\frac{|a_0|}{t^n|a_n|}\right\}\sin\alpha+\frac{\tau|a_0|}{t^{n-1}|a_n|}.$$

We next present the following result.

Theorem 3.9. *Let* $P(z)=\sum_{j=0}^{n}a_jz^j$ *be a polynomial of degree* n *with complex coefficients. If* $\operatorname{Re} a_j=\alpha_j$ *and* $\operatorname{Im} a_j=\beta_j$ *for* $j=0,1,2,\cdots,n$, *such that for some real* $\lambda>0$,

$$\lambda+\alpha_n \ge \alpha_{n-1} \ge \cdots \ge \alpha_1 \ge \alpha_0, \quad \alpha_n \ge 0,$$

and

$$\beta_n \ge \beta_{n-1} \ge \cdots \ge \beta_1 \ge \beta_0 > 0,$$

then all the zeros of $P(z)$ *lie in*

$$\left|z+\frac{\lambda}{a_n}\right| \le \frac{1}{|a_n|}\left\{\lambda+\alpha_n-\alpha_0+|\alpha_0|+\beta_n\right\}. \tag{4}$$

Proof of Theorem 1.3. Consider the polynomial

$$\begin{aligned}F(z)&=(1-z)P(z)\\&=(1-z)\big(a_nz^n+a_{n-1}z^{n-1}+\cdots+a_1z+a_0\big)\\&=-a_nz^{n+1}+(a_n-a_{n-1})z^n+(a_{n-1}-a_{n-2})z^{n-1}\\&\qquad+\cdots+(a_1-a_0)z+a_0\\&=\big\{-a_nz^{n+1}+(\alpha_n-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}+\cdots+(\alpha_1-\alpha_0)z\\&\quad+\alpha_0\big\}+i\big\{(\beta_n-\beta_{n-1})z^n+(\beta_{n-1}-\beta_{n-2})z^{n-1}\\&\qquad+\cdots+(\beta_1-\beta_0)z+\beta_0\big\}\\&=-z^n\big(a_nz+\lambda\big)+\big\{(\alpha_n+\lambda-\alpha_{n-1})z^n+(\alpha_{n-1}-\alpha_{n-2})z^{n-1}\\&\qquad+\cdots+(\alpha_1-\alpha_0)z+\alpha_0\big\}+i\big\{(\beta_n-\beta_{n-1})z^n\\&\qquad+(\beta_{n-1}-\beta_{n-2})z^{n-1}+\cdots+(\beta_1-\beta_0)z+\beta_0\big\}.\end{aligned}$$

This gives

$$
\begin{aligned}
|F(z)| &\geq |z|^n|a_nz+\lambda| - \big\{\big|\alpha_n+\lambda-\alpha_{n-1}\big||z|^n + \big|\alpha_{n-1}-\alpha_{n-2}\big||z|^{n-1} \\
&\quad + \cdots + \big|\alpha_1-\alpha_0\big||z| + |\alpha_0|\big\} - \big\{\big|\beta_n-\beta_{n-1}\big||z|^n + \big|\beta_{n-1}-\beta_{n-2}\big||z|^{n-1} \\
&\quad + \cdots + |\beta_1-\beta_0|\,|z| + |\beta_0|\big\} \\
&= |z|^n\Bigg[|a_nz+\lambda| - \Bigg\{\big|\alpha_n+\lambda-\alpha_{n-1}\big| + \frac{|\alpha_{n-1}-\alpha_{n-2}|}{|z|} \\
&\quad + \cdots + \frac{|\alpha_1-\alpha_0|}{|z|^{n-1}} + \frac{|\alpha_0|}{|z|^n}\Bigg\}\Bigg] \\
&\quad -|z|^n\Bigg[\big|\beta_n-\beta_{n-1}\big| + \frac{|\beta_{n-1}-\beta_{n-2}|}{|z|} + \cdots + \frac{|\beta_1-\beta_0|}{|z|^{n-1}} + \frac{|\beta_0|}{|z|^n}\Bigg].
\end{aligned}
$$

Now, let $|z| \geq 1$, so that $\frac{1}{|z|^{n-j}} \leq 1, 0 \leq j \leq n$, then we have

$$
\begin{aligned}
|F(z)| &\geq |z|^n\Big[\big|a_nz+\lambda\big| - \Big\{|\alpha_n+\lambda-\alpha_{n-1}| + |\alpha_{n-1}-\alpha_{n-2}| \\
&\quad + \cdots + \big|\alpha_1-\alpha_0\big| + |\alpha_0|\Big\}\Big] - |z|^n\Big[\Big\{|\beta_n-\beta_{n-1}| + |\beta_{n-1}-\beta_{n-2}| \\
&\quad + \cdots + |\beta_1-\beta_0| + |\beta_0|\Big\}\Big] \\
&= |z|^n\Big[|a_nz+\lambda| - \Big\{\alpha_n+\lambda-\alpha_{n-1}+\alpha_{n-1}-\alpha_{n-2} \\
&\quad + \cdots + \alpha_1-\alpha_0+|\alpha_0|\Big\}\Big] - |z|^n\Big[\Big\{\beta_n-\beta_{n-1}+\beta_{n-1}-\beta_{n-2} \\
&\quad + \cdots + \beta_1-\beta_0+\beta_0\Big\}\Big] \\
&= |z|^n\Big[|a_nz+\lambda| - \Big\{\alpha_n+\lambda-\alpha_0+|\alpha_0|+\beta_n\Big\}\Big] > 0, \quad \text{if}
\end{aligned}
$$

$$|a_nz+\lambda| > \alpha_n+\lambda-\alpha_0+|\alpha_0|+\beta_n$$

that is, if

$$\left|z+\frac{\lambda}{a_n}\right| > \frac{1}{|a_n|}\left\{\alpha_n+\lambda-\alpha_0+|\alpha_0|+\beta_n\right\}.$$

Thus all the zeros of $F(z)$ whose modulus is greater than or equal to 1 lie in

$$\left|z+\frac{\lambda}{a_n}\right| \leq \frac{1}{|a_n|}\left\{\alpha_n+\lambda-\alpha_0+|\alpha_0|+\beta_n\right\}.$$

But those zeros of $F(z)$ whose modulus is less than 1 satisfies the above inequality if $\alpha_n \geq 0$. Hence it follows that all the zeros of $F(z)$ lie in

$$\left|z + \frac{\lambda}{a_n}\right| \leq \frac{1}{|a_n|}\left\{\alpha_n + \lambda - \alpha_0 + |\alpha_0| + \beta_n\right\}$$

Since all the zeros of $P(z)$ are also the zeros of $F(z)$, Theorem 3.9 is proved completely.

□

Finally, as an application of Theorem 3.9, we present the following result.

Theorem 3.10. *Let* $P(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *with complex coefficients. If* $\operatorname{Re} a_j = \alpha_j$ *and* $\operatorname{Im} a_j = \beta_j$ *for* $j = 0, 1, 2, \cdots, n$, *such that for some real* $\lambda > 0$,

$$\alpha_n \geq \alpha_{n-1} \geq \cdots \geq \alpha_1 \geq \alpha_0 > 0,$$

and

$$\beta_n - \lambda \leq \beta_{n-1} \leq \cdots \leq \beta_1 \leq \beta_0, \quad \beta_n \leq 0$$

then all the zeros of $P(z)$ *lie in*

$$\left|z - \frac{\lambda i}{a_n}\right| \leq \frac{1}{|a_n|}\left\{\lambda - \beta_n + \beta_0 + |\beta_0| + \alpha_n\right\}. \tag{5}$$

Proof. Applying Theorem 3.9 to $iP(z)$, we get the desired result. □

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