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TWO INEQUALITIES INVOLVING CIRCUMRADIUS, INRADIUS AND MEDIANS OF AN ACUTE TRIANGLE

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Abstract. A new geometric inequality involving circumradius, inradius and medians of an acute triangle is established. Another similar inequality proposed by the author as an open problem many years ago is proved. Several conjectures are proposed after having been verified by the computer.

Keywords: triangle; median; circumradius; inradius; Euler's inequality.

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1. Introduction and Main Result

Let ABC be a triangle with circumradius R and inradius r, and let m_a, m_b, m_c be its medians.

In [2], the author proved the following inequality involving the reciprocal sum of the medians of an arbitrary triangle ABC:

$$\sum \frac{1}{m_a} \leq \frac{2}{3}\left(\frac{1}{R}+\frac{1}{r}\right). \tag{1.1}$$

where $\sum$ denotes the cyclic sum.

In [11], Wu and Shi considered improvements of inequality (1.1) and proved that the best constant k for the following inequality:

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$$\sum \frac{1}{m_a} \le \frac{1}{r} - k\left(\frac{1}{r} - \frac{2}{R}\right) \tag{1.2}$$

is the real root in the interval $\left(\frac{1}{3}, \frac{2}{5}\right)$ of equation

$$354294k^6 - 509571k^5 + 1927260k^4 - 2145600k^3 + 133376k^2 + 99328k + 12288 = 0 \tag{1.3}$$

Furthermore, the constant k is approximately equal to 0.3440653.

In the recent paper [3], the author established the following reverse inequality of (1.1):

$$\sum \frac{1}{m_a} \ge \frac{5}{2R + r}. \tag{1.4}$$

where the combined coefficients of the denominator are the best possible.

In this paper, for the acute triangle we shall establish two inequalities involving the sums $\sum \frac{1}{m_b + m_c}$ and $\sum \frac{1}{(m_b + m_c)^2}$. Our main results are the following:

Theorem 1. *In the acute triangle ABC the following inequality holds:*

$$\sum \frac{1}{m_b + m_c} \le \frac{2}{R + 2r}, \tag{1.5}$$

with equality if and only if triangle ABC is equilateral.

Theorem 2. *In the acute triangle ABC the following inequality holds:*

$$\sum \frac{1}{(m_b + m_c)^2} \le \frac{1}{6Rr}, \tag{1.6}$$

with equality if and only if triangle ABC is equilateral.

In fact, inequality (1.6) was proposed by the author as one of conjectures in a Chinese paper [4], where most conjectures have been solved. However, the author has not seen that anyone has proved inequality (1.6).

The aim of this paper is to prove Theorem 1 and 2. We also propose several conjectures checked by the computer.

2. PRELIMINARIES

In order to prove the main results, we need several lemmas.

As usual, we denote by a,b,c the sides of triangle ABC; s and S the semiperimeter and area respectively; h_a,h_b,h_c the altitudes; r_a,r_b,r_c the radii of excircles.

Lemma 1. *In any triangle ABC the following inequality holds:*

$$m_a m_b m_c \geq \frac{1}{8R}\sum b^2c^2, \tag{2.1}$$

with equality if and only if triangle ABC is isosceles.

In [9], the author pointed that inequality (2.1) can be obtained from the following known result (see [11]):

$$\sum h_a^2 \sum \frac{1}{m_a^2} \leq 9. \tag{2.2}$$

Lemma 2. *In any triangle ABC the following inequality holds:*

$$m_a \geq h_a + \frac{(b^2+c^2-a^2+14bc)(b-c)^2}{64aS}, \tag{2.3}$$

with equality if and only if $b=c$.

Inequality (2.3) is one of the equivalent form of Theorem 1.1 from the author's paper [7].

Lemma 3. *With the above notations, we have the following identities:*

$$\sum a^2 = 2s^2 - 8Rr - 2r^2, \tag{2.4}$$

$$\sum a^3 = 2s^3 - (12Rr + 6r^2)s, \tag{2.5}$$

$$\sum a^4 = 2s^4 - 4(4R+3r)rs^2 + 2(4R+r)^2r^2, \tag{2.6}$$

$$\sum a^5 = 2s^5 - 20(R+r)rs^3 + 10(2R+r)(4R+r)r^2s, \tag{2.7}$$

$$\begin{aligned}\sum a^6 =& 2s^6 - 6(4R+5r)rs^4 + 6(24R^2+24Rr+5r^2)r^2s^2 \\ & - 2(4R+r)^3r^3,\end{aligned} \tag{2.8}$$

$$\sum a^7 = 2s^7 - 14(2R+3r)rs^5 + 14(16R^2+20Rr+5r^2)r^2s^3$$
$$- 14(2R+r)(4R+r)^2r^3s, \tag{2.9}$$

$$\sum a^8 = 2s^8 - 8(4R+7r)rs^6 + 20(16R^2+24Rr+7r^2)r^2s^4$$
$$- 8(4R+r)(32R^2+32Rr+7r^2)r^3s^2 + 2(4R+r)^4r^4. \tag{2.10}$$

Lemma 4. *With the above notations, we have the following identities:*

$$\sum bc = s^2 + 4Rr + r^2, \tag{2.11}$$

$$\sum b^2c^2 = s^4 - 2(4R-r)rs^2 + (4R+r)^2r^2, \tag{2.12}$$

$$\sum b^3c^3 = s^6 - 3(4R-r)rs^4 + 3r^4s^2 + (4R+r)^3r^3 \tag{2.13}$$

$$\sum b^4c^4 = s^8 - 4(4R-r)rs^6 + 2(16R^2-8Rr+3r^2)r^2s^4$$
$$+ 4(4R+r)r^5s^2 + (4R+r)^4r^4. \tag{2.14}$$

In Lemma 3 and 4, identities (2.4)-(2.6), (2.11) and (2.12) can be found in the monograph [11]. The others have been proved in [5] and [6].

Lemma 5. *In any triangle ABC the following inequality holds:*

$$P_0 \equiv (m_b+m_c)(m_c+m_a)(m_a+m_b) \geq \frac{K_0}{32Rs^2}, \tag{2.15}$$

where

$$K_0 = 12s^6 + (86R^2+55Rr+13r^2)s^4 - (53R-4r)(4R+r)^2rs^2$$
$$+ 3(4R+r)^4r^2.$$

Equality in (2.15) *holds if and only if* $\triangle ABC$ *is equilateral.*

Proof. First, we note that

$$P_0 = 2m_am_bm_c + \sum m_a(m_b^2+m_c^2). \tag{2.16}$$

Using the known formula:

$$m_a = \frac{1}{2}\sqrt{2b^2+2c^2-a^2}, \tag{2.17}$$

we get

$$4(m_b^2+m_c^2)=4a^2+b^2+c^2. \tag{2.18}$$

Consequently, from identity (2.16) using Lemma 1 and 2, we get

$$P_0 \geq \frac{1}{4R}\sum b^2c^2+\frac{1}{4}\sum(4a^2+b^2+c^2)\left[h_a+\frac{(b^2+c^2-a^2+14bc)(b-c)^2}{64aS}\right].$$

Using $h_a = 2S/a$ again, we get

$$P_0 \geq \frac{1}{4R}\sum b^2c^2+\frac{P_1}{256abcS}, \tag{2.19}$$

where

$$P_1=\sum bc(4a^2+b^2+c^2)\left[128S^2+(b^2+c^2-a^2+14bc)(b-c)^2\right].$$

Letting $d = abc$ and applying the equivalent form of Heron's formula

$$16S^2=2\sum b^2c^2-\sum a^4, \tag{2.20}$$

we easily obtain the following identity:

$$\begin{aligned} P_1 = &\left(108\sum a^2-10\sum bc\right)d^2+\left(52\sum a^3\sum a^2-13\sum a\sum a^4-71\sum a^5\right)d\\ &12\sum a^8-7\sum a\sum a^7-17\sum a^3\sum a^5+12\sum a^2\sum a^6+24\sum b^4c^4. \end{aligned} \tag{2.21}$$

Further, with the help of software Maple, using $\sum a = 2s$, Lemma 3, Lemma 4 and the following identity

$$abc = 4Rrs, \tag{2.22}$$

we immediately obtain

$$\begin{aligned} P_1 =& 32r^2\left[4s^6+(86R^2+119Rr-3r^2)s^4-(53R+4r)(4R+r)^2rs^2\right.\\ &\left.+3(4R+r)^4r^2\right]. \end{aligned} \tag{2.23}$$

Finally, from (2.19) using identities (2.12), (2.22) and $S = rs$, we get inequality (2.15). This completes the proof of Lemma 5. □

Lemma 6. *In the acute triangle ABC the following inequality holds:*

$$s^2 \geq 4R^2 - Rr + 13r^2 + \frac{(R-2r)r^3}{R^2}, \tag{2.24}$$

with equality if and only if triangle ABC is equilateral or right isosceles.

Inequality (2.24) was obtained by the author in [8].

Lemma 7. *In the acute triangle ABC the following inequality holds:*

$$s^2 \geq 16Rr - 3r^2 - \frac{4r^3}{R}, \tag{2.25}$$

with equality if and only if $\triangle ABC$ is equilateral or right isosceles.

Inequality (2.25) was first established by the author in a Chinese paper [10]. Later, the author also gave a direct proof in [8].

The following lemma provides an interesting inequality involving the medians and altitudes of an acute triangles.

Lemma 8. *In the acute triangle ABC the following inequality holds:*

$$\sum m_b m_c \leq \frac{1}{3}\sum h_a^2 + \frac{2}{3}\sum m_a^2, \tag{2.26}$$

with equality if and only if $\triangle ABC$ is equilateral.

Proof. First, by the Cauchy inequality we have

$$\left(\sum m_b m_c\right)^2 \leq \sum \frac{1}{r_b + r_c} \sum (r_b + r_c) m_b^2 m_c^2. \tag{2.27}$$

Using the following known formula:

$$r_a = \frac{S}{s-a} \tag{2.28}$$

and $S = rs$, we easily get

$$\sum \frac{1}{r_b + r_c} = \frac{abc\sum a - \sum a \sum a^3 + \left(\sum a^2\right)^2}{4abcS}.$$

Further, using $\sum = 2s$, identities (2.4), (2.5) and (2.22), one obtains

$$\sum \frac{1}{r_b + r_c} = \frac{s^2 + (4R+r)^2}{4Rs^2}. \tag{2.29}$$

In addition, by applying (2.17), (2.28), $s=(a+b+c)/2$ and $S=rs$, one can get

$$\sum(r_b+r_c)m_b^2m_c^2=\frac{K_1}{32r}, \tag{2.30}$$

where

$$\begin{aligned}K_1 =&7abc\sum a\sum a^2-11abc\sum a^3+2\sum a\sum a^5-6\sum a^6\\&+4\sum b^3c^3-15(abc)^2.\end{aligned}$$

Then, using $\sum a=2s$, identities (2.4), (2.5), (2.7), (2.8), (2.13) and (2.22), we further obtain

$$\sum(r_b+r_c)m_b^2m_c^2=\frac{1}{4}K_2, \tag{2.31}$$

where

$$K_2=(5R+14r)s^4-(88R^2+59Rr+16r^2)rs^2+2(4R+r)^3r^2.$$

Consequently, it follows from (2.27), (2.29) and (2.31) that

$$\left(\sum m_bm_c\right)^2\le\frac{\left[s^2+(4R+r)^2\right]K_2}{16Rs^2}. \tag{2.32}$$

On the other hand, by the equality $2Rh_a=bc$ and the following known identity:

$$\sum m_a^2=\frac{3}{4}\sum a^2, \tag{2.33}$$

we have

$$\frac{1}{3}\sum h_a^2+\frac{2}{3}\sum m_a^2=\frac{1}{12R^2}\sum b^2c^2+\frac{1}{2}\sum a^2.$$

Using identities (2.4) and (2.12), we further get

$$\frac{1}{3}\sum h_a^2+\frac{2}{3}\sum m_a^2=\frac{K_3}{12R^2}, \tag{2.34}$$

where

$$K_3=s^4+(12R^2-8Rr+2r^2)s^2-(4R+r)(2R-r)(6R+r)r.$$

Now, by inequality (2.32) and identity (2.34), to prove inequality (2.26) we need to show

$$\frac{\left[s^2+(4R+r)^2\right]K_2}{16Rs^2}\le\frac{K_3^2}{144R^4},$$

i.e.

$$Q_0 \equiv s^2K_3^2 - 9K_2R^3\left[s^2+(4R+r)^2\right] \geq 0.$$

With the help of software Maple, by using the expressions of K_2 and K_3, the above inequality is transformed into

$$\begin{aligned}Q_0 \equiv & s^{10}+(24R^2-16Rr+4r^2)s^8+(99R^4-414R^3r+120R^2r^2\\ & -16Rr^3+6r^4)s^6+(-720R^6-2736R^5r+342R^4r^2-46R^3r^3\\ & -88R^2r^4+16Rr^5+4r^6)s^4+(792R^5+603R^4r+30R^3r^2\\ & -8R^2r^3+8Rr^4+r^5)(4R+r)^2rs^2-18(4R+r)^5R^3r^2 \geq 0,\end{aligned} \tag{2.35}$$

which needs to be proved.

According to Euler's inequality (valid for any triangle):

$$R \geq 2r \tag{2.36}$$

and Lemma 6, we know that for acute triangle ABC holds:

$$v_0 \equiv s^2-(4R^2-Rr-13r^2) \geq 0. \tag{2.37}$$

Based on this inequality, one can write inequality (2.35) as follows:

$$Q_0 \equiv v_0^5+m_4v_0^4+m_3v_0^3+m_2v_0^2+m_1v_0+m_0 \geq 0, \tag{2.38}$$

where

$$\begin{aligned}m_4 =& 44R^2-21Rr+69r^2,\\ m_3 =& 643R^4-846R^3r+2546R^2r^2-1124Rr^3+1904r^4,\\ m_2 =& 3412R^6-11169R^5r+29517R^4r^2-33890R^3r^3\\ & +54734R^2r^4-22544Rr^5+26264r^6,\\ m_1 =& 6416R^8-40008R^7r+134625R^6r^2-291952R^5r^3\\ & +442837R^4r^4-452028R^3r^5+519024R^2r^6\\ & -200816Rr^7+181104r^8,\end{aligned}$$

$$\begin{aligned} m_0 =&(R-2r)(1984R^9 - 26128R^8r + 127428R^7r^2 - 301051R^6r^3 \\ &+ 559631R^5r^4 - 791539R^4r^5 + 597875R^3r^6 - 811914R^2r^7 \\ &+ 210308Rr^8 - 249704r^9). \end{aligned}$$

Clearly, Euler's inequality shows that $m_4 > 0$ and $m_3 > 0$. If we set $e = R - 2r$ and substitute $R = 2r + e$ into the expression of m_2, then it is easy to obtain

$$\begin{aligned} m_2 =&3412e^6 + 29775e^5r + 122547e^4r^2 + 301406e^3r^3 + 485162e^2r^4 \\ &+ 495840er^5 + 262224r^6. \end{aligned} \tag{2.39}$$

As $e \geq 0$, so that $m_2 > 0$. Thus, to prove inequality (2.38) it remains to show that

$$m_1v_0 + m_0 \geq 0. \tag{2.40}$$

We shall consider the following two cases to finish the proof of inequality (2.40).

Case 1 R and r satisfy that $h_0 \equiv R^2 - 2Rr - r^2 > 0$.

Firstly, in the same way to prove $m_2 > 0$ one can easily show that $m_1 > 0$. Hence, by Lemma 6, to prove (2.40) we require the following inequality to be proved:

$$m_1\frac{(R-2r)r^3}{R^2} + m_0 \geq 0.$$

A direct calculation gives the equivalent inequality

$$x_1\frac{(R-2r)}{R^2} \geq 0, \tag{2.41}$$

where

$$\begin{aligned} x_1 =&1984R^{11} - 26128R^{10}r + 127428R^9r^2 - 294635R^8r^3 + 519623R^7r^4 \\ &- 656914R^6r^5 + 305923R^5r^6 - 369077R^4r^7 - 241720R^3r^8 \\ &+ 269320R^2r^9 - 200816Rr^{10} + 181104r^{11}. \end{aligned}$$

Since $R \geq 2r$, to prove (2.41) we need to show the strict inequality $x_1 > 0$. However, it is easy to verify the following identity:

$$48828125x_1 = eh_0x_2 + 97656250\left[2766994h_0 + R(147480R - 311143r)\right]r^9 \tag{2.42}$$

where

$$\begin{aligned} e =&R-2r,\\ x_2 =&96875000000e^8+661718750000e^7r+792382812500e^6r^2\\ &-5077880859375e^5r^3-11135107421875e^4r^4+55848583984375e^3r^5\\ &+313514160156250e^2r^6+659879296875000er^7+676876269531250r^8. \end{aligned}$$

It follows from the hypothesis $R^2-2Rr-r^2>0$ that $R\geq(1+\sqrt{2})r$. So, it is easy to find $147480R-311143r>0$. Thus, by (2.42) we only need to show $x_2>0$. Then, it is enough to show that

$$\begin{aligned} &661718750000e^7r+792382812500e^6r^2-5077880859375e^5r^3\\ &-11135107421875e^4r^4+55848583984375e^3r^5>0. \end{aligned}\tag{2.43}$$

One can only consider the case $r=1$, i.e.,

$$\begin{aligned} &661718750000e^7+792382812500e^6-5077880859375e^5\\ &-11135107421875e^4+55848583984375e^3>0. \end{aligned}$$

Dividing both sides by 66171875000 gives

$$10e^4+(11.974\cdots)e^3-(76.737\cdots)e^2-(168.275\cdots)e+(843.992\cdots)>0.$$

So, we only need to prove

$$10e^4+10e^3-80e^2-170e+840>0.$$

It suffices to show

$$e^4+e^3-8e^2-17e+48>0,$$

which can be rewritten as

$$e(e+5)(e-2)^2+8e^2-37e+48>0.$$

Notice that $8e^2-37e+48>0$. One sees that the desired inequality holds. Hence, inequalities (2.43) and (2.40) are proved in Case 1.

Case 2 R and r satisfy $h_0 \equiv R^2 - 2Rr - r^2 \le 0$.

By Lemma 7 we have

$$v_0 = s^2 - (4R^2 - Rr + 13r^2) \ge 16Rr - 3r^2 - \frac{4r^3}{R} - 4R^2 + Rr - 13r^2.$$

Simplifying gives

$$v_0 \ge \frac{(R-2r)(2r^2 + 9Rr - 4R^2)}{R}. \tag{2.44}$$

Since $m_1 > 0$, to prove inequality (2.40) we need to show that

$$m_1 \frac{(R-2r)(2r^2 + 9Rr - 4R^2)}{R} + m_0 \ge 0.$$

A direct computation gives the equivalent inequality:

$$x_3 \frac{(R-2r)}{R} \ge 0, \tag{2.45}$$

where

$$\begin{aligned} x_3 = & -23680R^{10} + 191648R^9 r - 758312R^8 r^2 + 1998366R^7 r^3 \\ & - 3570035R^6 r^4 + 4418202R^5 r^5 - 4660799R^4 r^6 \\ & + 3758510R^3 r^7 - 1283404R^2 r^8 + 978600Rr^9 + 362208r^{10}. \end{aligned}$$

By Euler's inequality, it remains to show that $x_3 > 0$. We can rewrite x_3 as follows:

$$x_3 = -eh_0x_4 - 529824h_0r^8 + 2R(769766r - 311143R)r^8, \tag{2.46}$$

where

$$\begin{aligned} x_4 = & 23680R^7 - 96928R^6 r + 299560R^5 r^2 - 556702R^4 r^3 \\ & + 638403R^3 r^4 - 793604R^2 r^5 + 684578Rr^6 + 83808r^7. \end{aligned}$$

If we set $R = 2r + e (e \ge 0)$ and substitute it into the expression of x_4, then we find that all the terms are nonnegative after expanding. Thus, we have $x_4 > 0$.

In addition, it follows from the hypothesis $R^2 - 2Rr - r^2 \le 0$ that $(1+\sqrt{2})r \ge R$. This yields $769766r - 311143R > 0$. Finally, from identity (2.46) we deduce that $x_4 > 0$ holds in the case $h_0 < 0$.

Combining the discussions of the above two cases, we conclude that inequality (2.40) holds for all acute triangles. Also, it is easy to determine that equality of (2.40) holds if and only if $\triangle ABC$ is equilateral. This completes the proof of Lemma 8. □

Remark 1. *Inspired by inequality* (2.26)*, the author has found and proved that for the acute triangle ABC the following inequality chain holds:*

$$\begin{aligned}&\frac{1}{2}\sum(m_b-m_c)^2+\frac{1}{3}\sum h_a^2\geq\frac{1}{4}\sum a^2\geq\frac{1}{2}\sum(m_b-m_c)^2+\frac{1}{3}\sum h_bh_c\\ &\geq\frac{1}{4}\sum bc,\end{aligned}\tag{2.47}$$

in which the first inequality is equivalent to inequality (2.26).

3. Proofs of Theorem 1 and Theorem 2

In this section, we shall give the proofs of Theorem 1 and 2.

3.1. Proof of Theorem 1.

Proof. We first note that inequality (1.5) it is equivalent to

$$2(m_b+m_c)(m_c+m_a)(m_a+m_b)\geq(R+2r)\sum(m_c+m_a)(m_a+m_b),$$

that is

$$2(m_b+m_c)(m_c+m_a)(m_a+m_b)\geq(R+2r)\left(\sum m_a^2+3\sum m_bm_c\right).\tag{3.1}$$

By Lemma 5 and 8, we only need to prove

$$\frac{K_0}{16Rs^2}\geq(R+2r)\left(3\sum m_a^2+\sum h_a^2\right).$$

Multiplying both sides by $16s^2R^2$ and using $2Rh_a=bc$ and the previous identity (2.33), one can see that the inequality is equivalent to

$$D_0\equiv RK_0-4(R+2r)s^2\left(9R^2\sum a^2+\sum b^2c^2\right)\geq0.$$

Using the expression of K_0 and simplifying gives

$$\begin{aligned} D_0 \equiv & 8(R-r)s^6+(14R^3-57R^2r+69Rr^2-16r^3)s^4 \\ & -(4R+r)(140R^3-91R^2r+32Rr^2+8r^3)rs^2 \\ & +3(4R+r)^4Rr^2 \geq 0, \end{aligned} \tag{3.2}$$

which needs to be proved.

We set $v_0 = s^2-4R^2+Rr-13r^2$. Inequality (2.37) shows that for acute triangle $v_0 \geq 0$ holds. We can rewrite D_0 as follows:

$$\begin{aligned} D_0 \equiv & 8(R-r)v_0^3+(110R^3-177R^2r+405Rr^2-328r^3)v_0^2 \\ & +(496R^5-1620R^4r+3966R^3r^2-4929R^2r^3+6442Rr^4 \\ & -4480r^5)v_0+(R-2r)(736R^6-2688R^5r+5350R^4r^2 \\ & -10933R^3r^3+11376R^2r^4-11348Rr^5+10192r^6) \geq 0. \end{aligned} \tag{3.3}$$

By Euler's inequality, one sees that $110R^3-177R^2r+405Rr^2-328r^3 > 0$. Thus, to prove $D_0 \geq 0$ we only need to show that

$$\begin{aligned} D_1 \equiv & (496R^5-1620R^4r+3966R^3r^2-4929R^2r^3+6442Rr^4-4480r^5)v_0 \\ & +(R-2r)(736R^6-2688R^5r+5350R^4r^2-10933R^3r^3+11376R^2r^4 \\ & -11348Rr^5+10192r^6) \geq 0. \end{aligned} \tag{3.4}$$

We shall consider the following two cases to finish the proof of inequality (3.4).

Case 1 R and r satisfy $5R-12r > 0$.

We set $e = R-2r$, then $e \geq 0$ and it is easy to obtain

$$\begin{aligned} & (496R^5-1620R^4r+3966R^3r^2-4929R^2r^3+6442Rr^4-4480r^5) \\ & = 496e^5+3340e^4r+10846e^3r^2+19667e^2r^3+22158er^4+10368r^5 > 0. \end{aligned} \tag{3.5}$$

Note that $v_0 \geq 0$ and $R \geq 2r$, to prove (3.4) it remains to prove the following strict inequality:

$$\begin{aligned} y_1 \equiv & 736R^6-2688R^5r+5350R^4r^2-10933R^3r^3+11376R^2r^4 \\ & -11348Rr^5+10192r^6 > 0. \end{aligned} \tag{3.6}$$

But, it is easy to check that

$$
\begin{aligned}
3125y_1 =&(5R-12r)(R-2r)(460000R^4+344000R^3r+2649350R^2r^2\\
&+3172815Rr^3+8353506r^4)+32(2255221R-5269817r)r^5.
\end{aligned}
\tag{3.7}
$$

By the assumption that $5R-12r>0$, it is easy to know $2255221R-5269817r>0$. Thus, from the above identity we deduce $y_1>0$. This completes the proof of (3.4) in Case 1.

Case 2 R and r satisfy $5R-12r\le 0$.

By the previous inequalities (2.43) and (3.5), we have

$$
\begin{aligned}
D_1 \ge&\frac{(R-2r)(2r^2+9Rr-4R^2)}{R}(496R^5-1620R^4r+3966R^3r^2\\
&-4929R^2r^3+6442Rr^4-4480r^5+(R-2r)(736R^6-2688R^5r\\
&+5350R^4r^2-10933R^3r^3+11376R^2r^4-11348Rr^5+10192r^6).
\end{aligned}
$$

Simplifying gives

$$
D_1 \ge \frac{R-2r}{R}y_2, \tag{3.8}
$$

where

$$
\begin{aligned}
y_2 =&-1248R^7+8256R^6r-24102R^5r^2+41237R^4r^3-50821R^3r^4\\
&+54692R^2r^5-17244Rr^6-8960r^7.
\end{aligned}
$$

Let $e=R-2r$, then it is easy to get

$$
78152y_2=5(12r-5R)ey_3+368812232(12r-5R)r^6+1287375536r^7, \tag{3.9}
$$

where

$$
\begin{aligned}
y_3 =&3900000e^5+30360000e^4r+105462750e^3r^2+210506975e^2r^3\\
&+280843415er^4+255843616r^5.
\end{aligned}
$$

As $e\ge 0$ and the assumption $12r-5R>0$ we deduce $y_2>0$ from (3.9). Thus, it follows from (3.8) that $D_1\ge 0$. Inequality (3.4) is proved.

Combining the arguments of the above two cases, we conclude that inequality (3.4) is valid for all acute triangles. And we completes the proof of inequality (1.5). It is easily shown that equality in (1.5) holds only when triangle ABC is an equilateral triangle. This completes the proof of Theorem 1. □

3.2. Proof of Theorem 2.

Proof. To prove inequality (1.6), note first that it is equivalent to

$$(m_b+m_c)^2(m_c+m_a)^2(m_a+m_b)^2 \geq 6Rr\sum(m_c+m_a)^2(m_a+m_b)^2. \tag{3.10}$$

Using the previous identity (2.33), the following known identity

$$\sum m_a^4 = \frac{9}{16}\sum a^4, \tag{3.11}$$

inequality (2.26) and identity (2.34), we have

$$\begin{aligned}
&\sum(m_c+m_a)^2(m_a+m_b)^2\\
&=\sum m_a^4+2\sum m_bm_c\sum m_a^2+3\left(\sum m_bm_c\right)^2\\
&=\frac{9}{16}\sum a^4+\frac{3}{2}\sum a^2\sum m_bm_c+3\left(\sum m_bm_c\right)^2\\
&\leq \frac{9}{16}\sum a^4+\frac{K_3}{8R^2}\sum a^2+\frac{K_3^2}{48R^4}.
\end{aligned}$$

Finally, with the help of Maple using identities (2.4), (2.6) and the expression of K_3, we immediately obtain

$$\sum(m_c+m_a)^2(m_a+m_b)^2 \leq \frac{K_4}{48R^4}, \tag{3.12}$$

where

$$\begin{aligned}
K_4 =& s^8+(36R^2-16Rr+4r^2)s^6+(342R^4-432R^3r+132R^2r^2\\
&-16Rr^3+6r^4)s^4-4(684R^5-207R^4r-8R^3r^2+25R^2r^3\\
&-4Rr^4-r^5)rs^2+(342R^4-144R^3r-20R^2r^2+8Rr^3\\
&+r^4)(4R+r)^2r^2.
\end{aligned}$$

By Lemma 5 and inequality (3.10), to prove inequality (3.8) we only need to show

$$\left(\frac{K_0}{32Rs^2}\right)^2 \geq 6Rr\cdot\frac{K_4}{48R^4},$$

that is

$$E_0 \equiv RK_0^2 - 128rs^4K_4 \geq 0.$$

With the help of Maple, it is easy to know that the above inequality is equivalent to

$$\begin{aligned}E_0 \equiv &(144R-128r)s^{12}+(2064R^3-3288R^2r+2360Rr^2-512r^3)s^{10}\\&+(7396R^5-54668R^4r+51917R^3r^2-15970R^2r^3+2313Rr^4\\&-768r^5)s^8-2(72928R^6-106720R^5r+76406R^4r^2+4083R^3r^3\\&-6923R^2r^4+936Rr^5+256r^6)rs^6+(53200R^5-18680R^4r\\&+22509R^3r^2+3218R^2r^3-930Rr^4-128r^5)(4R+r)^2r^2s^4\\&-6(53R-4r)(4R+r)^6Rr^3s^2+9(4R+r)^8Rr^4 \geq 0. \qquad (3.13)\end{aligned}$$

Note that for any triangle the following inequality holds (see [10]):

$$(4R+r)^2 \geq 3s^2, \qquad (3.14)$$

we only need to show

$$\begin{aligned}E_1 \equiv &(144R-128r)s^{10}+(2064R^3-3288R^2r+2360Rr^2-512r^3)s^8\\&+(7396R^5-54668R^4r+51917R^3r^2-15970R^2r^3+2313Rr^4\\&-768r^5)s^6-2(72928R^6-106720R^5r+76406R^4r^2+4083R^3r^3\\&-6923R^2r^4+936Rr^5+256r^6)rs^4+(53200R^5-18680R^4r\\&+22509R^3r^2+3218R^2r^3-930Rr^4-128r^5)(4R+r)^2r^2s^2\\&-6(53R-4r)(4R+r)^6Rr^3+27(4R+r)^6Rr^4 \geq 0, \qquad (3.15)\end{aligned}$$

According to the previous inequality $v_0 \equiv s^2-(4R^2-Rr+13r^2) \geq 0$, one can write the above inequality in the form:

$$E_1 \equiv n_5v_0^5+n_4v_0^4+n_3v_0^3+n_2v_0^2+n_1v_0+n_0 \geq 0, \qquad (3.16)$$

where

$$
\begin{aligned}
n_5 =& 144R-128r,\\
n_4 =& 4944R^3-6568R^2r+12360Rr^2-8832r^3,\\
n_3 =& 63460R^5-147532R^4r+371597R^3r^2-376418R^2r^3\\
&+403721Rr^4-243712r^5,\\
n_2 =& 379056R^7-1389820R^6r+3950876R^5r^2-6452911R^4r^3\\
&+9586583R^3r^4-8025731R^2r^5+6376191Rr^6-3361792r^7,\\
n_1 =& 1067712R^9-5554592R^8r+18099612R^7r^2-40657780R^6r^3\\
&+71938507R^5r^4-92811924R^4r^5+104317475R^3r^6\\
&-75553590R^2r^7+49149249Rr^8-23181312r^9,\\
n_0 =& (R-2r)(1149184R^{10}-5574592R^9r+18653296R^8r^2\\
&-46214692R^7r^3+81322316R^6r^4-131709673R^5r^5\\
&+143156617R^4r^6-153888527R^3r^7+103416673R^2r^8\\
&-58414647Rr^9+31962112r^{10}).
\end{aligned}
$$

Since $R \geq 2r$, we see that $n_5 > 0$ and $n_4 > 0$. Let $e = R-2r$, then we have

$$
\begin{aligned}
n_3 =& 63460e^5+487068e^4r+1729741e^3r^2+3389196e^2r^3\\
&+3712989er^4+1701042r^5 > 0.
\end{aligned}
\tag{3.17}
$$

Similarly, we have $n_2 > 0$. Thus, to prove inequality (3.16) it remains to prove that for the acute triangle ABC holds:

$$
n_1v_0+n_0 \geq 0. \tag{3.18}
$$

As the proof of inequality (3.4), we shall also consider the following two cases.

Case 1 R and r satisfy $5R-12r > 0$.

Substituting $R = 2r + e(e \geq 0)$ into the expression of n_1 and expanding, we find that all the terms are non-negative. So, we have $n_1 > 0$. To prove (3.18) it remains to show that $n_0 \geq 0$. Since $R \geq 2r$, we only need to prove the following strict inequality:

$$\begin{aligned} z_1 \equiv & 1149184R^{10} - 5574592R^9r + 18653296R^8r^2 - 46214692R^7r^3 \\ & + 81322316R^6r^4 - 131709673R^5r^5 + 143156617R^4r^6 - 153888527R^3r^7 \\ & + 103416673R^2r^8 - 58414647Rr^9 + 31962112r^{10} > 0. \end{aligned} \tag{3.19}$$

However, it is easy to obtain

$$\begin{aligned} 9765625z_1 = & 5(R-2r)(5R-12r)z_2 + 2432992422589223(5R-12r)r^9 \\ & + 573527481897196r^{10} \end{aligned} \tag{3.20}$$

where

$$\begin{aligned} z_2 = & 448900000000R^8 - 202415000000R^7r + 4241097750000R^6r^2 \\ & + 1579808037500R^5r^3 + 18360415852500R^4r^4 + 21753660155375R^3r^5 \\ & + 63506662107275R^2r^6 + 114899038666835Rr^7 + 241120929909779r^8. \end{aligned}$$

By Euler's inequality we have

$$448900000000R - 202415000000r > 0,$$

so that $z_2 > 0$. Hence, from (3.20) by Euler's inequality and the assumption $5R - 12r > 0$ we deduce $z_1 > 0$. This completes the proof of inequality (3.18) in Case 1.

Case 2 R and r satisfy $5R - 12r \leq 0$.

We have known $n_1 > 0$. Thus, by the previous inequality (2.44), for proving (3.18) we require the following inequality to be proved:

$$\frac{(R-2r)(2r^2 + 9Rr - 4R^2)}{R} n_1 + n_0 \geq 0.$$

Simplifying gives equivalent inequality:

$$\frac{R-2r}{R} z_3 \geq 0, \tag{3.21}$$

where

$$\begin{aligned} z_3 =& -3121664R^{11}+26253184R^{10}r-101601056R^9r^2 \\ &+268203752R^8r^3-536152508R^7r^4+805669026R^6r^5 \\ &-965543585R^5r^6+901559260R^4r^7-564527683R^3r^8 \\ &+325546662R^2r^9-78371198Rr^{10}-46362624r^{11}. \end{aligned}$$

Since $R \geq 2r$, it remains to show $z_3 > 0$. But it is easy to get

$$1953125z_3 = -(5R-12r)(R-2r)^3 z_4 + z_5, \tag{3.22}$$

where

$$\begin{aligned} e =& R-2r, \\ z_4 =& 1219400000000e^7+17059410000000e^6r+109676676500000e^5r^2 \\ &+417167004975000e^4r^3+1008575325427500e^3r^4 \\ &+1527468345014750e^2r^5+1236948010271525er^6 \\ &+338367127023 60r^7, \\ z_5 =& 45(106634517846909e^3+59569264062500e^2r \\ &+15351996093750er^2+7648551562500r^3)r^8. \end{aligned}$$

As $e \geq 0$, one sees that both strict inequalities $z_4 > 0$ and $z_5 > 0$ are valid. Thus, by identity (3.22) and the assumption that $5R-12r \leq 0$, we deduce that $z_3 > 0$. Therefore, inequality (3.18) is proved in Case 2.

Combining the arguments of the above two cases, we conclude that inequality (3.18) holds for all acute triangles. And, we finish the proof of inequality (1.6). Moreover, it is easy to determine that equality in (1.6) occurs if and only if triangle ABC is equilateral. This completes the proof of Theorem 2. □

4. SEVERAL CONJECTURES

In this section, we shall propose several conjectures for acute triangles.

Let k be a positive real number. Theorem 1 shows that for the acute triangle the following inequality

$$\sum \frac{1}{m_b+m_c} \le \frac{1}{2r+k(R-2r)} \tag{4.1}$$

holds for $k = 1/2$.

The author propose here the following related conjecture:

Conjecture 1. *Suppose that inequality* (4.1) *holds for acute triangle ABC, then the maximum k_{max} of k is given by*

$$k_{max} = \frac{23\sqrt{10}-42}{41} + \frac{92\sqrt{5}-166\sqrt{2}}{123} \approx 0.5134\cdots,$$

which arrives only when triangle ABC is right isosceles.

Many years ago, the author found that there exist a large number of acute triangle inequalities in which the equalities hold if and only if the triangle is right isosceles. It seems likely that this kind of inequalities are not easy to prove. Next, we introduce several such conjectured inequalities, which only involves the medians and sides of an acute triangle.

Conjecture 2. *If ABC is an acute triangle, then*

$$\frac{\sum m_a}{\sum a} \ge \frac{\sqrt{2}+2\sqrt{5}}{4+2\sqrt{2}}, \tag{4.2}$$

with equality if and only if triangle ABC is right isosceles.

Conjecture 3. *If ABC is an acute triangle, then*

$$\frac{\sum m_b m_c}{\sum bc} \ge \frac{5+2\sqrt{10}}{4+8\sqrt{2}}, \tag{4.3}$$

with equality if and only if triangle ABC is right isosceles.

Conjecture 4. *If ABC is an acute triangle, then*

$$\sum a^2 m_a \ge \frac{2+\sqrt{10}}{2} abc, \tag{4.4}$$

with equality if and only if triangle ABC is right isosceles.

Conjecture 5. *If ABC is an acute triangle, then*

$$\sum m_a^3 \leq \frac{\sqrt{2}+5\sqrt{5}}{8+8\sqrt{2}} \sum a^3, \tag{4.5}$$

with equality if and only if triangle ABC is right isosceles.

CONFLICT OF INTERESTS

The author declares that there is no conflict of interests.

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