Available online at http://scik.org
Adv. Inequal. Appl. 2024, 2024:1
https://doi.org/10.28919/aia/8682
ISSN: 2050-7461

INEQUALITIES CONCERNING THE CIRCUMRADIUS, INRADIUS AND SEMIPERIMETER OF A NON-OBTUSE TRIANGLE

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Abstract. In this paper, we provide two new proofs of a remarkable inequality involving the circumradius, inradius and semiperimeter of a non-obtuse triangle. Furthermore, we generalize this inequality and another similar result to the case with one parameter. We also give some applications of our new results.

Keywords: non-obtuse triangle; the fundamental triangle inequality; Ciamberlini's inequality; Walker's inequality; Euler's inequality.

2020 AMS Subject Classification: 51M16.

1. Introduction

Throughout this paper, we denote by R, r and s the circumradius, inradius and semiperimeter of a triangle ABC, respectively.

For any triangle ABC, we have the following double inequality:

$$\begin{aligned}&2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}\\ &\leq s^2 \leq 2R^2+10Rr-r^2+2(R-2r)\sqrt{R^2-2Rr}.\end{aligned} \tag{1.1}$$

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Received June 6, 2024

This was first proved by E. Rouché in 1851 and it was subsequently rediscovered in many different forms by several authors. The above inequalities are known today as the fundamental triangle inequality. For its history, variants, proofs, corollaries and related results, we refer the reader to the monograph [17] and articles [1-3, 6, 15, 20, 21].

In 1961, W. J. Blundon [3] proved that for any triangle the lower bound and the upper bound of s^2 given by the above inequalities are the best possible. However, we have already known that for a non-obtuse triangle, there is no best lower bound of s^2 (or s) in terms of R and r.

In 1944, C. Ciamberlini [5] first gave the following linear inequality for non-obtuse triangles (see also [4]):

$$s \geq 2R + r, \tag{1.2}$$

with equality if and only if the triangle is right-angled.

Ciamberlini's inequality is also a fundamental result in triangle inequalities. A number of triangle inequalities (not limited to non-obtuse triangles) can be proved by using (1.2) and other results.

In 1972, A. Walker [19] found that the following quadratic inequality (see also [17, p.248]):

$$s^2 \geq 2R^2 + 8Rr + 3r^2 \tag{1.3}$$

holds for any non-obtuse triangle. Equality in (1.3) holds if and only if the triangle is equilateral or right isosceles.

In 1996, X. Z. Yang [23] and X. L. Chen [18] independently generalized Walker's (1.3) to the case with one parameter. X. Z. Yang established the following inequality:

$$s^2 \geq 2(1+k)R^2 + 2\left[4-(3+\sqrt{2})k\right] + \left[3+4(1+\sqrt{2})k\right] r^2, \tag{1.4}$$

where k is a number and $-1 \leq k \leq 1$. X.L.Chen proved the equivalent inequality:

$$s^2 \geq kR^2 + \left[2(7+\sqrt{2})-(3+\sqrt{2})k\right] - \left[1+4\sqrt{2}-2(1+\sqrt{2})k\right] r^2, \tag{1.5}$$

where k is a real number and $2(1-\sqrt{2}) \leq k \leq 4$.

In 2010, J. Liu [7] established the following non-obtuse triangle inequality:

$$s^2 \geq 16Rr - 3r^2 - \frac{4r^3}{R}. \tag{1.6}$$

Equality condition is the same as Walker's inequality (1.3).

In 2020, J. Liu [11] applied inequality (1.6), Ciamberlini's inequality (1.2) and Euler's inequality

$$R \geq 2r \tag{1.7}$$

to obtain a further generalization of Walker's inequality, that is

$$s^2 \geq \frac{4mR^3 - 4(m-n)R^2r - (7m+3n)Rr^2 - 2(m+2n)r^3}{(m+n)R - 2mr}, \tag{1.8}$$

where m and n are arbitrary non-negative real numbers (not both zero). As applications of this result, a lot of new inequalities in the form $s^2 \geq f(R,r)$ were given in [11]. For example,

$$s^2 \geq 4R^2 + 9r^2 + \frac{4r^3}{R}, \tag{1.9}$$

and

$$s^2 \geq 4R^2 - Rr + 13r^2 + \frac{(R-2r)r^3}{R^2}. \tag{1.10}$$

In the recent articles [12-14, 16, 22], the author used Ciamberlini's inequality (1.2), inequalities (1.6) and (1.10) to establish some new triangle inequalities. We think that both inequalities (1.6) and (1.10) are remarkable.

In this paper, we shall give two new proofs of inequality (1.6) by establishing two identities. We also generalize both inequalities (1.6) and (1.10) to the case with one parameter. Our new results are the same as that inequality (1.8) in fact contains Walker's inequality (1.3), inequality (1.9) and other known inequalities.

2. Two New Proofs of Inequality (1.6)

In [11], the author pointed out that in any triangle the following identity holds:

$$\sum(c^2+a^2-b^2)(a^2+b^2-c^2)(b-c)^2 = 32r^2s^2(s^2-2R^2-8Rr-3r^2), \tag{2.1}$$

where a, b and c are the sides of the triangle and $\sum$ denotes the cyclic sum over the sides.

Clearly, we immediately deduce from (2.1) that Walker's inequality (1.3) holds for non-obtuse triangles. A natural question is whether an identity similar to (2.1) that is related to inequality (1.6) exist ? Upon reflection, we find two related identities. One of them is

$$\sum a(b+c-a)(b^2+c^2-a^2)(a-b)^2(a-c)^2$$
$$= 16rs^2(s^2-12Rr-3r^2)(Rs^2-16R^2r+3Rr^2+4r^3). \tag{2.2}$$

Next, we first prove this identity and then use it to prove inequality (1.6). Denote by P_0 the left hand side of (2.2). Expanding gives

$$\begin{aligned} P_0 =& -3c^7b-5b^2ca^5+2b^4ca^3+c^8-5a^2bc^5+8abc^6+10a^2b^3c^3 \\ &+2a^4bc^3-5a^2b^5c-5b^2c^5a+8a^6bc+2a^4b^3c-5ab^5c^2 \\ &+2ab^3c^4+10a^3b^3c^2+2a^3bc^4+2b^6c^2-11b^2c^4a^2 \\ &-5a^5bc^2+2b^4c^3a-11b^2c^2a^4-11b^4c^2a^2-6b^4c^4 \\ &+10b^2c^3a^3+8b^6ca+2b^2c^6-6c^4a^4+2c^2a^6+2c^6a^2 \\ &-6a^4b^4+2a^2b^6+2a^6b^2+3c^5a^3+3c^3a^5-3a^7b+3b^3a^5 \\ &-3a^7c+3b^5a^3+a^8-3b^7a-3b^7c+3c^5b^3+3b^5c^3+b^8-3c^7a, \end{aligned} \tag{2.3}$$

from which we further easily obtain

$$\begin{aligned} P_0 =& 2\sum a^8-3\sum a\sum a^7+2\sum a^2\sum a^6+11abc\sum a^5 \\ &-5abc\sum a\sum a^4+2abc\sum a^2\sum a^3-11(abc)^2\sum a^2 \\ &+3\sum a^2\sum b^3c^3+7(abc)^2\sum bc-6\sum b^4c^4. \end{aligned} \tag{2.4}$$

Recalling that in any triangle ABC we have the following known identities:

$$abc = 4Rrs, \tag{2.5}$$

$$\sum bc = s^2+4Rr+r^2, \tag{2.6}$$

$$\sum a^2 = 2s^2-8Rr-2r^2, \tag{2.7}$$

$$\sum a^3 = 2s^3-(12Rr+6r^2)s, \tag{2.8}$$

$$\sum a^4 = 2s^4-4(4R+3r)s^2r+2(4R+r)^2r^2, \tag{2.9}$$

$$\sum a^5 = 2s^5 - 20(R+r)s^3r + 10(2R+r)(4R+r)sr^2, \tag{2.10}$$

$$\sum a^6 = 2s^6 - 6(4R+5r)s^4r + 6(24R^2+24Rr+5r^2)s^2r^2 - 2(4R+r)^3r^3, \tag{2.11}$$

$$\begin{aligned}\sum a^7 = {} & 2s^7 - (28Rr+42r^2)s^5 + (224R^2r^2+280Rr^3+70r^4)s^3 \\ & - (448R^3r^3+448R^2r^4+140Rr^5+14r^6)s,\end{aligned} \tag{2.12}$$

$$\begin{aligned}\sum a^8 = {} & 2s^8 - (32Rr+56r^2)s^6 + (320R^2r^2+480Rr^3+140r^4)s^4 \\ & - (1024R^3r^3+1280R^2r^4+480Rr^5+56r^6)s^2 \\ & + 512R^4r^4+512R^3r^5+192R^2r^6+32Rr^7+2r^8,\end{aligned} \tag{2.13}$$

$$\sum b^3c^3 = s^6 - 3(4R-r)s^4r + 3s^2r^4 + (4R+r)^3r^3, \tag{2.14}$$

$$\begin{aligned}\sum b^4c^4 = {} & s^8 - 4(4R-r)s^6r + 4(4R+r)s^2r^5 + 2(16R^2-8Rr+3r^2)s^4r^2 \\ & + (4R+r)^4r^4.\end{aligned} \tag{2.15}$$

In fact, identities (2.5)-(2.9) are given in the monograph [17] and identities (2.10)-(2.15) have been proved by the author in [8] and [10]. From (2.4), using $\sum a = 2s$ and the above identities, simplifying and factoring we obtain identity (2.2).

Based on identity (2.2), we can give a proof of inequality (1.6) as follows:

Proof. If triangle ABC is equilateral, then it is easy to know that (1.6) becomes an equality. If triangle ABC is not equilateral. Using (2.6) and (2.7), we get

$$\sum (b-c)^2 = 2(s^2 - 12Rr - 3r^2). \tag{2.16}$$

Thus if triangle ABC is not equilateral then we have the strict inequality $s^2 - 12Rr - 3r^2 > 0$. Note that the left hand side of (2.2) is non-negative for non-obtuse triangle ABC, we immediately deduce

$$Rs^2 - 16R^2r + 3Rr^2 + 4r^3 \geq 0,$$

which proves inequality (1.6). Also, from identity (2.2) we easily conclude that the equality in (1.6) occurs if and only if the triangle is equilateral or right isosceles. □

Next, we give another proof of inequality (1.6):

Proof. Firstly, we prove the following identity:

$$\sum a(a-b)(a-c)(b+c-a)(b^2+c^2-a^2)=16rs^2\left(Rs^2-16R^2r+3Rr^2+4r^3\right). \tag{2.17}$$

Denote by Q_0 the left hand side of (2.17). Expanding and arranging gives

$$\begin{aligned} Q_0=&4\sum a^6-2\sum a\sum a^5-\sum a^2\sum a^4+6abc\sum a^3\\ &-abc\sum a\sum a^2+4\sum b^3c^3-6(abc)^2. \end{aligned} \tag{2.18}$$

Then using $\sum a=2s$, identities (2.5), (2.14) and (2.7)-(2.11), we easily obtain identity (2.17).

We now prove that for non-obtuse triangle ABC the following inequality holds:

$$\sum a(a-b)(a-c)(b+c-a)(b^2+c^2-a^2)\geq 0. \tag{2.19}$$

By symmetry, we may assume that $a\geq b\geq c$, then

$$a(a-b)(a-c)(b+c-a)(b^2+c^2-a^2)\geq 0.$$

Thus, it remains to show that

$$b(b-c)(b-a)(c+a-b)(c^2+a^2-b^2)+c(c-a)(c-b)(a+b-c)(a^2+b^2-c^2)\geq 0.$$

Since $a\geq b\geq c, c^2+a^2-b^2>0$ and $a^2+b^2-c^2>0$, we need to show

$$c(a-c)(a+b-c)(a^2+b^2-c^2)-b(a-b)(c+a-b)(c^2+a^2-b^2)\geq 0.$$

Note that

$$(a+b-c)c-(c+a-b)b=(b-c)(b+c-a)\geq 0.$$

It remains to show

$$(a-c)(a^2+b^2-c^2)-(a-b)(c^2+a^2-b^2)\geq 0,$$

which can be rewritten as

$$a^2(2a-b-c)+(b+c)(b-c)^2\geq 0.$$

This is true under the assumption. Thus inequality (2.19) is proved. Therefore, from identity (2.17) we deduce that inequality (1.6) holds and its equality condition is easy to be determined.

□

Remark 1. *Inequality* (2.19) *can be generalized to*

$$\sum a(b+c-a)(b^2+c^2-a^2)(a-b)^n(a-c)^n \geq 0, \tag{2.20}$$

where n is a natural number.

Remark 2. *In Remark 2 of the article [11], the author gave a simple proof of inequality* (1.6), *but the typographical errors were appeared here. Both expressions* $R^2 - Rr - r^2$ *in (2.10) and (2.11) should be corrected to* $R^2 - 2Rr - r^2$.

3. A Generalization of Inequality (1.6) and Its Applications

In this section, we shall give a generalization of inequality (1.6), which is actually an unified generalization of Walker's inequality (1.3), inequality (1.10) and other known inequalities. We shall use Ciamberlini's inequality (1.2) and the first inequality of the fundamental triangle inequality (1.1) to prove the following Theorem 1 and Theorem 2 given in the next section.

Theorem 1. *Let k be a real number such that* $0 \leq k \leq 3+\sqrt{2}$*, then for non-obtuse triangle ABC the following inequality holds:*

$$s^2 \geq (4-k)R^2 + 4kRr + 3(3-k)r^2 + \frac{2(2-k)r^3}{R}, \tag{3.1}$$

with equality if and only if the triangle ABC is equilateral or right isosceles.

Proof. We consider the following two cases to complete the proof.

Case 1. R and r satisfy $R^2 - 2Rr - r^2 \geq 0$.

In this case, according to Ciamberlini's inequality (1.3), to prove inequality (3.1) we need to show

$$(2R+r)^2 - (4-k)R^2 - 4kRr - 3(3-k)r^2 - \frac{2(2-k)r^3}{R} \geq 0.$$

Simplifying and factoring gives equivalent inequality

$$\frac{(R^2-2Rr-r^2)\,[(R-2r)k+4r]}{R} \geq 0,$$

which is clearly true since $k \geq 0$ and Euler's inequality $R \geq 2r$. Thus, inequality (3.1) is proved under Case 1. Moreover, one sees that equality in (3.1) holds only when $s = 2R + r$ and $R^2 - 2Rr - r^2 = 0$, and then we further conclude that the triangle must be right isosceles.

Case 2. R and r satisfy $R^2-2Rr-r^2<0$.

In this case, by the first inequality of the fundamental triangle inequality (1.1), to prove inequality (3.1) we need to show

$$2R^2+10Rr-r^2-2(R-2r)\sqrt{R^2-2Rr}$$
$$-(4-k)R^2-4kRr-3(3-k)r^2-\frac{2(2-k)r^3}{R}\geq 0.$$

Multiplying both side by R and factoring, it becomes the following equivalent inequality:

$$(R-2r)\left[(R^2-2Rr-r^2)k-2R^2+2r^2+6Rr-2R\sqrt{R^2-2Rr}\right]\geq 0.$$

Since we have Euler's inequality (1.7), it remains to show the following strict inequality:

$$(R^2-2Rr-r^2)k-2R^2+2r^2+6Rr-2R\sqrt{R^2-2Rr}>0. \tag{3.2}$$

If $k=0$, note that $-2R^2+2r^2+6Rr>0$ follows from the assumption, so we have to show

$$(-2R^2+2r^2+6Rr)^2-4R^2(R^2-2Rr)>0.$$

This is equivalent to

$$r(4R+r)(r^2+2Rr-R^2)>0,$$

which is true under the assumption.

If $0<0\leq k\leq 3+\sqrt{2}$, in this setting, to prove (3.2) we need to show

$$(3+\sqrt{2})(R^2-2Rr-r^2)-2R^2+2r^2+6Rr-2R\sqrt{R^2-2Rr}>0,$$

that is

$$\sqrt{2}R(R-2r)+R^2-(\sqrt{2}+1)r^2-2R\sqrt{R^2-2Rr}>0.$$

By Euler's inequality, we only need to prove

$$\left[\sqrt{2}R(R-2r)+R^2-(\sqrt{2}+1)r^2\right]^2-4R^2(R^2-2Rr)>0,$$

Expanding and factoring, we know that the above inequality is equivalent to

$$\frac{1}{7}(2\sqrt{2}-1)(7R+5r+3\sqrt{2}r)(R+\sqrt{2}r-r)(R-\sqrt{2}r-r)^2>0.$$

By the assumption $R^2 - 2Rr - r^2 < 0$, we know that $R \neq (\sqrt{2}+1)r$. Thus, the above inequality holds strictly. We thus proved that when $0 \leq k \leq 3+\sqrt{2}$ inequality (3.2) holds strictly. This completes the proof of inequality (3.1) under Case 2.

Combining the arguments of the above two cases, we proved that when $0 \leq k \leq 3+\sqrt{2}$ inequality (3.1) holds for all non-obtuse triangles.

We have known that the equality in the first inequality of (1.1) holds if and only if the triangle is isosceles with the vertex angle greater than or equal to $\pi/3$ (see [21]). Also, we have known that the equality in Euler's inequality $R \geq 2r$ holds if and only if the triangle is equilateral. Note that inequality (3.2) is strict. We thus conclude that the equality condition of (3.1) is the same as Euler's inequality under Case 2.

Finally, combining the equality conditions of (3.1) under Case 1 and Case 2, we determine that equality in (3.1) holds if and only if triangle ABC is equilateral or right isosceles. This completes the proof of Theorem 1. □

Remark 3. *In fact, it is easy to know that inequality* (3.1) *is equivalent to*

$$(R-2r)(R^2-2Rr-r^2)k + Rs^2 - 4R^3 - 9r^2R - 4r^3 \geq 0, \tag{3.3}$$

which could be proved by showing the previous inequality (1.9) *and the case $k = 3+\sqrt{2}$ of inequality* (3.3) *(We omit the details here).*

Remark 4. *For any non-acute triangle, we have Emmerich's inequality (see [17, p.251]):*

$$R \geq (1+\sqrt{2})r. \tag{3.4}$$

Thus, from the above proof of inequality (3.1) *under Case 2, one can see that if $k > 3+\sqrt{2}$ then inequality* (3.1) *holds for non-acute triangles.*

In what follows, we shall give some applications of Theorem 1.

In Theorem 1, taking $k = 2, 4, 0$ respectively, we immediately obtain the following corollary:

Corollary 3.1. *For any non-obtuse triangle ABC Walker's inequality* (1.3)*, inequalities* (1.6) *and* (1.9) *hold.*

In inequality (3.1) we can take $k = r/R$ and then it is easy to obtain

Corollary 3.2. *For any non-obtuse triangle ABC inequality* (1.10) *holds.*

We have seen that inequality (3.1) is an unified generalization of Walker's inequality (1.3), inequalities (1.6), (1.9) and (1.10).

Since $R \geq 2r$, we have

$$\frac{2r(2R+r)}{R^2} < 4.$$

Note that $4 < 3+\sqrt{2}$, we can take $k = 2r(2R+r)/R^2$ in Theorem 1 and then it is easy to obtain the following improvement of Walker's inequality:

Corollary 3.3. *Let ABC be a non-obtuse triangle, then*

$$s^2 \geq 2R^2 + 8Rr + 3r^2 + \frac{2(R-2r)(R^2-2Rr-r^2)^2}{R^3}. \tag{3.5}$$

If n is a positive number, then the previous inequality (1.8) is equivalent to

$$s^2 \geq \frac{4mR^3 - 4(m-1)R^2r - (7m+3)Rr^2 - 2(m+2)r^3}{(m+1)R - 2mr}. \tag{3.6}$$

To consider the relation between inequalities (3.1) and (3.6), we assume that

$$\frac{4mR^3 - 4(m-1)R^2r - (7m+3)Rr^2 - 2(m+2)r^3}{(m+1)R - 2mr}$$
$$= (4-k)R^2 + 4kRr + 3(3-k)r^2 + \frac{2(2-k)r^3}{R}. \tag{3.7}$$

Solving k gives

$$k = \frac{4(R-mr)}{(m+1)R - 2mr}.$$

By Euler's inequality, one sees that if $0 \leq m \leq 2$ then $k \geq 0$. Also, we note that

$$0 < \frac{4(R-r)}{(m+1)R-2mr} = 4 - \frac{4m(R-r)}{(m+1)R-2mr} < 4.$$

Therefore, from Theorem 1 we can obtain the following corollary:

Corollary 3.4. *When* $0 \leq m \leq 2$*, inequality* (3.6) *holds for any non-obtuse triangle ABC.*

Remark 5. *From equation* (3.7)*, one can solve m as follows:*

$$m = \frac{4-k}{R + k(R-2r)}.$$

Thus, from inequality (3.6) *we can deduce that if* $0 \leq k \leq 4$ *then inequality* (3.1) *holds for non-obtuse triangles.*

In Theorem 1, we take $k = 3/2$. Simplifying and factoring gives the following inequality (which was given in [11]):

Corollary 3.5. *Let ABC be a non-obtuse triangle, then*

$$s^2 \geq \frac{(5R+2r)(R+r)^2}{2R}. \tag{3.8}$$

In Theorem 1, we can take $k = 4(R-2r)/(5R-4r)$ since it is easy to show $0 \leq k < 4$. A simple calculation gives the following inequality (which was given in [11]):

Corollary 3.6. *Let ABC be a non-obtuse triangle, then*

$$s^2 \geq \frac{R(4R+r)^2}{5R-4r}. \tag{3.9}$$

Remark 6. *In any triangle the following identity holds:*

$$\sum a(a-b)(a-c)(b^2+c^2-a^2)(b+c-a)^{-1} = 4r\left[(5R-4r)s^2 - R(4R+r)^2\right], \tag{3.10}$$

which could be used to prove (3.9) *(we omit the details here).*

In inequality (3.1), we can take $k = 2(2R+r)/(R+r)$ and then it is easy to obtain the following new inequality:

Corollary 3.7. *Let ABC be a non-obtuse triangle, then*

$$s^2 \geq \frac{r(18R^2+5Rr-r^2)}{R+r}. \tag{3.11}$$

Remark 7. *In any triangle the following identity holds:*

$$\sum a(b+c)(a-b)(a-c)(b^2+c^2-a^2) = 16rs^2\left[(R+r)s^2 - r(18R^2+5Rr-r^2)\right], \tag{3.12}$$

which could be used to prove inequality (3.11).

Clearly, we can take

$$k = \frac{2(4R^2+4Rr+3r^2)}{2R^2+2Rr+3r^2}$$

in (3.1) and further obtain

Corollary 3.8. *Let ABC be a non-obtuse triangle, then*

$$s^2 \geq \frac{r(2R+r)(4R+r)^2}{2R^2+2Rr+3r^2}. \tag{3.13}$$

Remark 8. *In any triangle the following identity holds:*

$$\begin{aligned} &\sum(a-b)(a-c)(c+a-b)(a+b-c)(b^2+c^2-a^2)a^2 \\ &= 64sr^2\left[(2R^2+2Rr+3r^2)s^2-r(2R+r)(4R+r)^2\right], \end{aligned} \tag{3.14}$$

which could be used to prove inequality (3.13).

In inequality (3.1), we take

$$k = \frac{2r}{(\sqrt{2}+1)R+r},$$

then it is easy to obtain

Corollary 3.9. *Let ABC be a non-obtuse triangle, then*

$$s^2 \geq 4R^2 - 2(\sqrt{2}-1)Rr + (7+4\sqrt{2})r^2. \tag{3.15}$$

Note that

$$3+\sqrt{2}-\frac{4R+2(\sqrt{2}-1)r}{R+(\sqrt{2}-1)r} = \frac{(\sqrt{2}-1)(R+r+\sqrt{2}r)}{R-r+\sqrt{2}r} > 0.$$

We can take in (3.1) that

$$k = \frac{4R+2(\sqrt{2}-1)r}{R+(\sqrt{2}-1)r}.$$

Simplifying gives us the following corollary:

Corollary 3.10. *Let ABC be a non-obtuse triangle, then*

$$s^2 \geq 2(7+\sqrt{2})Rr - (1+4\sqrt{2})r^2. \tag{3.16}$$

Putting

$$k_0 = \frac{2(2R^2-4Rr+2\sqrt{2}Rr+r^2-r^2\sqrt{2})}{(R-r)(R-r+\sqrt{2}r)},$$

then we have $k_0 > 0$ by Euler's inequality. Also, we have

$$3+\sqrt{2}-k_0 = \frac{(\sqrt{2}-1)(R^2-2Rr+\sqrt{2}Rr-r^2-r^2\sqrt{2})}{(R-r)(R-r+\sqrt{2}r)} > 0.$$

Thus, we can take $k = k_0$ in Theorem 1 and the following inequality follows after the simplification:

Corollary 3.11. *Let ABC be a non-obtuse triangle, then*

$$s^2 \geq 16Rr - 5r^2 + \frac{2(2-\sqrt{2})(R-2r)r^2}{R-r}. \tag{3.17}$$

Remark 9. *The constant* $2(2-\sqrt{2})$ *in* (3.17) *is the best possible in the sense that can not be replaced by a larger constant. In addition, we remark that inequality* (3.17) *can not be obtained from inequality* (3.6) *and inequality* (4.1) *below.*

4. A Generalization of Inequality (1.10) and Its Applications

Motivated and inspired by Theorem 1, we considered generalizations of inequality (1.10) and found the following Theorem 2 in connection with Walker's inequality (1.3), inequalities (1.6), (1.9), (1.10) and other known inequalities.

Theorem 2. *Let k be a real number such that* $0 \leq k \leq 9+3\sqrt{3}$, *then for non-obtuse triangle ABC the following inequality holds:*

$$s^2 \geq 4R^2 + (4-k)Rr + (4k-7)r^2 + [(16-3k)R + 2(4-k)r]\,\frac{r^3}{R^2}. \tag{4.1}$$

Equality occurs only when the following three cases: (i) The triangle ABC is equilateral; (ii) The triangle ABC is right isosceles; (iii) The triangle ABC is isosceles with the ratio $1:1:(3-\sqrt{3})$ *of the three sides and* $k = 9+3\sqrt{3}$.

Proof. We consider the following two cases to complete the proof.

Case 1. R and r satisfy $R^2 - 2Rr - r^2 \geq 0$.

In this setting, by Ciamberlini's inequality (1.2) we need to show

$$(2R+r)^2 - 4R^2 - (4-k)Rr - (4k-7)r^2 - [(3k-16)R + 2(k-4)r]\,\frac{r^3}{R^2} \geq 0.$$

Simplifying and factoring gives the following equivalent inequality:

$$\frac{r(R^2-2Rr-r^2)\,[k(R-2r)+8r]}{R^2} \geq 0,$$

which is true under the assumption since we have $k \geq 0$ and Euler's inequality. Also, we see that equality in (4.1) occurs only when $s = 2R+r$ and $R^2 - 2Rr - r^2 = 0$. Then it is easily determined that the triangle must be right isosceles.

Case 2. R and r satisfy $R^2 - 2Rr - r^2 < 0$.

We set $d_0 = \sqrt{R^2 - 2Rr}$. According to the first inequality of the fundamental triangle inequality (1.1), for proving (4.1) we need to prove

$$\begin{aligned} &2R^2 + 10Rr - r^2 - 2(R-2r)d_0 \\ &- 4R^2 + (k-4)Rr - (4k-7)r^2 + [(3k-16)R + 2(k-4)r]\frac{r^3}{R^2} \geq 0. \end{aligned}$$

It is not difficult to know that this inequality is equivalent to

$$\frac{R-2r}{R^2}\left[r(R^2-2Rr-r^2)k - 2R^3 + 2R^2r + 10Rr^2 + 4r^3 - 2d_0R^2\right] \geq 0. \tag{4.2}$$

By Euler's inequality $R \geq 2r$, it remains to show that

$$r(R^2-2Rr-r^2)k - 2R^3 + 2R^2r + 10Rr^2 + 4r^3 - 2d_0R^2 \geq 0. \tag{4.3}$$

When $k = 0$, we shall show the above inequality is strict, i.e.

$$-2R^3 + 2R^2r + 10Rr^2 + 4r^3 - 2d_0R^2 > 0. \tag{4.4}$$

Under the assumption $R^2 - 2Rr - r^2 < 0$, it is clear that $-2R^3 + 2R^2r + 10Rr^2 + 4r^3 > 0$. So we only need to show

$$(-2R^3 + 2R^2r + 10Rr^2 + 4r^3)^2 - (2d_0R^2)^2 > 0.$$

Using $d_0 = \sqrt{R^2 - 2Rr}$ and simplifying, the above inequality becomes

$$4r^2(r^2 + 2Rr - R^2)(3R + 2r)^2 > 0,$$

which is clearly true under the assumption.

When $0 < k \leq 9 + 3\sqrt{3}$, to prove inequality (4.3) we need to prove

$$(9+3\sqrt{3})(R^2-2Rr-r^2)r - 2R^3 + 2R^2r + 10Rr^2 + 4r^3 - 2d_0R^2 \geq 0. \tag{4.5}$$

To do this, we first prove the following inequality:

$$(9+3\sqrt{3})(R^2-2Rr-r^2)r - 2R^3 + 2R^2r + 10Rr^2 + 4r^3 > 0, \tag{4.6}$$

which can be rewritten as

$$\begin{aligned} &\left[(7+3\sqrt{3}-4\sqrt{2})r - 2R\right]\left[(1+\sqrt{2})r - R\right]^2 \\ &+ 2\left[(3\sqrt{6}+5\sqrt{2}-2)R - (3\sqrt{6}+6\sqrt{3}+\sqrt{2}+5)r\right]r^2 \geq 0. \end{aligned} \tag{4.7}$$

We now note it follows from the assumption $R^2-2Rr-r^2<0$ that

$$r>(\sqrt{2}-1)R. \tag{4.8}$$

So we have

$$(7+3\sqrt{3}-4\sqrt{2})r-2R>(7+3\sqrt{3}-4\sqrt{2})(\sqrt{2}-1)R-2R\approx(1.708\cdots)R>0,$$

In addition, by Euler's inequality we have

$$\begin{aligned}&(3\sqrt{6}+5\sqrt{2}-2)R-(3\sqrt{6}+6\sqrt{3}+\sqrt{2}+5)r\\ &\geq 2(3\sqrt{6}+5\sqrt{2}-2)r-(3\sqrt{6}+6\sqrt{3}+\sqrt{2}+5)r\\ &\approx(0.684\cdots)r>0.\end{aligned}$$

Thus, inequality (4.7) and then (4.6) are proved.

We turn back to (4.5). To prove this inequality it remains to show that

$$\left[(9+3\sqrt{3})(R^2-2Rr-r^2)r-2R^3+2R^2r+10Rr^2+4r^3\right]-(2d_0R^2)^2\geq 0.$$

Using $d_0=\sqrt{R^2-2Rr}$, simplifying and factoring gives the following equivalent inequality:

$$-\frac{1}{9}(3+\sqrt{3})r(R^2-2Rr-r^2)(12R+3r+r\sqrt{3})\left[3R-(3+2\sqrt{3})r\right]^2\geq 0, \tag{4.9}$$

which is true under the assumption. Thus inequality (4.5) is proved and we have proved that inequality (4.3) holds for $0\leq k\leq 9+3\sqrt{3}$. Therefore, we finish the proof of inequality (4.1) under Case 2.

Combining the arguments of the above two cases, we proved that inequality (4.1) holds for all non-obtuse triangles.

We now discuss the equality condition of (4.1) under Case 2.

Clearly, equality in (4.9) occurs if and only if

$$3R-(3+2\sqrt{3})r=0. \tag{4.10}$$

With this and the following known identity

$$\frac{R}{r}=\frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}, \tag{4.11}$$

we get

$$\frac{2abc}{(b+c-a)(c+a-b)(a+b-c)}=\frac{3+2\sqrt{3}}{3}. \tag{4.12}$$

If we set $b=c=1$, then it is easy to obtain $a=3-\sqrt{3}$ or $a=\sqrt{3}-1$ from (4.12). Thus, it is seen that the equality in (4.3) holds only when $k=9+3\sqrt{3}$ and its sides are in the ratio $1:1:(3-\sqrt{3})$ or $1:1:(\sqrt{3}-1)$. Note that the equality conditions of the first inequality of (1.1) and Euler's inequality (we have stated in the proof of Theorem 1). Therefore, we conclude that the equality in (4.1) holds if and only if the triangle is equilateral or its sides are in the ratio $1:1:(3-\sqrt{3})$ and $k=9+3\sqrt{3}$ under Case 2.

Combining the equality conditions of (4.1) under Case 1 and Case 2, we deduce that the statement for the equality condition of (4.1) in Theorem 2 is true. This completes the proof of Theorem 2. □

In what follows, we shall discuss some applications of Theorem 2.

In Theorem 2, we take $k=4,5$ respectively to obtain

Corollary 4.1. *Both inequalities* (1.9) *and* (1.10) *hold for non-obtuse triangle ABC.*

We see that inequality (4.1) is not only generalization of inequality (1.10) but also inequality (1.9).

If taking $k=0,1,2,3$ in inequality (4.1) respectively, then we get the following corollary:

Corollary 4.2. *In the non-obtuse triangle ABC, the following inequalities hold:*

$$s^2 \ge \frac{4R^4+4R^3r-7R^2r^2+16Rr^3+8r^4}{R^2}, \tag{4.13}$$

$$s^2 \ge \frac{4R^4+3R^3r-3R^2r^2+13Rr^3+6r^4}{R^2}, \tag{4.14}$$

$$s^2 \ge \frac{4R^4+2R^3r+R^2r^2+10Rr^3+4r^4}{R^2}, \tag{4.15}$$

$$s^2 \ge \frac{4R^4+R^3r+5R^2r^2+7Rr^3+2r^4}{R^2}. \tag{4.16}$$

Note that $14<9+3\sqrt{3}$ and

$$14-\frac{4(4R+r)}{R}=\frac{2(R-2r)}{R}\ge 0.$$

In Theorem 2, we can take

$$t = \frac{4(4R+r)}{R},$$

then a simple calculation gives

$$s^2 \geq \frac{4R^5 - 8R^4 r + 37R^3 r^2 - 4R^2 r^3 - 28Rr^4 - 8r^5}{R^3}. \tag{4.17}$$

It is easy to know that this inequality is equivalent to the following inequality:

Corollary 4.3. *Let ABC be a non-obtuse triangle, then*

$$s^2 \geq 16Rr - 3r^2 - \frac{4r^3}{R} + \frac{4(R-2r)(R^2 - 2Rr - r^2)^2}{R^3}. \tag{4.18}$$

Clearly, Euler's inequality shows that inequality (4.18) is stronger than the previous inequality (1.6).

In fact, it is easy to know that inequality (4.1) is equivalent to

$$r(R-2r)(R^2 - 2Rr - r^2)k + R^2 s^2 - (4R^4 + 4R^3 r - 7R^2 r^2 + 16Rr^3 + 8r^4) \geq 0. \tag{4.19}$$

Note that $R \geq 2r$ and inequality (4.13), one sees that if $R^2 - 2Rr - r^2 \geq 0$ and $k > 0$ then inequality (4.19) and (4.1) hold. Therefore, by Theorem 2 we conclude that inequalities (4.19) and (4.1) hold for non-obtuse triangle ABC when $R^2 - 2Rr - r^2 < 0$ and $0 < k \leq 9 + \sqrt{3}$. Note that $9 + 3\sqrt{3} > 14$, we obtain the following conclusion:

Corollary 4.4. *Assume that $r^2 + 2Rr - R^2 > 0$ and $0 < k < 14$, then inequality (4.1) holds for all non-obtuse triangles.*

Next, we give several applications of Corollary 4.4.

Assume that $r^2 + 2Rr - R^2 > 0$, then $r > (\sqrt{2} - 1)R$ follows and it is easy to show

$$\frac{2(R+2r)}{r} < 14.$$

According to Corollary 4.4, one can take $k = 2(R+2r)/r$ in (4.1), then a simple calculation gives us the following conclusion:

Corollary 4.5. *Walker's inequality (1.3) holds for non-obtuse triangle ABC.*

Putting

$$k_1 = \frac{R^2 + 4Rr - 8r^2}{2r(R-r)},$$

Euler's inequality shows $k_1 > 0$. Again, when $r^2 + 2Rr - R^2 > 0$ we have

$$14 - k_1 = \frac{r^2 + 2Rr - R^2 + r(22R - 21r)}{2r(R-r)} > 0.$$

Thus, by Corollary 4.4, we can take $k = k_1$ in (4.1) and easily obtain the following inequality given in [11]:

Corollary 4.6. *Let ABC be a non-obtuse triangle, then*

$$s^2 \geq \frac{R(7R^2 - r^2)}{2(R-r)}. \tag{4.20}$$

Putting

$$k_2 = \frac{2(12R^3 + 36R^2r - 13Rr^2 - 10r^3)}{r(22R^2 - 4Rr - 5r^2)},$$

then we have $k_2 > 0$. When $r^2 + 2Rr - R^2 > 0$, by Euler's inequality $R \geq 2r$ we have

$$\begin{aligned} &14r(22R^2 - 4Rr - 5r^2) - 2(12R^3 + 36R^2r - 13Rr^2 - 10r^3) \\ &= -24R^3 + 236R^2r - 30Rr^2 - 50r^3 \\ &= 24R(r^2 + 2Rr - R^2) + 2r(94R^2 - 27Rr - 25r^2) > 0. \end{aligned}$$

Hence, if $r^2 + 2Rr - R^2 > 0$ then $0 < k_2 < 14$. This means that we can take $k = k_2$ in (4.1) and further obtain the following corollary:

Corollary 4.7. *Let ABC be a non-obtuse triangle, then*

$$s^2 \geq \frac{(2R+r)^2(4R+r)^2}{22R^2 - 4Rr - 5r^2}. \tag{4.21}$$

Now, we set

$$k_3 = \frac{R^3 + 6R^2r + 6Rr^2 - 4r^3}{r(R^2 + 2Rr - r^2)}.$$

Note that

$$\begin{aligned} &14r(R^2 + 2Rr - r^2) - (R^3 + 6R^2r + 6Rr^2 - 4r^3) \\ &= (r^2 + 2Rr - R^2)(R - 2r) + r(4R^2 + 25Rr - 8r^2). \end{aligned}$$

Euler's inequality shows that if $r^2+2Rr-R^2>0$ then $0<k_3<14$. Thus by Corollary 4.4 we can take $k=k_3$ in (4.1) and further calculations gives us the following corollary:

Corollary 4.8. *Let ABC be a non-obtuse triangle, then*

$$s^2 \geq \frac{(3R+r)(R+r)^3}{R^2+2Rr-r^2}. \tag{4.22}$$

5. Two Open Problems

Considering generalizations of the previous inequality (2.19), we propose the following problem:

Open Problem 1. *What conditions do the real numbers m and n satisfy? For any non-obtuse triangle ABC the following inequality holds:*

$$\sum(a-b)(a-c)(b^2+c^2-a^2)a^m(b+c-a)^n \geq 0. \tag{5.1}$$

Another similar problem is the following:

Open Problem 2. *What conditions do the real numbers m and n satisfy? For any non-obtuse triangle ABC the following inequality holds:*

$$\sum(a-b)(a-c)(b^2+c^2-a^2)a^m(b+c)^n \geq 0. \tag{5.2}$$

Conflict of Interests

The author declares that there is no conflict of interests.

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