Available online at http://scik.org
Adv. Inequal. Appl. 2023, 2023:2
https://doi.org/10.28919/aia/7794
ISSN: 2050-7461

INTEGRAL INEQUALITIES OF HERMITE-HADAMARD TYPE FOR *L*-BOUNDED NORM WEAK CONVEX MAPPINGS

S. S. DRAGOMIR[1,2,*]

[1]Mathematics, School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia

[2]DST-NRF Centre of Excellence in the Mathematical and Statistical Sciences, School of Computer Science and Applied Mathematics, University of the Witwatersrand, Private Bag 3, Johannesburg 2050, South Africa

Abstract. In this paper we introduce a class of functions that extends the concept of Lipschitzian function and called them L-bounded norm weak convex functions. Integral inequalities of Hermite-Hadamard type are obtained and applications for discrete inequalities of Jensen type are provided as well.

Keywords: Banach spaces; Banach algebras; Lipschitz type inequalities; Ostrowski-type inequalities; mid-point inequalities; Hermite-Hadamard type inequalities.

2020 AMS Subject Classification: 46B20, 26D15, 47A99.

1. Introduction

Let $\mathscr{B}(H)$ be the Banach algebra of bounded linear operators on a complex Hilbert space H. The absolute value of an operator A is the positive operator $|A|$ defined as $|A| := (A^*A)^{1/2}$.

One of the central problems in perturbation theory is to find bounds for

$$\|f(A) - f(B)\|$$

*Corresponding author

E-mail address: sever.dragomir@vu.edu.au

Received November 07, 2023

in terms of $\|A-B\|$ for different classes of measurable functions f for which the function of operator can be defined. For some results on this topic, see [5], [34] and the references therein.

It is known that [4] in the infinite-dimensional case the map $f(A) := |A|$ is not *Lipschitz continuous* on $\mathscr{B}(H)$ with the usual operator norm, i.e. there is no constant $L > 0$ such that

$$\||A|-|B|\| \leq L\|A-B\|$$

for any $A, B \in \mathscr{B}(H)$.

However, as shown by Farforovskaya in [32], [33] and Kato in [39], the following inequality holds

$$\||A|-|B|\| \leq \frac{2}{\pi}\|A-B\|\left(2+\log\left(\frac{\|A\|+\|B\|}{\|A-B\|}\right)\right) \tag{1.1}$$

for any $A, B \in \mathscr{B}(H)$ with $A \neq B$.

If the operator norm is replaced with *Hilbert-Schmidt norm* $\|C\|_{HS} := (\operatorname{tr} C^* C)^{1/2}$ of an operator C, then the following inequality is true [2]

$$\||A|-|B|\|_{HS} \leq \sqrt{2}\|A-B\|_{HS} \tag{1.2}$$

for any $A, B \in \mathscr{B}(H)$.

The coefficient $\sqrt{2}$ is best possible for a general A and B. If A and B are restricted to be selfadjoint, then the best coefficient is 1.

It has been shown in [4] that, if A is an invertible operator, then for all operators B in a neighborhood of A we have

$$\||A|-|B|\| \leq a_1\|A-B\| + a_2\|A-B\|^2 + O\left(\|A-B\|^3\right) \tag{1.3}$$

where

$$a_1 = \left\|A^{-1}\right\|\|A\| \text{ and } a_2 = \left\|A^{-1}\right\| + \left\|A^{-1}\right\|^3\|A\|^2.$$

In [3] the author also obtained the following *Lipschitz type inequality*

$$\|f(A)-f(B)\| \leq f'(a)\|A-B\| \tag{1.4}$$

where f is an *operator monotone function* on $(0,\infty)$ and $A, B \geq aI_H > 0$.

Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two Banach spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. For any mapping $F:C\subset X\to Y$ we can consider the associated functions $\Phi_{F,x,y,\lambda}, \Psi_{F,x,y,\lambda}:[0,1]\to Y$, where $x,y\in C$, $\lambda\in[0,1]$, defined by [25]

$$\Phi_{F,x,y,\lambda}(t):=(1-\lambda)F[(1-t)((1-\lambda)x+\lambda y)+ty] \\ +\lambda F[(1-t)x+t((1-\lambda)x+\lambda y)] \tag{1.5}$$

and

$$\Psi_{F,x,y,\lambda}(t):=(1-\lambda)F[(1-t)((1-\lambda)x+\lambda y)+ty] \\ +\lambda F[tx+(1-t)((1-\lambda)x+\lambda y)]. \tag{1.6}$$

We say that the mapping $F:B\subset X\to Y$ is *Lipschitzian* with the constant $L>0$ on the subset B of X if

$$\|F(x)-F(y)\|_Y\le L\|x-y\|_X \text{ for any } x,y\in B. \tag{1.7}$$

The following result holds [25]:

Theorem 1. *Let $F:C\subset X\to Y$ be a Lipschitzian mapping with the constant $L>0$ on the convex subset C of X. If $x,y\in C$, then we have*

$$\left\|\Lambda_{F,x,y,\lambda}(t)-\int_0^1 F[sy+(1-s)x]\,ds\right\|_Y \\ \le 2L\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\left[\frac{1}{4}+\left(\lambda-\frac{1}{2}\right)^2\right]\|x-y\|_X \tag{1.8}$$

for any $t\in[0,1]$ and $\lambda\in[0,1]$, where $\Lambda_{F,x,y,\lambda}=\Phi_{F,x,y,\lambda}$ or $\Lambda_{F,x,y,\lambda}=\Psi_{F,x,y,\lambda}$.

If we take in (1.8) $\Lambda_{F,x,y,\lambda}=\Phi_{F,x,y,\lambda}$, $\lambda=\frac{1}{2}$, then we get

$$\left\|\frac{1}{2}\left(F\left[(1-t)\frac{x+y}{2}+ty\right]+F\left[(1-t)x+t\frac{x+y}{2}\right]\right)\right. \\ \left.-\int_0^1 F[sy+(1-s)x]\,ds\right\|\le\frac{1}{2}L\left[\frac{1}{4}+\left(t-\frac{1}{2}\right)^2\right]\|x-y\|_X \tag{1.9}$$

for any $x,y\in C$ and $t\in[0,1]$.

If we take in (1.8) $\Lambda_{F,x,y,\lambda} = \Psi_{F,x,y,\lambda}$, $\lambda = \frac{1}{2}$, then we get

$$\left\| \frac{1}{2}\left(F\left[(1-t)\frac{x+y}{2} + ty \right] + F\left[tx + (1-t)\frac{x+y}{2} \right] \right) - \int_0^1 F[sy+(1-s)x]\,ds \right\|_Y \le \frac{1}{2}L\left[\frac{1}{4} + \left(t - \frac{1}{2} \right)^2 \right] \|x-y\|_X \tag{1.10}$$

for any $t \in [0,1]$ and $x, y \in C$.

We also have the simpler inequalities

$$\left\| \frac{1}{2}\left[F\left(\frac{3x+y}{4} \right) + F\left(\frac{x+3y}{4} \right) \right] - \int_0^1 F[sy+(1-s)x]\,ds \right\|_Y \le \frac{1}{8}L\|x-y\|_X, \tag{1.11}$$

$$\left\| F\left(\frac{x+y}{2} \right) - \int_0^1 F[sy+(1-s)x]\,ds \right\|_Y \le \frac{1}{4}L\|x-y\|_X \tag{1.12}$$

and

$$\left\| \frac{1}{2}[F(x)+F(y)] - \int_0^1 F[sy+(1-s)x]\,ds \right\|_Y \le \frac{1}{4}L\|x-y\|_X \tag{1.13}$$

for any $x, y \in C$. The constants $\frac{1}{8}$ and $\frac{1}{4}$ are best possible.

The inequalities (1.12) and (1.13) are the corresponding versions of Hermite-Hadamard inequalities for Lipschitzian functions. The scalar cases were obtained in [12] and [43]. For Hermite-Hadamard's type inequalities, see for instance [10], [12], [13], [35], [37], [38], [40], [42], [43], [46], [47], [48], [49], [50] and the references therein.

From (1.8) we also have the Ostrowski's inequality

$$\left\| F[ty+(1-t)x] - \int_0^1 F[sy+(1-s)x]\,ds \right\|_Y \le L\left[\frac{1}{4} + \left(t - \frac{1}{2} \right)^2 \right] \|x-y\|_X \tag{1.14}$$

for any $t \in [0,1]$ and x, $y \in C$. For Ostrowski's type inequalities for the Lebesgue integral, see [1], [8]-[9] and [15]-[30]. Inequalities for the Riemann-Stieltjes integral may be found in [17], [19] while the generalization for isotonic functionals was provided in [20]. For the case of functions of self-adjoint operators on complex Hilbert spaces, see the recent monograph [23].

Motivated by the above results, we introduce here a class of functions that extends the concept of Lipschitzian function and called them L-bounded norm weak convex functions. Integral inequalities of Hermite-Hadamard type are obtained and applications for discrete inequalities of Jensen type are provided as well.

2. L-BOUNDED NORM WEAK CONVEX MAPPINGS

Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Let C be a convex set in X. We consider the following class of functions:

Definition 1. *A mapping $F : C \subset X \to Y$ is called L-bounded norm weak convex, for some given $L > 0$, if it satisfies the condition*

$$\|(1-\lambda)F(x)+\lambda F(y)-F((1-\lambda)x+\lambda y)\|_Y \le L\lambda(1-\lambda)\|x-y\|_X \tag{2.1}$$

for any $x, y \in C$ and $\lambda \in [0,1]$. For simplicity, we denote this by $F \in \mathcal{BNW}_L(C)$.

We have from (2.1) for $\lambda = \frac{1}{2}$ the Jensen's inequality

$$\left\|\frac{F(x)+F(y)}{2}-F\left(\frac{x+y}{2}\right)\right\|_Y \le \frac{1}{4}L\|x-y\|_X \tag{2.2}$$

for any $x, y \in C$.

We observe that $\mathcal{BNW}_L(C)$ is a convex subset in the linear space of all functions defined on C and with values in Y.

The following simple result holds:

Lemma 1. *If the function $F : C \subset X \to Y$ is Lipschitzian with the constant $K > 0$, then $F \in \mathcal{BNW}_L(C)$ with $L = 2K$.*

Proof. Since F is Lipschitzian, we have

$$\|F((1-\lambda)x+\lambda y)-F(x)\|_Y \le K\lambda\|x-y\|_X$$

and

$$\|F((1-\lambda)x+\lambda y)-F(y)\|_Y \le K(1-\lambda)\|x-y\|_X$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

If we multiply the first inequality by $1-\lambda$ and the second inequality by λ and add these inequalities, we get

$$\begin{aligned}&(1-\lambda)\left\|F((1-\lambda)x+\lambda y)-F(x)\right\|_Y+\lambda\left\|F((1-\lambda)x+\lambda y)-F(y)\right\|_Y\\&\quad\leq 2K\lambda(1-\lambda)\left\|x-y\right\|_X\end{aligned}$$

for any $x,\ y\in C$ and $\lambda\in[0,1]$.

We also have

$$\begin{aligned}&(1-\lambda)\left\|F((1-\lambda)x+\lambda y)-F(x)\right\|_Y+\lambda\left\|F((1-\lambda)x+\lambda y)-F(y)\right\|_Y\\&\quad\geq\left\|(1-\lambda)F((1-\lambda)x+\lambda y)-(1-\lambda)F(x)+\lambda F((1-\lambda)x+\lambda y)-\lambda F(y)\right\|_Y\\&\quad=\left\|F((1-\lambda)x+\lambda y)-(1-\lambda)F(x)-\lambda F(y)\right\|,\end{aligned}$$

which proves that

$$\left\|(1-\lambda)F(x)+\lambda F(y)-F((1-\lambda)x+\lambda y)\right\|\leq 2K\lambda(1-\lambda)\left\|x-y\right\|_X$$

for any $x,\ y\in C$ and $\lambda\in[0,1]$, namely $F\in\mathscr{BNW}_L(C)$ with $L=2K$. □

We observe also that, by the triangle inequality, we have

$$\begin{aligned}&\left\|F((1-\lambda)x+\lambda y)\right\|_Y-\left\|(1-\lambda)F(x)+\lambda F(y)\right\|_Y\\&\quad\leq\left\|(1-\lambda)F(x)+\lambda F(y)-F((1-\lambda)x+\lambda y)\right\|_Y\end{aligned}\tag{2.3}$$

and by (2.1) we get

$$\left\|F((1-\lambda)x+\lambda y)\right\|_Y-\left\|(1-\lambda)F(x)+\lambda F(y)\right\|_Y\leq L\lambda(1-\lambda)\left\|x-y\right\|_X,$$

which, again, by the triangle inequality gives

$$\left\|F((1-\lambda)x+\lambda y)\right\|_Y\leq L\lambda(1-\lambda)\left\|x-y\right\|_X+(1-\lambda)\left\|F(x)\right\|_Y+\lambda\left\|F(y)\right\|_Y\tag{2.4}$$

for any $x,y\in C$ and $\lambda\in[0,1]$.

Now, if the function $t \mapsto \|F((1-\lambda)x+\lambda y)\|_Y$, for some $x, y \in C$, is Lebesgue integrable on $[0,1]$, then by taking the integral in (2.4) we get

$$\int_0^1 \|F((1-\lambda)x+\lambda y)\|_Y d\lambda \le L\|x-y\|_X \int_0^1 \lambda(1-\lambda)d\lambda + \|F(x)\|_Y \int_0^1 (1-\lambda)d\lambda + \|F(y)\|_Y \int_0^1 \lambda d\lambda \tag{2.5}$$

and since

$$\int_0^1 \lambda(1-\lambda)d\lambda = \frac{1}{6}, \ \int_0^1 (1-\lambda)d\lambda = \int_0^1 \lambda d\lambda = \frac{1}{2},$$

then we get from (2.5) that

$$\int_0^1 \|F((1-\lambda)x+\lambda y)\|_Y d\lambda \le \frac{1}{6}L\|x-y\|_X + \frac{1}{2}\left[\|F(x)\|_Y + \|F(y)\|_Y\right]. \tag{2.6}$$

If we assume continuity for the function F on C in the norm topology of $(X;\|\cdot\|_X)$, then the inequality (2.6) holds for any $x, y \in C$. Moreover, if we assume that $(Y;\|\cdot\|_Y)$ is a Banach space and F is continuos on C, then we have the generalized triangle inequality

$$\left\|\int_0^1 F((1-\lambda)x+\lambda y)d\lambda\right\|_Y \le \int_0^1 \|F((1-\lambda)x+\lambda y)\|_Y d\lambda,$$

and by (2.6) we get

$$\left\|\int_0^1 F((1-\lambda)x+\lambda y)d\lambda\right\|_Y \le \frac{1}{6}L\|x-y\|_X + \frac{1}{2}\left[\|F(x)\|_Y + \|F(y)\|_Y\right] \tag{2.7}$$

for any $x, y \in C$.

We have the following results:

Theorem 2. *Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$ with Y complete. Assume that the mapping $F: C \subset X \to Y$ is continuous on the convex set C in the norm topology. If $F \in \mathcal{BNW}_L(C)$ for some $L > 0$, then we have*

$$\left\|\frac{F(x)+F(y)}{2} - \int_0^1 F((1-\lambda)x+\lambda y)d\lambda\right\|_Y \le \frac{1}{6}L\|x-y\|_X \tag{2.8}$$

and

$$\left\|\int_0^1 F((1-\lambda)x+\lambda y)d\lambda - F\left(\frac{x+y}{2}\right)\right\|_Y \le \frac{1}{8}L\|x-y\|_X \tag{2.9}$$

for any $x, y \in C$.

The constants $\frac{1}{6}$ and $\frac{1}{8}$ are best possible.

Proof. From (2.1) we have successively

$$\begin{aligned}&\left\|\int_0^1 \left[(1-\lambda) F(x)+\lambda F(y)-F((1-\lambda) x+\lambda y)\right] d\lambda\right\|_Y \\ &\leq \int_0^1\|(1-\lambda) F(x)+\lambda F(y)-F((1-\lambda) x+\lambda y)\|_Y d\lambda \\ &\leq L\|x-y\|_X \int_0^1 \lambda(1-\lambda) d\lambda=\frac{1}{6} L\|x-y\|_X\end{aligned}$$

which produces the desired result (2.8).

Utilising (2.2) we have

$$\begin{aligned}&\left\|\frac{F((1-\lambda) x+\lambda y)+F(\lambda x+(1-\lambda) y)}{2}-F\left(\frac{x+y}{2}\right)\right\|_Y \\ &\leq \frac{1}{4} L\|(1-\lambda) x+\lambda y-\lambda x-(1-\lambda) y\|_X \\ &=\frac{1}{2} K\left|\lambda-\frac{1}{2}\right|\|x-y\|_X\end{aligned} \tag{2.10}$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

Integrating in (2.10) we get

$$\begin{aligned}&\left\|\int_0^1\left[\frac{F((1-\lambda) x+\lambda y)+F(\lambda x+(1-\lambda) y)}{2}-F\left(\frac{x+y}{2}\right)\right] d\lambda\right\|_Y \\ &\leq \int_0^1\left\|\frac{F((1-\lambda) x+\lambda y)+F(\lambda x+(1-\lambda) y)}{2}-F\left(\frac{x+y}{2}\right)\right\|_Y d\lambda \\ &\leq \frac{1}{2} K\|x-y\|_X \int_0^1\left|\lambda-\frac{1}{2}\right| d\lambda=\frac{1}{8} K\|x-y\|_X\end{aligned} \tag{2.11}$$

and since

$$\int_0^1 F((1-\lambda) x+\lambda y) d\lambda=\int_0^1 F(\lambda x+(1-\lambda) y) d\lambda,$$

then from (2.11) we get (2.9).

Now, consider the function $F_0: H \to \mathbb{R}$, $F_0(x)=\|x\|^2$ where $(H,\langle .,.\rangle)$ is a complex inner product space. If $x, y \in H$ and $\lambda \in [0,1]$, then

$$\begin{aligned}
&(1-\lambda) F_0(x)+\lambda F_0(y)-F_0((1-\lambda) x+\lambda y) \\
&\quad=(1-\lambda)\|x\|^2+\lambda\|y\|^2-\|(1-\lambda) x+\lambda y\|^2 \\
&\quad=(1-\lambda)\|x\|^2+\lambda\|y\|^2-(1-\lambda)^2\|x\|^2-2(1-\lambda) \lambda \operatorname{Re}\langle x, y\rangle-\lambda^2\|y\|^2 \\
&\quad=(1-\lambda) \lambda\left[\|x\|^2-2 \operatorname{Re}\langle x, y\rangle+\|y\|^2\right]=(1-\lambda) \lambda\|x-y\|^2 .
\end{aligned}$$

Consider C_0 a convex subset of H such that $\|x-y\| \leq 1$ for any $x, y \in C$. For instance $C_0 = B\left(0, \frac{1}{2}\right)$ is the closed ball centered in 0 and with a radius $\frac{1}{2}$. Then for all $x,\ y \in B\left(0, \frac{1}{2}\right)$ we have $\|x-y\| \leq \|x\|+\|y\| \leq \frac{1}{2}+\frac{1}{2}=1$.

Therefore, if we consider $F_0(x)=\|x\|^2$ defined on $C_0=B\left(0, \frac{1}{2}\right)$, we have

$$0 \leq(1-\lambda) F_0(x)+\lambda F_0(y)-F_0((1-\lambda) x+\lambda y) \leq(1-\lambda) \lambda\|x-y\|$$

which shows that $F_0 \in \mathscr{B N W}_L(C_0)$ with $L=1$.

We have

$$\begin{aligned}
\int_0^1 F_0((1-\lambda) x+\lambda y) d \lambda & =\int_0^1\|(1-\lambda) x+\lambda y\|^2 d \lambda \\
& =\int_0^1\left[(1-\lambda)^2\|x\|^2+2(1-\lambda) \lambda \operatorname{Re}\langle x, y\rangle+\lambda^2\|y\|^2\right] d \lambda \\
& =\frac{1}{3}\left[\|x\|^2+\operatorname{Re}\langle x, y\rangle+\|y\|^2\right]
\end{aligned}$$

for any $x,\ y \in H$.

Therefore

$$\begin{aligned}
&\frac{F_0(x)+F_0(y)}{2}-\int_0^1 F_0((1-\lambda) x+\lambda y) d \lambda \\
&\qquad=\frac{1}{2}\left[\|x\|^2+\|y\|^2\right]-\frac{1}{3}\left[\|x\|^2+\operatorname{Re}\langle x, y\rangle+\|y\|^2\right]=\frac{1}{6}\|x-y\|^2 .
\end{aligned}$$

Now, assume that the inequality (2.8) holds with a constant $A>0$, namely

$$\left\|\frac{F(x)+F(y)}{2}-\int_0^1 F((1-\lambda) x+\lambda y) d \lambda\right\|_Y \leq A L\|x-y\|_X,$$

then by taking $F_0 \in \mathscr{B N W}_L(C_0)$ with $L=1$ defined above, we get

$$\frac{1}{6}\|x-y\|^2 \leq A\|x-y\|_X$$

namely

$$\frac{1}{6}\|x-y\| \leq A. \tag{2.12}$$

If $e \in H$ with $\|e\|=1$, then $x=\frac{1}{2}e$ and $y=-\frac{1}{2}e \in B\left(0,\frac{1}{2}\right)$ giving that $x-y=e$ and by (2.12) we get $A \geq \frac{1}{6}$.

Now, consider the function $F_0 : X \to [0,\infty)$, $F_0(x) = \left\|x-\frac{a+b}{2}\right\|$, with $a,b \in X$ with $a \neq b$. Then

$$|F_0(x)-F_0(y)| = \left|\left\|x-\frac{a+b}{2}\right\| - \left\|y-\frac{a+b}{2}\right\|\right| \leq \|x-y\|,$$

for any x, $y \in X$, which shows that F_0 is Lipschitzian with the constant $K=1$.

By utilising Lemma 1 we conclude that $F_0 \in \mathscr{BNW}_L(C)$ with $L=2$.

We have

$$\int_0^1 F_0((1-\lambda)a+\lambda b)\,d\lambda - F_0\left(\frac{a+b}{2}\right) = \int_0^1 \left\|(1-\lambda)a+\lambda b-\frac{a+b}{2}\right\| d\lambda = \frac{1}{4}\|b-a\|,$$

which shows that the inequality (2.9) holds with equality. □

3. Related Inequalities

We have the following result as well:

Theorem 3. *Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$ with Y complete. Assume that the mapping $F : C \subset X \to Y$ is continuous on the convex set C in the norm topology. If $F \in \mathscr{BNW}_L(C)$ for some $L>0$, then we have*

(3.1)
$$\left\|\int_0^1 F(uy+(1-u)x)\,du - \frac{1}{2\lambda-1}\int_{1-\lambda}^{\lambda} F(sx+(1-s)y)\,ds\right\|_F \leq \frac{1}{2}L\lambda(1-\lambda)\|y-x\|_X$$

for any $\lambda \in [0,1]$, $\lambda \neq \frac{1}{2}$ and x, $y \in C$.

Proof. Since $F \in \mathscr{BNW}_L(C)$ for $K>0$, then

$$\|(1-\lambda)F(u)+\lambda F(v)-F((1-\lambda)u+\lambda v)\|_Y \leq L\lambda(1-\lambda)\|u-v\|_X \tag{3.2}$$

for any u, $v \in C$ and $\lambda \in [0,1]$.

Let $t \in [0,1]$ and for x, $y \in C$, take

$$u = (1-t)((1-\lambda)x + \lambda y) + ty, \; v = tx + (1-t)((1-\lambda)x + \lambda y) \in C$$

in (3.2) to get

$$\begin{aligned} &\|(1-\lambda)F((1-t)((1-\lambda)x+\lambda y)+ty) + \lambda F(tx+(1-t)((1-\lambda)x+\lambda y)) \\ &\quad -F((1-\lambda)[(1-t)((1-\lambda)x+\lambda y)+ty] + \lambda[tx+(1-t)((1-\lambda)x+\lambda y)])\|_Y \\ &\leq L\lambda(1-\lambda)\|(1-t)((1-\lambda)x+\lambda y)+ty-[tx+(1-t)((1-\lambda)x+\lambda y)]\|_X . \end{aligned} \tag{3.3}$$

Observe that

$$\begin{aligned} &(1-\lambda)[(1-t)((1-\lambda)x+\lambda y)+ty] + \lambda[tx+(1-t)((1-\lambda)x+\lambda y)] \\ &= (1-\lambda)(1-t)((1-\lambda)x+\lambda y) + (1-\lambda)ty + \lambda tx + \lambda(1-t)((1-\lambda)x+\lambda y) \\ &= (1-t)((1-\lambda)x+\lambda y) + (1-\lambda)ty + \lambda tx \\ &= [(1-t)(1-\lambda)+\lambda t]x + [(1-t)\lambda + (1-\lambda)t]y \end{aligned}$$

and

$$\begin{aligned} &(1-t)((1-\lambda)x+\lambda y)+ty-[tx+(1-t)((1-\lambda)x+\lambda y)] \\ &\quad = (1-t)(1-\lambda)x + (1-t)\lambda y + ty - tx - (1-t)(1-\lambda)x - (1-t)\lambda y = t(y-x). \end{aligned}$$

Then by (3.3) we have

$$\begin{aligned} &\|(1-\lambda)F((1-t)((1-\lambda)x+\lambda y)+ty) + \lambda F(tx+(1-t)((1-\lambda)x+\lambda y)) \\ &\quad -F([(1-t)(1-\lambda)+\lambda t]x + [(1-t)\lambda+(1-\lambda)t]y)\|_Y \\ &\leq L\lambda(1-\lambda)t\|y-x\|_X , \end{aligned} \tag{3.4}$$

for any t, $\lambda \in [0,1]$ and $x, y \in C$.

Integrating the inequality (3.4) over t on $[0,1]$ and using the generalized triangle inequality for norms and integrals, we get

$$\begin{aligned}(3.5)\qquad &\left\|(1-\lambda)\int_0^1 F((1-t)((1-\lambda)x+\lambda y)+ty)\,dt\right.\\ &+\lambda\int_0^1 F(tx+(1-t)((1-\lambda)x+\lambda y))\,dt\\ &\left.-\int_0^1 F([(1-t)(1-\lambda)+\lambda t]x+[(1-t)\lambda+(1-\lambda)t]y)\,dt\right\|_Y\\ &\leq\frac{1}{2}L\lambda(1-\lambda)\|y-x\|_X,\end{aligned}$$

for any $\lambda\in[0,1]$ and x, $y\in C$.

Observe that

$$(3.6)\qquad \int_0^1 F[(1-t)(\lambda y+(1-\lambda)x)+ty]\,dt = \int_0^1 F[((1-t)\lambda+t)y+(1-t)(1-\lambda)x]\,dt$$

and

$$\begin{aligned}(3.7)\qquad &\int_0^1 F(tx+(1-t)((1-\lambda)x+\lambda y))\,dt\\ &=\int_0^1 F((1-t)x+t((1-\lambda)x+\lambda y))\,dt=\int_0^1 F[t\lambda y+(1-\lambda t)x]\,dt.\end{aligned}$$

If we make the change of variable $u:=(1-t)\lambda+t$ then we have $1-u=(1-t)(1-\lambda)$ and $du=(1-\lambda)\,du$. Then

$$\int_0^1 F[((1-t)\lambda+t)y+(1-t)(1-\lambda)x]\,dt=\frac{1}{1-\lambda}\int_\lambda^1 F[uy+(1-u)x]\,du.$$

If we make the change of variable $u:=\lambda t$ then we have $du=\lambda dt$ and

$$\int_0^1 F[t\lambda y+(1-\lambda t)x]\,dt=\frac{1}{\lambda}\int_0^\lambda F[uy+(1-u)x]\,du.$$

Therefore

$$\begin{aligned}&(1-\lambda)\int_0^1 F[(1-t)(\lambda y+(1-\lambda)x)+ty]\,dt+\lambda\int_0^1 F[t(\lambda y+(1-\lambda)x)+(1-t)x]\,dt\\ &=\int_\lambda^1 F[uy+(1-u)x]\,du+\int_0^\lambda F[uy+(1-u)x]\,du=\int_0^1 F[uy+(1-u)x]\,du,\end{aligned}$$

and we have the simple equality

$$(1-\lambda)\int_0^1 F((1-t)((1-\lambda)x+\lambda y)+ty)\,dt + \lambda\int_0^1 F(tx+(1-t)((1-\lambda)x+\lambda y))\,dt = \int_0^1 F[uy+(1-u)x]\,du \tag{3.8}$$

for any $\lambda\in[0,1]$ and $x, y\in C$.

Consider now the integral

$$\int_0^1 F([(1-t)(1-\lambda)+\lambda t]x+[(1-t)\lambda+(1-\lambda)t]y)\,dt.$$

Put

$$s=(1-t)(1-\lambda)+\lambda t = 1-\lambda+(2\lambda-1)t.$$

Then

$$1-s=(1-t)\lambda+(1-\lambda)t.$$

If $\lambda\neq\frac{1}{2}$, then $s=1-\lambda+(2\lambda-1)t$ is a change of variable with $dt=\frac{1}{2\lambda-1}$ and we have

$$\int_0^1 F([(1-t)(1-\lambda)+\lambda t]x+[(1-t)\lambda+(1-\lambda)t]y)\,dt = \frac{1}{2\lambda-1}\int_{1-\lambda}^{\lambda} F(sx+(1-s)y)\,ds.$$

Now, making use of (3.5) we get the desired result (3.1). □

Remark 1. *We observe that for $\lambda\to\frac{1}{2}$ we recapture from (3.1) the inequality (2.9). If we take in (3.1) $\lambda=\frac{3}{4}$, then we get*

$$\left\|\int_0^1 F[uy+(1-u)x]\,du-2\int_{1/4}^{3/4} F(sx+(1-s)y)\,ds\right\|_F \le \frac{3}{32}L\|y-x\|_X. \tag{3.9}$$

4. Applications for Gâteaux Differentiable Functions

Following [11, p. 59], let $(X,\|\cdot\|_X)$ and $(Y,\|\cdot\|_Y)$ be two normed linear spaces, Ω an open subset of X and $f:\Omega\to Y$. If $a\in\Omega$, $u\in X\setminus\{0\}$ and if the limit

$$\lim_{t\to0}\frac{1}{t}[f(a+tu)-f(a)]$$

exists, then we denote this derivative $\partial_u f(a)$. It is called the directional derivative of f at a in the direction u. If the directional derivative is defined in all directions and there is a continuous linear mapping Φ from X into Y such that for all $u \in X$

$$\partial_u f(a) = \Phi(u),$$

then we say that f is *Gâteaux-differentiable* at a and that Φ is the *Gâteaux differential* of f at a. If a mapping f is differentiable at a point a, then clearly all its directional derivatives exist and we have

$$\partial_u f(a) = f'(a)u, \ u \in X.$$

Thus f is Gâteaux-differentiable at a. However, the Gâteaux differential may exist without the differential existing. The existence of directional derivatives at a point does not imply that the mapping is Gâteaux-differentiable. To distinguish the differential from the Gâteaux differential, the differential is often referred as the Fréchet differential.

Theorem 4. *Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Assume that the mapping $F : C \subset X \to Y$ is defined on the open convex set C and $F \in \mathscr{BNW}_L(C)$ for some $L > 0$. If $x_k \in C$, $p_k \geq 0$ for $k \in \{1,...,n\}$ with $\sum_{k=1}^n p_k = 1$ and F is Gâteaux-differentiable at $\sum_{k=1}^n p_k x_k \in C$, then for any $y_j \in C$ and $q_j \geq 0$ for $j \in \{1,...,m\}$ with $\sum_{j=1}^m q_j = 1$ and $\sum_{j=1}^m q_j y_j = \sum_{k=1}^n p_k x_k$ we have*

$$\left\| \sum_{j=1}^m q_j F(y_j) - F\left(\sum_{k=1}^n p_k x_k\right) \right\|_Y \leq L \sum_{j=1}^m q_j \left\| y_j - \sum_{k=1}^n p_k x_k \right\|_X. \tag{4.1}$$

In particular, we have

$$\left\| \sum_{j=1}^n p_j F(x_j) - F\left(\sum_{k=1}^n p_k x_k\right) \right\|_Y \leq L \sum_{j=1}^n p_j \left\| x_j - \sum_{k=1}^n p_k x_k \right\|_X. \tag{4.2}$$

Proof. Since $F \in \mathscr{BNW}_L(C)$ then we have

$$\|\lambda [F(y) - F(x)] + F(x) - F((1-\lambda)x + \lambda y)\|_Y \leq L\lambda(1-\lambda)\|x-y\|_X$$

for any $x, y \in C$ and $\lambda \in [0,1]$.

This implies that

$$\left\| F(y) - F(x) - \frac{F(x + \lambda(y-x)) - F(x)}{\lambda} \right\|_Y \leq L(1-\lambda)\|x-y\|_X \tag{4.3}$$

for any $x, y \in C$ and $\lambda \in (0,1)$.

If we assume that F is Gâteaux-differentiable at x, then by taking the limit over $\lambda \to 0+$ in (4.3) we get

$$\left\| F(y) - F(x) - \partial_{y-x} F(x) \right\|_Y \le L \|x - y\|_X \tag{4.4}$$

for any $x, y \in C$.

Now, if F is Gâteaux-differentiable at $\sum_{k=1}^{n} p_k x_k \in C$, then

$$\left\| F(y) - F\left(\sum_{k=1}^{n} p_k x_k\right) - \partial_{y-\sum_{k=1}^{n} p_k x_k} F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \le L \left\| \sum_{k=1}^{n} p_k x_k - y \right\|_X \tag{4.5}$$

for any $y \in C$.

If $y_j \in C$ and $q_j \ge 0$ for $j \in \{1, ..., m\}$ with $\sum_{j=1}^{m} q_j = 1$, then by (4.5) we have

$$\begin{aligned} \sum_{j=1}^{m} q_j \left\| F(y_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) - \partial_{y_j-\sum_{k=1}^{n} p_k x_k} F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \\ \le L \sum_{j=1}^{m} q_j \left\| \sum_{k=1}^{n} p_k x_k - y_j \right\|_X . \end{aligned} \tag{4.6}$$

By the generalized triangle inequality we have

$$\begin{aligned} \left\| \sum_{j=1}^{m} q_j F(y_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) - \partial_{\sum_{j=1}^{m} q_j y_j-\sum_{k=1}^{n} p_k x_k} F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \\ \le \sum_{j=1}^{m} q_j \left\| F(y_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) - \partial_{y_j-\sum_{k=1}^{n} p_k x_k} F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \end{aligned} \tag{4.7}$$

and by (4.6) and (4.7) we have the following inequality of interest

$$\begin{aligned} \left\| \sum_{j=1}^{m} q_j F(y_j) - F\left(\sum_{k=1}^{n} p_k x_k\right) - \partial_{\sum_{j=1}^{m} q_j y_j-\sum_{k=1}^{n} p_k x_k} F\left(\sum_{k=1}^{n} p_k x_k\right) \right\|_Y \\ \le L \sum_{j=1}^{m} q_j \left\| \sum_{k=1}^{n} p_k x_k - y_j \right\|_X . \end{aligned} \tag{4.8}$$

If we take $\sum_{j=1}^{m} q_j y_j = \sum_{k=1}^{n} p_k x_k$ in (4.8), then we get the desired inequality (4.1).

The inequality (4.2) follows by (4.1) on taking $m = n$ and $q_j = p_j,\ j \in \{1, ..., n\}$. □

We also have:

Theorem 5. *Let $(X;\|\cdot\|_X)$ and $(Y;\|\cdot\|_Y)$ be two normed linear spaces over the complex number field $\mathbb{C}$. Assume that the mapping $F: C \subset X \to Y$ is defined on the open convex set C and $F \in \mathcal{BNW}_L(C)$ for some $L > 0$. Let $x_k \in C$, $p_k \geq 0$ for $k \in \{1,...,n\}$ with $\sum_{k=1}^n p_k = 1$ and F is Gâteaux-differentiable at x_k for any $k \in \{1,...,n\}$. If there exists $z \in C$ such that*

$$\sum_{k=1}^{n} p_k \partial_z F(x_k) = \sum_{k=1}^{n} p_k \partial_{x_k} F(x_k), \tag{4.9}$$

then we have

$$\left\| F(z) - \sum_{k=1}^{n} p_k F(x_k) \right\|_Y \leq L \sum_{k=1}^{n} p_k \|x_k - z\|_X. \tag{4.10}$$

Proof. From (4.4) we have

$$\left\| F(y) - F(x_k) - \partial_{y-x_k} F(x_k) \right\|_Y \leq L \|x_k - y\|_X \tag{4.11}$$

for any $y \in C$ and for any $k \in \{1,...,n\}$.

If we multiply (4.11) by $p_k \geq 0$ for $k \in \{1,...,n\}$ and sum, we get

$$\sum_{k=1}^{n} p_k \left\| F(y) - F(x_k) - \partial_{y-x_k} F(x_k) \right\|_Y \leq L \sum_{k=1}^{n} p_k \|x_k - y\|_X \tag{4.12}$$

for any $y \in C$.

By the generalized triangle inequality we get

$$\sum_{k=1}^{n} p_k \left\| F(y) - F(x_k) - \partial_{y-x_k} F(x_k) \right\|_Y \geq \left\| F(y) - \sum_{k=1}^{n} p_k F(x_k) - \sum_{k=1}^{n} p_k \partial_{y-x_k} F(x_k) \right\|_Y. \tag{4.13}$$

By the linearity of the Gâteaux differential we have

$$\sum_{k=1}^{n} p_k \partial_{y-x_k} F(x_k) = \sum_{k=1}^{n} p_k \partial_y F(x_k) - \sum_{k=1}^{n} p_k \partial_{x_k} F(x_k)$$

and by (4.12) and (4.13) we have the inequality of interest

$$\left\| F(y) - \sum_{k=1}^{n} p_k F(x_k) - \sum_{k=1}^{n} p_k \partial_y F(x_k) + \sum_{k=1}^{n} p_k \partial_{x_k} F(x_k) \right\|_Y \leq L \sum_{k=1}^{n} p_k \|x_k - y\|_X \tag{4.14}$$

for any $y \in C$.

Now, if $z \in C$ is such that (4.9) holds, then by (4.14) we get the desired result (4.10). $\square$

Remark 2. *Let $x_k \in C$, $p_k \geq 0$ for $k \in \{1,...,n\}$ with $\sum_{k=1}^{n} p_k = 1$ and F is differentiable at x_k for any $k \in \{1,...,n\}$. If there exists $z \in C$ such that*

$$\sum_{k=1}^{n} p_k F'(x_k) z = \sum_{k=1}^{n} p_k F(x_k) x_k, \tag{4.15}$$

then we have the inequality (4.10).

Moreover, if the operator $\sum_{k=1}^{n} p_k F'(x_k)$ is invertible and

$$z := \left(\sum_{k=1}^{n} p_k F'(x_k)\right)^{-1} \left(\sum_{k=1}^{n} p_k F(x_k) x_k\right) \in C, \tag{4.16}$$

then we have the inequality

$$\left\| F\left(\left(\sum_{k=1}^{n} p_k F'(x_k)\right)^{-1} \left(\sum_{k=1}^{n} p_k F(x_k) x_k\right)\right) - \sum_{k=1}^{n} p_k F(x_k) \right\|_Y \leq L \sum_{k=1}^{n} p_k \left\| x_k - \left(\sum_{k=1}^{n} p_k F'(x_k)\right)^{-1} \left(\sum_{k=1}^{n} p_k F(x_k) x_k\right) \right\|_X. \tag{4.17}$$

Conflict of Interests

The author declares that there is no conflict of interests.

References

[1] G. A. Anastassiou, Univariate Ostrowski inequalities, revisited. Monatsh. Math. 135 (2002), 175–189.

[2] H. Araki and S. Yamagami, An inequality for Hilbert-Schmidt norm, Commun. Math. Phys. 81 (1981), 89-96.

[3] R. Bhatia, First and second order perturbation bounds for the operator absolute value, Linear Algebra Appl. 208/209 (1994), 367-376.

[4] R. Bhatia, Perturbation bounds for the operator absolute value. Linear Algebra Appl. 226/228 (1995), 639–645.

[5] R. Bhatia, D. Singh and K. B. Sinha, Differentiation of operator functions and perturbation bounds. Comm. Math. Phys. 191 (1998), 603–611.

[6] R. Bhatia, Matrix Analysis, Springer Verlag, 1997.

[7] P. Cerone and S. S. Dragomir, Midpoint-type rules from an inequalities point of view, Ed. G. A. Anastassiou, Handbook of Analytic-Computational Methods in Applied Mathematics, CRC Press, New York. 135-200.

[8] P. Cerone and S. S. Dragomir, New bounds for the three-point rule involving the Riemann-Stieltjes integrals, in Advances in Statistics Combinatorics and Related Areas, C. Gulati, et al. (Eds.), World Science Publishing, 2002, 53-62.

[9] P. Cerone, S. S. Dragomir and J. Roumeliotis, Some Ostrowski type inequalities for n-time differentiable mappings and applications, Demonstr. Math. 32 (1999), 697—712.

[10] L. Ciurdariu, A note concerning several Hermite-Hadamard inequalities for different types of convex functions. Int. J. Math. Anal. 6 (2012), 1623–1639.

[11] R. Coleman, Calculus on Normed Vector Spaces, Springer New York Heidelberg Dordrecht London, 2012.

[12] S. S. Dragomir, Y. J. Cho and S. S. Kim, Inequalities of Hadamard's type for Lipschitzian mappings and their applications. J. Math. Anal. Appl. 245 (2000), 489–501.

[13] S. S. Dragomir and C. E. M. Pearce, Selected Topics on Hermite-Hadamard Inequalities and Applications, RGMIA Monographs, 2000. http://rgmia.org/monographs/hermite_hadamard.html.

[14] S. S. Dragomir, Semi-Inner Products and Applications, Nova Science Publishers, Inc., Hauppauge, NY, 2004. x+222 pp. ISBN: 1-59033-947-9, Preprint RGMIA Monographs, Victoria University, 2000. http://rgmia.org/monographs/SIP.html.

[15] S. S. Dragomir, Ostrowski's inequality for monotonous mappings and applications, J. KSIAM, 3 (1999), 127-135.

[16] S. S. Dragomir, The Ostrowski's integral inequality for Lipschitzian mappings and applications, Comp. Math. Appl. 38 (1999), 33-37.

[17] S. S. Dragomir, On the Ostrowski's inequality for Riemann-Stieltjes integral, Korean J. Appl. Math. 7 (2000), 477-485.

[18] S. S. Dragomir, On the Ostrowski's inequality for mappings of bounded variation and applications, Math. Ineq. Appl. 4 (2001), 33-40.

[19] S. S. Dragomir, On the Ostrowski inequality for Riemann-Stieltjes integral $\int_a^b f(t)\,du(t)$ where f is of Hölder type and u is of bounded variation and applications, J. KSIAM, 5 (2001), 35-45.

[20] S. S. Dragomir, Ostrowski type inequalities for isotonic linear functionals, J. Inequal. Pure Appl. Math. 3 (2002), 68.

[21] S. S. Dragomir, An inequality improving the first Hermite-Hadamard inequality for convex functions defined on linear spaces and applications for semi-inner products. J. Inequal. Pure Appl. Math. 3 (2002), 31.

[22] S. S. Dragomir, An Ostrowski like inequality for convex functions and applications, Rev. Math. Compl. 16 (2003), 373-382.

[23] S. S. Dragomir, Operator Inequalities of Ostrowski and Trapezoidal Type. Springer Briefs in Mathematics. Springer, New York, 2012. x+112 pp. ISBN: 978-1-4614-1778-1.

[24] S. S. Dragomir, Inequalities for power series in Banach algebras, SUT J. Math. 50 (2014), 25-45.

[25] S. S. Dragomir, Integral inequalities for Lipschitzian mappings between two Banach spaces and applications, Preprint RGMIA Res. Rep. Coll. 17(2014), 144. http://rgmia.org/papers/v17/v17a144.pdf.

[26] S. S. Dragomir, P. Cerone, J. Roumeliotis and S. Wang, A weighted version of Ostrowski inequality for mappings of Hölder type and applications in numerical analysis, Bull. Math. Soc. Sci. Math. Romanie, 42 (1999), 301-314.

[27] S. S. Dragomir and Th. M. Rassias (Eds), Ostrowski Type Inequalities and Applications in Numerical Integration, Kluwer Academic Publisher, 2002.

[28] S. S. Dragomir and S. Wang, A new inequality of Ostrowski's type in L_1-norm and applications to some special means and to some numerical quadrature rules, Tamkang J. Math. 28 (1997), 239-244.

[29] S. S. Dragomir, S. Wang, Applications of Ostrowski's inequality to the estimation of error bounds for some special means and some numerical quadrature rules, Appl. Math. Lett. 11(1998), 105-109.

[30] S. S. Dragomir, S. Wang, A new inequality of Ostrowski's type in L_p-norm and applications to some special means and to some numerical quadrature rules, Indian J. Math. 40 (1998), 245-304.

[31] R. Douglas, Banach Algebra Techniques in Operator Theory, Academic Press, 1972.

[32] Yu. B. Farforovskaya, Estimates of the closeness of spectral decompositions of self-adjoint operators in the Kantorovich-Rubinshtein metric (in Russian), Vesln. Leningrad. Gos. Univ. Ser. Mat. Mekh. Astronom. 4 (1967), 155-156.

[33] Yu. B. Farforovskaya, An estimate of the norm $\|f(B)-f(A)\|$ for self-adjoint operators A and B (in Russian) Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. 56 (1976), 143-162 .

[34] Yu. B. Farforovskaya and L. Nikolskaya, Modulus of continuity of operator functions. Algebra i Analiz 20 (2008), no. 3, 224–242; translation in St. Petersburg Math. J. 20 (2009), no. 3, 493–506.

[35] Y. Feng and W. Zhao, Refinement of Hermite-Hadamard inequality. Far East J. Math. Sci. 68 (2012), 245–250.

[36] A. M. Fink, Bounds on the deviation of a function from its averages, Czechoslovak Math. J. 42 (1992), 298-310.

[37] X. Gao, A note on the Hermite-Hadamard inequality. J. Math. Inequal. 4 (2010), 587–591.

[38] S.-R. Hwang, K.-L. Tseng and K.-C. Hsu, Hermite-Hadamard type and Fejér type inequalities for general weights (I). J. Inequal. Appl. 2013 (2013), 170.

[39] T. Kato, Continuity of the map $S\to|S|$ for linear operators, Proc. Japan Acad. 49 (1973), 143-162.

[40] U. S. Kırmacı and R. Dikici, On some Hermite-Hadamard type inequalities for twice differentiable mappings and applications. Tamkang J. Math. 44 (2013), 41–51.

[41] J. Mikusiński, The Bochner Integral, Birkhäuser Verlag, 1978.

[42] M. Muddassar, M. I. Bhatti and M. Iqbal, Some new s-Hermite-Hadamard type inequalities for differentiable functions and their applications. Proc. Pak. Acad. Sci. 49 (2012), 9–17.

[43] M. Matić and J. Pečarić, Note on inequalities of Hadamard's type for Lipschitzian mappings. Tamkang J. Math. 32 (2001), 127–130.

[44] A. Ostrowski, Über die Absolutabweichung einer differentienbaren Funktionen von ihren Integralmittelwert, Comment. Math. Hel. 10 (1938), 226-227.

[45] W. Rudin, Functional Analysis, McGraw Hill, 1973.

[46] M. Z. Sarikaya, On new Hermite Hadamard Fejér type integral inequalities. Stud. Univ. Babeş-Bolyai Math. 57 (2012), 377–386.

[47] S. Wasowicz and A. Witkowski, On some inequality of Hermite-Hadamard type. Opuscula Math. 32 (2012), 591–600.

[48] B.-Y. Xi and F. Qi, Some integral inequalities of Hermite-Hadamard type for convex functions with applications to means. J. Funct. Spaces Appl. 2012 (2012) 980438.

[49] G. Zabandan, A. Bodaghi and A. Kılıçman, The Hermite-Hadamard inequality for r-convex functions. J. Inequal. Appl. 2012 (2012), 215.

[50] C.J. Zhao, W.S. Cheung, X.Y. Li, On the Hermite-Hadamard type inequalities. J. Inequal. Appl. 2013 (2013), 228.