Available online at http://scik.org
Adv. Inequal. Appl. 2024, 2024:2
https://doi.org/10.28919/aia/8810
ISSN: 2050-7461

FRACTAL AND TEMPERED-FRACTAL GRONWALL'S INEQUALITIES TYPE

AHMED M. A. EL-SAYED, WAGDY G. EL-SAYED, SHAYMAA I. NASIM*

Faculty of Science, Department of Mathematics and Computer Science, Alexandria University, Alexandria, Egypt

Abstract. In this paper, we generalize the main forms of Gronwall's differential and integral inequalities to the fractal and tempered-fractal differential and integral operators.

Keywords: fractal operator; tempered-fractal operator; Gronwall's inequality.

2010 AMS Subject Classification: 34A12, 34A34, 45G05.

1. Introduction

The main forms of Gronwall's differential and integral inequalities can be given by

Lemma 1. [6] *Let x be a real continuous function defined in $[0,T]$ satisfies the differential inequality*

$$x^{'} \leq a(t)\,x + b(t),\ t \in (0,T] \tag{1}$$

for some $a, b \in L_1[0,T]$, then x satisfies the pointwise estimate

$$x(t) \leq x(0)\, e^{\int_0^t a(\theta)d\theta} + \int_0^t b(s)\, e^{\int_s^t a(\theta)d\theta}\, ds,\ \forall t \in (0,T]. \tag{2}$$

Lemma 2. [14] *Let x, a and b be real continuous functions defined in $[0,T]$, $a(t) \geq 0$ for $t \in [0,T]$. We suppose that on $[0,T]$ we have the inequality*

*Corresponding author

E-mail address: shaymaa.nasim_pg@alexu.edu.eg

Received August 04, 2024

$$x(t) \le b(t) + \int_0^t a(s)\, x(s)\, ds, \tag{3}$$

then

$$x(t) \le b(t) + \int_0^t a(s)\, b(s)\, e^{\int_s^t a(\theta)d\theta}\, ds. \tag{4}$$

The fractal and tempered-fractal forms of Gronwall's inequality will be an extensions of (1), (3) that incorporate concepts from the theory of fractals and tempered operators.

Definition 1. *Let f be defined on $[a,b]$, $x \in [a,b]$, and $t \in (a,b)$, $t \neq x$, and $\beta \in (0,1)$. The fractal derivative is defined by*

$$D_\beta\, f(x) \;=\; \frac{df(x)}{dx^\beta} \;=\; \lim_{x\to t} \frac{f(x)-f(t)}{x^\beta - t^\beta},$$

if the limit exists.

- *Let f be differentiable, then*

  $$\frac{df(x)}{dx^\beta} \;=\; \lim_{x\to t} \frac{f(x)-f(t)}{x-t}\,\frac{x-t}{x^\beta - t^\beta} \;=\; \frac{df(x)}{dx}\,\frac{x^{1-\beta}}{\beta},$$

  and

  $$\lim_{\beta\to 1} \frac{df(x)}{dx^\beta} \;=\; \lim_{\beta\to 1} \frac{df(x)}{dx}\,\frac{x^{1-\beta}}{\beta} \;=\; \frac{df(x)}{dx}.$$

- *Let $\frac{f(x)}{dx^\beta}$ be exists, then*

  $$\lim_{x\to t}(\, f(x) - f(t)\,) \;=\; \lim_{x\to t} \frac{f(x)-f(t)}{x^\beta - t^\beta}\,(x^\beta - t^\beta) \;=\; \frac{f(x)}{dx^\beta}\, 0 \;=\; 0,$$

  which implies that f is continuous at x.

Example 1. Let $f(x) \;=\; \sqrt{x}$. Differentiating f with respect to x, we obtain

$$\frac{d\sqrt{x}}{dx} = \frac{1}{2\sqrt{x}}$$

which proves that f is not differentiable at zero. But the fractal derivative $D_\beta\; f(x)$ of order $\beta = 1/2$ exists and

$$\frac{d\sqrt{x}}{d\sqrt{x}} = \lim_{x\to t} \frac{\sqrt{x}-\sqrt{t}}{\sqrt{x}-\sqrt{t}} = 1.$$

Definition 2. *Let f be a bounded measurable function on $[a,b]$. Then the fractal integral of the function f can be defined as*

$$I_\beta\, f(x) \;=\; \beta \int_0^x s^{\beta-1}\, f(s)\, ds,$$

and

$$\lim_{\beta\to 1} I_\beta\, f(x) \;=\; \int_0^x f(s)\, ds,$$

then

$$\frac{d}{dx}\left(I_\beta\, f(x)\right) \;=\; \beta\, \frac{d}{dx} \int_0^x s^{\beta-1}\, f(s)\, ds \;=\; \beta\, x^{\beta-1}\, f(x) \text{ a.e., } x\in[a,b]$$

and

$$\frac{x^{1-\beta}}{\beta}\frac{d}{dx}\left(I_\beta\, f(x)\right) \;=\; \frac{d}{dx^\beta}\left(I_\beta\, f(x)\right) \;=\; f(x).$$

Definition 3. *Let f be defined on $[a,b]$, $x\in[a,b]$, $\lambda>0$, and $\beta\in(0,1)$. Then the tempered-fractal derivative of the function f is defined by*

$$ {}^{T}_{\lambda}D_\beta\, f(x) \;=\; e^{-\lambda x}\frac{d}{dx^\beta}(f(x)\, e^{\lambda x}). \tag{5}$$

Definition 4. *Let f be a bounded measurable function on $[a,b]$. Then the tempered-fractal integral of the function f can be defined as*

$$ {}^{T}_{\lambda}I_\beta\, f(x) \;=\; \beta \int_0^x s^{\beta-1}\, e^{-\lambda(x-s)}\, f(s)\, ds.$$

The paper organizes as follows. Firstly, we generalize the differential form of Gronwall's lemma to the fractal differential form

$$D_\beta\, x(t) \le a(t)\, x(t) + b(t),\ t\in(0,T]$$

and the tempered-fractal differential form

$$ {}^{T}_{\lambda}D_\beta\, x(t) \le a(t)\, x(t) + b(t),\ t\in(0,T].$$

Moreover, some corollaries and the non-linear case will be given.

Secondly, we generalize the integral form of Gronwall's lemma to the fractal integral form

$$x(t) \le b(t) + \int_0^t \beta\, s^{\beta-1}\, a(s)\, x(s)\, ds$$

and the tempered-fractal integral form

$$x(t) \leq b(t) + \int_0^t \beta\ s^{\beta-1}\ e^{-\lambda(t-s)}\ a(s)\ x(s)\ ds.$$

Additionally, some corollaries and the non-linear case will be given.

2. Fractal Differential Form

Lemma 3. *Let x be a real continuous function defined in $[0,T]$ satisfies the fractal differential inequality*

$$D_\beta\ x(t) \leq a(t)\ x(t) + b(t),\ t \in (0,T] \tag{6}$$

for some bounded measurable functions a and b, then x satisfies the pointwise estimate

$$x(t) \leq x(0)\ e^{\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta} + \beta \int_0^t s^{\beta-1}\ b(s)\ e^{\beta \int_s^t \theta^{\beta-1} a(\theta) d\theta}\ ds,\ \forall t \in (0,T]. \tag{7}$$

Proof. we have

$$\begin{aligned}
\frac{dx(t)}{dt^\beta} &\leq a(t)\ x(t) + b(t), \\
\frac{t^{1-\beta}}{\beta} \frac{dx(t)}{dt} &\leq a(t)\ x(t) + b(t), \\
\frac{dx(t)}{dt} &\leq \beta\ t^{\beta-1}\ a(t)\ x(t) + \beta\ t^{\beta-1}\ b(t), \\
\frac{dx(t)}{dt} - \beta\ t^{\beta-1}\ a(t)\ x(t) &\leq \beta\ t^{\beta-1}\ b(t).
\end{aligned}$$

Multiplying by $e^{-\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta}$, we obtain

$$\begin{aligned}
e^{-\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta}\ x'(t) - \beta\ t^{\beta-1}\ e^{-\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta}\ a(t)\ x(t) &\leq \beta\ t^{\beta-1}\ e^{-\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta}\ b(t), \\
\frac{d}{dt}\left(\ e^{-\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta}\ x(t)\ \right) &\leq \beta\ t^{\beta-1}\ e^{-\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta}\ b(t).
\end{aligned}$$

Integrating, we get

$$\begin{aligned}
e^{-\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta}\ x(t) - x(0) &\leq \beta \int_0^t s^{\beta-1}\ e^{-\beta \int_0^s \theta^{\beta-1} a(\theta) d\theta}\ b(s)\ ds, \\
e^{-\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta}\ x(t) &\leq x(0) + \beta \int_0^t s^{\beta-1}\ e^{-\beta \int_0^s \theta^{\beta-1} a(\theta) d\theta}\ b(s)\ ds, \\
x(t) &\leq e^{\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta}\ x(0) + \beta \int_0^t s^{\beta-1}\ e^{\beta \int_s^t \theta^{\beta-1} a(\theta) d\theta}\ b(s)\ ds.
\end{aligned}$$

□

Remark 1. *If $\beta = 1$ in Lemma 3, then the fractal differential inequality (6) will be a differential inequality as in Lemma 1.*

Now, we shall present some important special cases resulting from lemma 3

2.1. Special cases of fractal differential form.

(i) If a is constant, then the fractal differential inequality (6) will be

$$D_\beta\, x(t) \leq a\, x(t) + b(t)$$

and by Lemma 3, we obtain

$$x(t) \leq x(0)\, e^{at^\beta} + \beta \int_0^t s^{\beta-1}\, b(s)\, e^{a(t^\beta - s^\beta)}\, ds.$$

(ii) If b is constant, then the fractal differential inequality (6) will be

$$D_\beta\, x(t) \leq a(t)\, x(t) + b$$

and this implies that

$$x(t) \leq x(0)\, e^{\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta} + \beta\, b \int_0^t s^{\beta-1}\, e^{\beta \int_s^t \theta^{\beta-1} a(\theta) d\theta}\, ds.$$

(iii) If $b = 0$, then the fractal differential inequality (6) will be

$$D_\beta\, x(t) \leq a(t)\, x(t)$$

and this implies that

$$x(t) \leq x(0)\, e^{\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta}.$$

(iv) If a and b are constants, then the fractal differential inequality (6) will be

$$D_\beta\, x(t) \leq a\, x(t) + b$$

and this implies that

$$x(t) \leq x(0)\, e^{at^\beta} + \frac{b}{a}\, (e^{at^\beta} - 1).$$

2.2. The non-linear fractal differential form.

Lemma 4. *Let $f : [0,T] \times R^+ \to R^+$ be measurable in $t \in [0,T]$ and continuous in $x \in R^+$, and there exist two bounded measurable functions a and b such that $f(t,x(t)) \leq a(t)\, x(t) + b(t)$. If the function f satisfies the non-linear fractal differential inequality*

$$D_\beta\, x(t) \leq f(t, x(t)),$$

then x satisfies the inequality (7).

Proof. By applying Lemma 3, we get the result. □

Lemma 5. *Let $f : [0,T] \times R^+ \to R^+$ be measurable in $t \in [0,T]$ and continuous in $x \in R^+$, and there exist a bounded measurable function a and a positive constant b such that $f(t,x(t)) \leq a(t) + b\, x(t)$. If the function f satisfies the non-linear fractal differential inequality*

$$D_\beta\, x(t) \leq f(t, D_\alpha\, x(t)),$$

then x satisfies the inequality

$$x(t) \leq x(0) + \int_0^t \frac{\beta\, s^{\beta-1}\, a(s)}{1 - b\, \frac{\beta}{\alpha}\, T^{\beta-\alpha}}\, ds.$$

Proof. let $\frac{dx(t)}{dt} = y(t)$, we get

$$x(t) = x(0) + \int_0^t y(s)\, ds \tag{8}$$

and

$$\begin{aligned}
\frac{dx(t)}{dt^\beta} &\leq f(t, \frac{dx(t)}{dt^\alpha}),\\
\frac{t^{1-\beta}}{\beta}\, y(t) &\leq f(t, \frac{t^{1-\alpha}}{\alpha}\, y(t)),\\
y(t) &\leq \beta\, t^{\beta-1}\, f(t, \frac{t^{1-\alpha}}{\alpha}\, y(t))\\
&\leq \beta\, t^{\beta-1}\, (\, a(t) + b\, \frac{t^{1-\alpha}}{\alpha}\, y(t)\,)\\
&\leq \beta\, t^{\beta-1}\, a(t) + b\, \frac{\beta}{\alpha}\, T^{\beta-\alpha}\, y(t).
\end{aligned}$$

Thus

$$\begin{aligned}(1-b\,\frac{\beta}{\alpha}\,T^{\beta-\alpha})\,y(t) &\leq \beta\,t^{\beta-1}\,a(t),\\ y(t) &\leq \frac{\beta\,t^{\beta-1}\,a(t)}{1-b\,\frac{\beta}{\alpha}\,T^{\beta-\alpha}},\end{aligned}$$

then, by substituting in (8), we get the result. □

3. TEMPERED-FRACTAL DIFFERENTIAL FORM

Lemma 6. *Let x be a real continuous function defined in $[0,T]$ satisfies the tempered-fractal differential inequality*

$$ {}_{\lambda}^{T}D_{\beta}\,x(t) \leq a(t)\,x(t)+b(t),\; t\in(0,T] \tag{9}$$

for some bounded measurable functions a and b, then x satisfies the pointwise estimate

$$x(t)\leq x(0)\,e^{-\lambda t+\beta\int_0^t\theta^{\beta-1}a(\theta)d\theta}+\beta\int_0^t s^{\beta-1}\,b(s)\,e^{-\lambda(t-s)+\beta\int_s^t\theta^{\beta-1}a(\theta)d\theta}\,ds,\;\forall t\in(0,T]. \tag{10}$$

Proof. We have

$$\begin{aligned} e^{-\lambda t}\,\frac{d}{dt^{\beta}}\,(e^{\lambda t}\,x(t)) &\leq a(t)\,x(t)+b(t),\\ e^{-\lambda t}\,\frac{t^{1-\beta}}{\beta}\frac{d}{dt}\,(e^{\lambda t}\,x(t)) &\leq a(t)\,x(t)+b(t),\\ e^{-\lambda t}\,\frac{t^{1-\beta}}{\beta}\,(e^{\lambda t}\,\frac{dx(t)}{dt}+\lambda\,e^{\lambda t}\,x(t)) &\leq a(t)\,x(t)+b(t),\\ \frac{dx(t)}{dt}+\lambda\,x(t) &\leq \beta\,t^{\beta-1}\,a(t)\,x(t)+\beta\,t^{\beta-1}\,b(t),\\ \frac{dx(t)}{dt} &\leq (\,\beta\,t^{\beta-1}\,a(t)-\lambda\,)\,x(t)+\beta\,t^{\beta-1}\,b(t).\end{aligned}$$

Now, by using Gronwall's lemma 1, we get

$$\begin{aligned} x(t) &\leq x(0)\,e^{\int_0^t(\,\beta\,\theta^{\beta-1}\,a(\theta)-\lambda\,)d\theta}+\int_0^t\beta\,s^{\beta-1}\,b(s)\,e^{\int_s^t(\,\beta\,\theta^{\beta-1}\,a(\theta)-\lambda\,)d\theta}\,ds,\\ &\leq x(0)\,e^{-\lambda t+\beta\int_0^t\theta^{\beta-1}a(\theta)d\theta}+\beta\int_0^t s^{\beta-1}\,b(s)\,e^{-\lambda(t-s)+\beta\int_s^t\theta^{\beta-1}a(\theta)d\theta}\,ds.\end{aligned}$$

□

Remark 2. *If $\lambda = 0$ in Lemma 6, then the tempered-fractal differential inequality (9) will be a fractal differential inequality as in Lemma 3.*

Now, we shall present some important special cases resulting from lemma 6.

3.1. Special cases of tempered-fractal differential form.

(i) If a is constant, then the tempered-fractal differential inequality (9) will be

$$ {}_{\lambda}^{T}D_{\beta}\, x(t) \le a\, x(t) + b(t) $$

and by Lemma 6, we obtain

$$ x(t) \le x(0)\, e^{-\lambda t + a t^{\beta}} + \beta \int_0^t s^{\beta-1}\, b(s)\, e^{-\lambda(t-s) + a(t^{\beta} - s^{\beta})}\, ds. $$

(ii) If b is constant, then the tempered-fractal differential inequality (9) will be

$$ {}_{\lambda}^{T}D_{\beta}\, x(t) \le a(t)\, x(t) + b $$

and this implies that

$$ x(t) \le x(0)\, e^{-\lambda t + \beta \int_0^t \theta^{\beta-1} a(\theta) d\theta} + \beta\, b \int_0^t s^{\beta-1}\, e^{-\lambda(t-s) + \beta \int_s^t \theta^{\beta-1} a(\theta) d\theta}\, ds. $$

(iii) If $b = 0$, then the tempered-fractal differential inequality (9) will be

$$ {}_{\lambda}^{T}D_{\beta}\, x(t) \le a(t)\, x(t) $$

and this implies that

$$ x(t) \le x(0)\, e^{-\lambda t + \beta \int_0^t \theta^{\beta-1} a(\theta) d\theta}. $$

(iv) If a and b are constants, then the tempered-fractal differential inequality (9) will be

$$ {}_{\lambda}^{T}D_{\beta}\, x(t) \le a\, x(t) + b $$

and this implies that

$$ x(t) \le x(0)\, e^{-\lambda t + a t^{\beta}} + \beta\, b \int_0^t s^{\beta-1}\, e^{-\lambda(t-s) + a(t^{\beta} - s^{\beta})}\, ds. $$

(v) Let x be a real continuous function defined in $[0,T]$ satisfies the tempered differential inequality

$$ {}_{\lambda}^{T}D\, x(t) \leq a(t)\, x(t) + b(t),\ t \in (0,T] $$

for some bounded measurable functions a and b, then x satisfies

$$ x(t) \leq x(0)\, e^{-\lambda t + \int_0^t a(\theta)d\theta} + \int_0^t b(s)\, e^{-\lambda(t-s) + \int_s^t a(\theta)d\theta}\, ds,\ \forall t \in (0,T]. $$

3.2. The non-linear tempered-fractal differential form.

Lemma 7. *Let $f : [0,T] \times R^+ \to R^+$ be measurable in $t \in [0,T]$ and continuous in $x \in R^+$, and there exist two bounded measurable functions a and b such that $f(t,x(t)) \leq a(t)\, x(t) + b(t)$. If the function f satisfies the non-linear tempered-fractal differential inequality*

$$ {}_{\lambda}^{T}D_{\beta}\, x(t) \leq f(t,x(t)), $$

then x satisfies the inequality (10).

Proof. By applying Lemma 6, we get the result. □

4. FRACTAL INTEGRAL FORM

Lemma 8. *Let x, a and b be real continuous functions defined in $[0,T]$, $a(t) \geq 0$ for $t \in [0,T]$. We suppose that on $[0,T]$ we have the fractal integral inequality*

$$ x(t) \leq b(t) + \int_0^t \beta\, s^{\beta-1}\, a(s)\, x(s)\, ds, \tag{11} $$

then

$$ x(t) \leq b(t) + \int_0^t \beta\, s^{\beta-1}\, a(s)\, b(s)\, e^{\beta \int_s^t \theta^{\beta-1} a(\theta)d\theta}\, ds. \tag{12} $$

Proof. Let us consider the function

$$ H(t) = \int_0^t \beta\, s^{\beta-1}\, a(s)\, x(s)\, ds,\ t \in [0,T] $$

then we have $H(0)=0$, and

$$\begin{aligned} H^{'}(t) &= \beta\ t^{\beta-1}\ a(t)\ x(t) \\ &\leq \beta\ t^{\beta-1}\ a(t)\ (\ b(t)+\int_0^t \beta\ s^{\beta-1}\ a(s)\ x(s)\ ds\ ) \\ &\leq \beta\ t^{\beta-1}\ a(t)\ b(t)+\beta\ t^{\beta-1}\ a(t)\ H(t),\ t\in[0,T]. \end{aligned}$$

Then

$$H^{'}(t)-\beta\ t^{\beta-1}\ a(t)\ H(t)\leq \beta\ t^{\beta-1}\ a(t)\ b(t).$$

Multiplying by $e^{-\int_0^t \beta\theta^{\beta-1}a(\theta)d\theta}>0$, we obtain

$$\begin{aligned} H^{'}(t)\ e^{-\int_0^t \beta\theta^{\beta-1}a(\theta)d\theta}-\beta\ t^{\beta-1}\ a(t)\ H(t)\ e^{-\int_0^t \beta\theta^{\beta-1}a(\theta)d\theta} &\leq \beta\ t^{\beta-1}\ a(t)\ b(t)e^{-\int_0^t \beta\theta^{\beta-1}a(\theta)d\theta}, \\ \frac{d}{dt}(H(t)\ e^{-\int_0^t \beta\theta^{\beta-1}a(\theta)d\theta}) &\leq \beta\ t^{\beta-1}\ a(t)\ b(t)e^{-\int_0^t \beta\theta^{\beta-1}a(\theta)d\theta}. \end{aligned}$$

Integrating, we get

$$\begin{aligned} H(t)\ e^{-\int_0^t \beta\theta^{\beta-1}a(\theta)d\theta}-H(0) &\leq \int_0^t \beta\ s^{\beta-1}\ a(s)\ b(s)\ e^{-\int_0^s \beta\theta^{\beta-1}a(\theta)d\theta}\ ds, \\ H(t) &\leq e^{\int_0^t \beta\theta^{\beta-1}a(\theta)d\theta}\int_0^t \beta\ s^{\beta-1}\ a(s)\ b(s)\ e^{-\int_0^s \beta\theta^{\beta-1}a(\theta)d\theta}\ ds \\ &\leq \int_0^t \beta\ s^{\beta-1}\ a(s)\ b(s)\ e^{\int_s^t \beta\theta^{\beta-1}a(\theta)d\theta}\ ds,\ t\in[0,T]. \end{aligned}$$

Since

$$x(t)\leq b(t)+H(t)\ \Rightarrow\ x(t)-b(t)\leq H(t).$$

Thus

$$\begin{aligned} x(t)-b(t) &\leq \int_0^t \beta\ s^{\beta-1}\ a(s)\ b(s)\ e^{\int_s^t \beta\theta^{\beta-1}a(\theta)d\theta}\ ds, \\ x(t) &\leq b(t)+\int_0^t \beta\ s^{\beta-1}\ a(s)\ b(s)\ e^{\int_s^t \beta\theta^{\beta-1}a(\theta)d\theta}\ ds. \end{aligned}$$

□

Remark 3. *If $\beta=1$ in Lemma 8, then the fractal integral inequality (11) will be the integral form of Gronwall's inequality as in Lemma 2.*

Now, we shall present some important corollaries resulting from lemma 8.

4.1. Some corollaries of the fractal integral form.

(i) If a is constant, then the fractal integral inequality (11) will be

$$x(t) \leq b(t) + a \int_0^t \beta\, s^{\beta-1}\, x(s)\, ds,$$

and by Lemma 8, we obtain

$$x(t) \leq b(t) + a \int_0^t \beta\, s^{\beta-1}\, b(s)\, e^{a(t^\beta - s^\beta)}\, ds.$$

(ii) If b is differentiable, then the fractal integral inequality (11) follows that

$$x(t) \leq b(0)\, e^{\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta} + \int_0^t b^{'}(s)\, e^{\beta \int_s^t \theta^{\beta-1} a(\theta) d\theta}\, ds.$$

Proof. We already know that the fractal integral inequality (11) follows that

$$\begin{aligned} x(t) &\leq b(t) + \int_0^t \beta\, s^{\beta-1}\, a(s)\, b(s)\, e^{\beta \int_s^t \theta^{\beta-1} a(\theta) d\theta}\, ds \\ &\leq b(t) + \int_0^t b(s)\, \frac{d}{ds}(-\, e^{\int_s^t \beta \theta^{\beta-1} a(\theta) d\theta})\, ds, \end{aligned}$$

then

$$\begin{aligned} x(t) &\leq b(t) - (\, b(s)\, e^{\int_s^t \beta \theta^{\beta-1} a(\theta) d\theta}\,)_{s=0}^{s=t} + \int_0^t b^{'}(s)\, e^{\int_s^t \beta \theta^{\beta-1} a(\theta) d\theta}\, ds \\ &\leq b(0)\, e^{\int_0^t \beta \theta^{\beta-1} a(\theta) d\theta} + \int_0^t b^{'}(s)\, e^{\int_s^t \beta \theta^{\beta-1} a(\theta) d\theta}\, ds. \end{aligned}$$

□

(iii) If b is constant, then the fractal integral inequality (11) will be

$$x(t) \leq b + \int_0^t \beta\, s^{\beta-1}\, a(s)\, x(s)\, ds,$$

and it follows that

$$x(t) \leq b\, e^{\beta \int_0^t \theta^{\beta-1} a(\theta) d\theta}.$$

(iv) If $b = 0$, then the fractal integral inequality (11) will be

$$x(t) \leq \int_0^t \beta\, s^{\beta-1}\, a(s)\, x(s)\, ds,$$

and it follows that

$$x(t) \leq 0.$$

(v) If a and b are constants, then the fractal integral inequality (11) will be

$$x(t) \leq b + a \int_0^t \beta\ s^{\beta-1}\ x(s)\ ds,$$

and it follows that

$$x(t) \leq b\ e^{at^\beta}.$$

4.2. The non-linear fractal integral form.

Lemma 9. *Let $f : [0,T] \times R^+ \to R^+$ be measurable in $t \in [0,T]$ and continuous in $x \in R^+$, and there exist two bounded measurable functions a and c such that $f(t,x(t)) \leq a(t)\ x(t) + c(t)$. If the function f satisfies the non-linear fractal integral inequality*

$$x(t) \leq b(t) + \int_0^t \beta\ s^{\beta-1}\ f(s,x(s))\ ds,$$

then x satisfies the inequality

$$x(t) \leq B(t) + \int_0^t \beta\ s^{\beta-1}\ a(s)\ B(s)\ e^{\beta \int_s^t \theta^{\beta-1}\ a(\theta)\ d\theta}\ ds,$$

where $B(t) = b(t) + \beta \int_0^t s^{\beta-1}\ c(s)\ ds$.

Proof.

$$\begin{aligned} x(t) &\leq b(t) + \int_0^t \beta\ s^{\beta-1}\ f(s,x(s))\ ds \\ &\leq b(t) + \int_0^t \beta\ s^{\beta-1}\ (a(s)\ x(s) + c(s))\ ds \\ &= b(t) + \int_0^t \beta\ s^{\beta-1}\ c(s)\ ds + \int_0^t \beta\ s^{\beta-1}\ a(s)\ x(s)\ ds \\ &= B(t) + \int_0^t \beta\ s^{\beta-1}\ a(s)\ x(s)\ ds. \end{aligned}$$

By applying Lemma 8, we get the result. □

5. Tempered-Fractal Integral Form

Lemma 10. *Let x, a and b be real continuous functions defined in $[0,T]$, $a(t) \geq 0$ for $t \in [0,T]$. We suppose that on $[0,T]$ we have the tempered-fractal integral inequality*

$$x(t) \leq b(t) + \int_0^t \beta\ s^{\beta-1}\ e^{-\lambda(t-s)}\ a(s)\ x(s)\ ds, \tag{13}$$

then x satisfies

$$x(t) \leq b(t) + \int_0^t \beta\, s^{\beta-1}\, e^{-\lambda(t-s)}\, a(s)\, b(s)\, e^{\beta \int_s^t \theta^{\beta-1} a(\theta) d\theta}\, ds. \tag{14}$$

Remark 4. *If $\lambda = 0$ in Lemma 10, then the tempered-fractal integral inequality (13) will be a fractal integral inequality as in Lemma 8.*

Now, we shall present some important corollaries resulting from lemma 10

5.1. Some corollaries of the tempered-fractal integral form.

(i) If a is constant, then the tempered-fractal integral inequality (13) will be

$$x(t) \leq b(t) + a \int_0^t \beta\, s^{\beta-1}\, e^{-\lambda(t-s)}\, x(s)\, ds,$$

and by Lemma 10, we obtain

$$x(t) \leq b(t) + a \int_0^t \beta\, s^{\beta-1}\, e^{-\lambda(t-s)}\, b(s)\, e^{a(t^\beta - s^\beta)}\, ds.$$

(ii) If b is differentiable, then the tempered-fractal integral inequality (13) follows that

$$x(t) \leq b(0)\, e^{-\lambda t + \beta \int_0^t \theta^{\beta-1} a(\theta) d\theta} + \int_0^t (b'(s) + \lambda\, b(s))\, e^{-\lambda(t-s) + \beta \int_s^t \theta^{\beta-1} a(\theta) d\theta}\, ds.$$

Proof. We already know that the tempered-fractal integral inequality (13) follows that

$$\begin{aligned} x(t) &\leq b(t) + \int_0^t \beta\, s^{\beta-1}\, e^{-\lambda(t-s)}\, a(s)\, b(s)\, e^{\beta \int_s^t \theta^{\beta-1} a(\theta) d\theta}\, ds \\ &\leq b(t) + \int_0^t e^{-\lambda(t-s)}\, b(s)\, \frac{d}{ds}(-\, e^{\int_s^t \beta \theta^{\beta-1} a(\theta) d\theta})\, ds, \end{aligned}$$

then we get

$$\begin{aligned} x(t) &\leq b(t) - (\, b(s)\, e^{-\lambda(t-s)}\, e^{\int_s^t \beta \theta^{\beta-1} a(\theta) d\theta}\,)_{s=0}^{s=t} \\ &\quad + \int_0^t (b'(s) + \lambda\, b(s))\, e^{-\lambda(t-s)}\, e^{\int_s^t \beta \theta^{\beta-1} a(\theta) d\theta}\, ds \\ &\leq b(0)\, e^{-\lambda t}\, e^{\int_0^t \beta \theta^{\beta-1} a(\theta) d\theta} + \int_0^t (b'(s) + \lambda\, b(s))\, e^{-\lambda(t-s)}\, e^{\int_s^t \beta \theta^{\beta-1} a(\theta) d\theta}\, ds. \end{aligned}$$

□

(iii) If b is constant, then the tempered-fractal integral inequality (13) will be

$$x(t) \leq b + \int_0^t \beta\ s^{\beta-1}\ e^{-\lambda(t-s)}\ a(s)\ x(s)\ ds,$$

and it follows that

$$x(t) \leq b\ e^{-\lambda t + \beta \int_0^t \theta^{\beta-1} a(\theta) d\theta} + \lambda\ b \int_0^t e^{-\lambda(t-s) + \beta \int_s^t \theta^{\beta-1} a(\theta) d\theta}\ ds.$$

(iv) If $b = 0$, then the tempered-fractal integral inequality (13) will be

$$x(t) \leq \int_0^t \beta\ s^{\beta-1}\ e^{-\lambda(t-s)}\ a(s)\ x(s)\ ds,$$

and it follows that

$$x(t) \leq 0.$$

(v) If a and b are constants, then the tempered-fractal integral inequality (13) will be

$$x(t) \leq b + a \int_0^t \beta\ s^{\beta-1}\ e^{-\lambda(t-s)}\ x(s)\ ds,$$

and it follows that

$$x(t) \leq b\ e^{-\lambda t + a t^\beta} + \lambda\ b \int_0^t e^{-\lambda(t-s) + a(t^\beta - t^s)}\ ds.$$

(vi) Let x, a and b be real continuous functions defined in $[0,T]$, $a(t) \geq 0$ for $t \in [0,T]$. We suppose that on $[0,T]$ we have the tempered integral inequality

$$x(t) \leq b(t) + \int_0^t\ e^{-\lambda(t-s)}\ a(s)\ x(s)\ ds,$$

then x satisfies

$$x(t) \leq b(t) + \int_0^t e^{-\lambda(t-s)}\ a(s)\ b(s)\ e^{\int_s^t a(\theta) d\theta}\ ds.$$

5.2. The non-linear tempered-fractal integral form.

Lemma 11. *Let $f : [0,T] \times R^+ \to R^+$ be measurable in $t \in [0,T]$ and continuous in $x \in R^+$, and there exist two bounded measurable functions a and c such that $f(t,x(t)) \leq a(t)\ x(t) + c(t)$. If the function f satisfies the non-linear tempered-fractal integral inequality*

$$x(t) \leq b(t) + \int_0^t \beta\ s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s,x(s))\ ds,$$

then x satisfies the inequality

$$x(t) \leq B(t) + \int_0^t \beta\ s^{\beta-1}\ e^{-\lambda(t-s)}\ a(s)\ B(s)\ e^{\beta \int_s^t \theta^{\beta-1}\ a(\theta)\ d\theta}\ ds,$$

where $B(t) = b(t) + \beta \int_0^t s^{\beta-1}\ e^{-\lambda(t-s)}\ c(s)\ ds.$

Proof.

$$\begin{aligned} x(t) &\leq b(t) + \int_0^t \beta\ s^{\beta-1}\ e^{-\lambda(t-s)}\ f(s,x(s))\ ds \\ &\leq b(t) + \int_0^t \beta\ s^{\beta-1}\ e^{-\lambda(t-s)}\ (a(s)\ x(s) + c(s))\ ds \\ &= b(t) + \int_0^t \beta\ s^{\beta-1}\ e^{-\lambda(t-s)}\ c(s)\ ds + \int_0^t \beta\ s^{\beta-1}\ e^{-\lambda(t-s)}\ a(s)\ x(s)\ ds \\ &= B(t) + \int_0^t \beta\ s^{\beta-1}\ e^{-\lambda(t-s)}\ a(s)\ x(s)\ ds. \end{aligned}$$

By applying Lemma 10, we obtain the result. □

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Deppman, E. Megías, R. Pasechnik, Fractal derivatives, fractional derivatives and q-deformed calculus, Entropy 25 (2023), 1008.

[2] A.M.A. El-Sayed, W.G. El-Sayed, S.I. Nasim, On the solvability of a delay tempered-fractal differential equation, J. Fract. Calc. Appl. 15 (2024), 1–14.

[3] A.M.A. El-Sayed, W.G. El-Sayed, S.I. Nasim, Solvability of an initial-value problem of non-linear implicit fractal differential equation, Alex. J. Sci. Technol. 1 (2024), 76–79.

[4] A. Parvate, A.D. Gangal, Fractal differential equations and fractal-time dynamical systems, Pramana 64(2005), 389–409.

[5] B.B. Mandelbrot, Fractal geometry: What is it, and what does it do? Proc. R. Soc. Lond. A. Math. Phys. Sci. 423 (1989), 3–16.

[6] E. Emmrich, Discrete versions of Gronwall's lemma and their application to the numerical analysis of parabolic problems, Techn. Univ. 1999.

[7] H.O. Peitgen, H. Jurgens, D. Saupe, Chaos and fractals, Springer, New York, 2004.

[8] J.H. He, A new fractal derivation, Therm. Sci. 15 (2011), S145–S147.

[9] L.E. Elsgolts, Differential equations and the calculus of variations, Mir Publishers, 1970.

[10] M.F. Barnsley, R.L. Devaney, B.B. Mandelbrot, et al. The science of fractal images, Springer, New York, 1988.

[11] N. Dunford, J.T. Schwartz, Linear operators, Part I: General theory, Interscience, New York, 1957.

[12] R.P. Agarwal, D. O'Regan, An introduction to ordinary differential equations, Springer, New York, 2008.

[13] S. Pilipovic, The a-tempered derivative and some spaces of exponential distributions, Publ. Inst. Math. Nouv. Ser. 34 (1983), 183–192.

[14] S.S. Dragomir, Some Gronwall type inequalities and applications, Science Direct Working Paper, (2003).

[15] W.H. Steeb, Problems and solutions, World Scientific, 2016.