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# SCHUR CONVEXITY OF RELATED FUNCTION FOR SIERPINSKI'S INEQUALITY AND ITS APPLICATION

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**Abstract.** In this paper, we using the theory of majorization discuss the Schur convexity about related function of Sierpinski's inequality, the Sierpinski's inequality is generalized and some applications are established.

**Keywords:** Sierpinski's inequality; Schur convexity; majorization.

**2010 AMS Subject Classification:** 26E60, 26D15, 26A51, 34K38.

## 1. Introduction

Throughout the paper we assume that the set of $n$-dimensional row vector on the real number field by $\mathbb{R}^n$.

$$\mathbb{R}^n_+ = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, i = 1, \dots, n\},$$

$$\mathbb{R}^n_{++} = \{x = (x_1, \dots, x_n) \in \mathbb{R}^n : x_i > 0, i = 1, \dots, n\},$$

In particular, $\mathbb{R}^1$, $\mathbb{R}^1_+$ and $\mathbb{R}^1_{++}$ denoted by $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{R}_{++}$ respectively.

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In this paper, for $x \in \mathbb{R}^n_{++}$, we defined

$$A_n(x) = \frac{1}{n}\sum_{i=1}^{n} x_i$$

is arithmetic mean of $n$ variables.

$$G_n(x) = \prod_{i=1}^{n} x_i^{\frac{1}{n}}$$

is geometric mean of $n$ variables.

$$H_n(x) = \frac{n}{\sum_{i=1}^{n} x_i^{-1}}$$

is harmonic mean of $n$ variables.

$$M_n^{[m]}(x) = \left(\frac{\sum_{i=1}^{n} x_i^m}{n}\right)^{\frac{1}{m}} \ (m \neq 0)$$

is $m$-order power mean of $n$ variables.

Sierpinski's inequality [1]: Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$. Then

$$\prod_{i=1}^{n} x_i \sum_{i=1}^{n} x_i^{-1} \leq \frac{1}{n^{n-2}}\left(\sum_{i=1}^{n} x_i\right)^{n-1} \tag{1}$$

There are many improvements and generalizations to the Sierpinski's inequality of the related arithmetic mean, geometric mean and harmonic mean (see[2],[3],[4],[5],[6]).

In recent years, majorization theory is used to study all kinds of means active, appeared a large number of research results (see[13]-[25]).

In this paper we using the majorization theory to study on the other hand, we discuss the Schur convexity of the correlation function: $L(x) = \prod_{i=1}^{n} x_i^{\alpha} \sum_{i=1}^{n} x_i^{\beta}$ for the Sierpinski's inequality and get some new results.

Our main result is as follows:

**Theorem 1.** *Let* $L(x) = \prod_{i=1}^{n} x_i^{\alpha} \sum_{i=1}^{n} x_i^{\beta}$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$.

$(i)$ *When* $\alpha > 0$*, if one of the following conditions is satisfied: (1)* $0 < \beta \leq \frac{1}{2}(1 + \sqrt{1+8\alpha})$*;(2)* $-1 \leq \beta \leq 1$ *and* $\alpha + \beta \geq 0$*, then* $L(x)$ *is Schur concave with* $x_1, \ldots, x_n$ *on* $\mathbb{R}^n_{++}$.

$(ii)$ *When* $\alpha < 0$*, if one of the following conditions is satisfied: (1)* $-1 \leq \beta < 0$*;(2)* $0 \leq \beta \leq \min(-\alpha, 1)$*;(3)* $\beta \geq 1$*, then* $L(x)$ *is Schur convex with* $x_1, \ldots, x_n$ *on* $\mathbb{R}^n_{++}$.

**Theorem 2.** *Let* $L(x)=\prod_{i=1}^{n}x_i^{\alpha}\sum_{i=1}^{n}x_i^{\beta}$, $x=(x_1,\ldots,x_n)\in\mathbb{R}_{++}^{n}$. *For any* $\alpha$ *and* $\beta$, *then* $L(x)$ *is Schur geometricall convex with* $x_1,\ldots,x_n$ *on* $\mathbb{R}_{++}^{n}$.

**Theorem 3.** *Let* $L(x)=\prod_{i=1}^{n}x_i^{\alpha}\sum_{i=1}^{n}x_i^{\beta}$, $x=(x_1,\ldots,x_n)\in\mathbb{R}_{++}^{n}$.

- $(i)$ *When* $\alpha\geq 0$, *if one of the following conditions is satisfied: (1)* $\beta\geq 0$*; (2)* $\beta\leq -1$*; (3)* $-1\leq\beta\leq 1$ *and* $\alpha+\beta\geq 0$, *then* $L(x)$ *is Schur harmonicall convex with* $x_1,\ldots,x_n$ *on* $\mathbb{R}_{++}^{n}$.
- $(ii)$ *When* $\alpha\leq 0$, *if one of the following conditions is satisfied: (1)* $-1\leq\beta\leq 0$*; (2)* $-1\leq\beta\leq 1$ *and* $\alpha+\beta\leq 0$*; (3)* $\beta\geq 1$ *and* $\alpha+\beta^2\leq 0$, *then* $L(x)$ *is Schur harmonicall concave with* $x_1,\ldots,x_n$ *on* $\mathbb{R}_{++}^{n}$.

## 2. Preliminaries

We introduce some definitions and lemmas, which will be used in the proofs of the main results in subsequent sections.

**Definition 1** ([7, 8])**.** Let $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)\in\mathbb{R}^n$.

- $(i)$ $x$ is said to be majorized by $y$ (in symbols $x\prec y$) if $\sum_{i=1}^{k}x_{[i]}\leq\sum_{i=1}^{k}y_{[i]}$ for $k=1,2,\ldots,n-1$ and $\sum_{i=1}^{n}x_i=\sum_{i=1}^{n}y_i$, where $x_{[1]}\geq\cdots\geq x_{[n]}$ and $y_{[1]}\geq\cdots\geq y_{[n]}$ are rearrangements of $x$ and $y$ in a descending order.
- $(ii)$ $\Omega\subset\mathbb{R}^n$ is called a convex set if $(\alpha x_1+\beta y_1,\ldots,\alpha x_n+\beta y_n)\in\Omega$ for any $x$ and $y\in\Omega$, where $\alpha$ and $\beta\in[0,1]$ with $\alpha+\beta=1$.
- $(iii)$ let $\Omega\subset\mathbb{R}^n$, $\varphi$: $\Omega\to\mathbb{R}$ is said to be a Schur convex function on $\Omega$ if $x\prec y$ on $\Omega$ implies $\varphi(x)\leq\varphi(y)$. $\varphi$ is said to be a Schur concave function on $\Omega$ if and only if $-\varphi$ is Schur convex function.

**Definition 2** ([7])**.** Let $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)\in\mathbb{R}_{+}^{n}$.

- $(i)$ $\Omega\in\mathbb{R}_{+}^{n}$ is called a geometrically convex set if $(x_1^{\alpha}y_1^{\beta},\ldots,x_n^{\alpha}y_1^{\beta})\in\Omega$ for any $x$ and $y\in\Omega$, where $\alpha$ and $\beta\in[0,1]$ with $\alpha+\beta=1$.
- $(ii)$ let $\Omega\subset\mathbb{R}_{+}^{n}$, $\varphi:\Omega\to\mathbb{R}_{+}$ is said to be a Schur-geometrically convex function on if $(\log x_1,\ldots,\log x_n)\prec(\log y_1,\ldots,\log y_n)$ on $\Omega$ implies $\varphi(x)\leq\varphi(y)$. $\varphi$ is said to be a

Schur geometrically concave function on $\Omega$ if and only if $-\varphi$ is Schur geometrically convex function.

**Definition 3** ([7, 8])**.** Let $\Omega \subset \mathbb{R}_+^n$.

($i$) A set $\Omega$ is said to be a harmonically convex set if $\frac{xy}{\lambda x+(1-\lambda)y}$ for every $x, y \in \Omega$ and $\lambda \in [0,1]$, where $xy = \sum_{i=1}^n x_i y_i$ and $\frac{1}{x} = \left(\frac{1}{x_1}, \ldots, \frac{1}{x_n}\right)$.

($ii$) A function $\varphi : \Omega \to \mathbb{R}_+$ is said to be a Schur harmonically convex function on $\Omega$ if $x \prec y$ implies $\varphi(x) \leq \varphi(y)$ A function $\varphi$ is said to be a Schur harmonically concave function on $\Omega$ if and only if $-\varphi$ is a Schur harmonically convex function.

**Lemma 1** ([7, 8])**.** *Let $\Omega \subset \mathbb{R}^n$ is convex set, and has a nonempty interior set $\Omega^0$ . Let $\varphi : \Omega \to \mathbb{R}$ is continuous on $\Omega$ and differentiable in $\Omega^\circ$. Then $\varphi$ is the Schur convex (Schur concave) function, if and only if it is symmetric on $\Omega$ and if*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \geq 0 (or \leq 0; respectively) \tag{2}$$

*holds for any $x = (x_1, \cdots, x_n) \in \Omega^\circ$.*

*Remark* 1. Lemma1 equivalent to

$$\frac{\partial \varphi}{\partial x_i} \geq \frac{\partial \varphi}{\partial x_{i+1}} (or \leq 0; respectively), i = 1, 2, \ldots, n-1 \tag{3}$$

for all $x \in D \cap \Omega$. Where $D = \{x : x_1 \geq \cdots \geq x_n\}$.

**Lemma 2** ([9])**.** *Let $\Omega \subset \mathbb{R}_n$ is convex set, and has a nonempty interior set $\Omega^\circ$, let $\varphi : \Omega \to \mathbb{R}$ is continuous on $\Omega$ and differentiable in $\Omega^\circ$. Then $\varphi$ is the Schur geometrically convex(Schur geometrically concave)function, if and only if it is symmetric on $\Omega$ and if*

$$(\log x_1 - \log x_2)\left(x_1 \frac{\partial \phi}{\partial x_1} - x_2 \frac{\partial \phi}{\partial x_2}\right) \geq 0 (or \leq 0; respectively) \tag{4}$$

*holds for any $x = (x_1, \ldots, x_n) \in \Omega^\circ$.*

**Lemma 3** ([7, 9])**.** *Let $\Omega \subset \mathbb{R}_n$ be a symmetric harmonically convex set with a nonempty interior $\Omega^\circ$, let $\varphi : \Omega \to \mathbb{R}$ be continuous on $\Omega$ and differentiable on $\Omega$. Then $\varphi$ is a Schur harmonically convex (Schur harmonically concave) function if and only if $\varphi$ is symmetric on $\Omega$ and*

$$(x_1 - x_2)\left(x_1^2 \frac{\partial \phi}{\partial x_1} - x_2^2 \frac{\partial \phi}{\partial x_2}\right) \geq 0 (or \leq 0; respectively) \tag{5}$$

*holds for any* $x = (x_1, \dots, x_n) \in \Omega^\circ$.

**Lemma 4.** *Let* $f(z) = -\alpha z^{\beta+1} + (\alpha+\beta)z^\beta - (\alpha+\beta)z + \alpha (z \ge 1)$. *If* $\alpha > 0$, $0 < \beta \le \frac{1}{2}(1 + \sqrt{1+8\alpha})$, *then* $f(z) \le 0$.

*Proof.* By computing, we have

$$f(1) = -\alpha + (\alpha+\beta) - (\alpha+\beta) + \alpha = 0,$$

$$f'(z) = -\alpha(\beta+1)z^\beta + (\alpha+\beta)\beta z^{\beta-1} - (\alpha+\beta),$$

$$f'(1) = -\alpha(\beta+1) + (\alpha+\beta)\beta - (\alpha+\beta) = \beta^2 - \beta - 2\alpha,$$

$$\begin{aligned} f''(z) &= -\alpha\beta(\beta+1)z^{\beta-1} + (\alpha+\beta)\beta(\beta-1)z^(\beta-2) \\ &= z^{\beta-2}m(z), \end{aligned}$$

where

$$m(z) = -\alpha\beta(\beta+1)z + (\alpha+\beta)\beta(\beta-1).$$

When $0 < \beta \le \frac{1}{2}(1+\sqrt{1+8\alpha})$, we have $\beta^2 - \beta - 2\alpha \le 0$, so, $m(1) = -\alpha\beta(\beta+1)\beta(\beta-1) = \beta(\beta^2 - \beta - 2\alpha) \le 0$. And $m'(z) = -\alpha\beta(\beta+1) \le 0$, so, $m(z) \le 0$, therefore $f''(z) \le 0$. And $f'(1) = \beta^2 - \beta - 2\alpha \le 0$, we have $f'(z) \le 0$, and $f(1) = 0$, so, $f(z) \le 0$.

The proof of Lemma 4 is complete. □

**Lemma 5.** *Let* $g(z) = (\alpha+\beta)z^{\beta+1} - \alpha z^\beta + \alpha z - \alpha - \beta (z \ge 1)$. *If* $\alpha \le 0$, $\beta \ge 1$, *and* $\alpha + \beta^2 \le 0$, *then* $g(z) \le 0$.

*Proof.* By computing, we have

$$g(1) = (\alpha+\beta) - \alpha + \alpha - \alpha - \beta = 0,$$

$$g'(z) = (\alpha+\beta)(\beta+1)z^\beta - \alpha\beta z^{\beta-1} + \alpha,$$

$$g'(1) = (\alpha+\beta)(\beta+1) - \alpha\beta + \alpha = \beta^2 + \beta + 2\alpha,$$

$$\begin{aligned} g''(z) &= (\alpha+\beta)(\beta+1)\beta z^{\beta-1} - \alpha\beta(\beta-1)z^{\beta-2} \\ &= z^{\beta-2}h(z), \end{aligned}$$

where

$$h(z) = (\alpha+\beta)(\beta+1)\beta z - \alpha\beta(\beta-1).$$

$$h(1) = (\alpha+\beta)(\beta+1)\beta - \alpha\beta(\beta-1) = \beta(\beta^2+\beta+2\alpha),$$

$$h'(z) = (\alpha+\beta)(\beta+1)\beta.$$

If $\alpha \le 0$, $\beta \ge 1$ and $\alpha+\beta^2 \le 0$, then

$$h'(z) = (\alpha+\beta^2)\beta + (\alpha+1)\beta^2 \le (\alpha+\beta^2)\beta + (\alpha+\beta^2)\beta^2 \le 0,$$

and

$$h(1) = \beta(\beta^2+\beta+2\alpha) \le 2\beta(\alpha+\beta^2) \le 0,$$

so, $h(z) \le 0$, we have $g''(z) \le 0$, and then $g'(z) \le g'(1) = \beta^2+\beta+2\alpha \le 0$. Therefore $g(z) \le g(1) = 0$.

The proof of Lemma 5 is complete. □

**Lemma 6** ([8]). *Let* $x = (x_1,\dots,x_n) \in \mathbb{R}^n_{++}$. *Then*

(*i*)

$$\Big(\underbrace{A_n(x),\cdots,A_n(x)}_{n}\Big) \prec (x_1,\cdots,x_n). \tag{6}$$

(*ii*)

$$\Big(\underbrace{\log G_n(x),\cdots,\log G_n(x)}_{n}\Big) \prec (\log x_1,\cdots,\log x_n). \tag{7}$$

(*iii*)

$$\Bigg(\underbrace{\frac{1}{(H_n(x))},\cdots,\frac{1}{(H_n(x))}}_{n}\Bigg) \prec \left(\frac{1}{x_1},\cdots,\frac{1}{x_n}\right). \tag{8}$$

**Lemma 7** ([1]). *If* $p > 0$, *then*

$$\frac{1}{p+1}n^{p+1} < \sum_{k=1}^{n} k^p < \frac{1}{p+1}[(n+1)^{p+1}-1]. \tag{9}$$

## 3. PROOF OF MAIN RESULTS

**Proof of Theorem 1.**

*Proof.* Let $L(x)=\prod_{i=1}^{n}x_i^{\alpha}\sum_{i=1}^{n}x_i^{\beta}$, then

$$\frac{\partial L}{\partial x_1}=\alpha x_1^{\alpha-1}x_2^{\alpha}\cdots x_n^{\alpha}\sum_{i=1}^{n}x_i^{\beta}+\beta x_1^{\beta-1}\prod_{i=1}^{n}x_i^{\alpha}$$
$$=\prod_{i=1}^{n}x_i^{\alpha}\left(\frac{\alpha}{x_1}\sum_{i=1}^{n}x_i^{\beta}+\beta x_1^{\beta-1}\right),$$

$$\frac{\partial L}{\partial x_2}=\prod_{i=1}^{n}x_i^{\alpha}\left(\frac{\alpha}{x_2}\sum_{i=1}^{n}x_i^{\beta}+\beta x_2^{\beta-1}\right).$$

It is easy to see $L(x)$ is symmetry with $x_1,\dots,x_n$, without loss of generality, we may assume that $x_1\geq x_2>0$, then $z:=\frac{x_1}{x_2}\geq 1$.

$(i)$ (1)If $\alpha>0$, $0<\beta<\frac{1}{2}(1+\sqrt{1+8\alpha})$, then by Lemma 4 we have

$$\Delta_1:=(x_1-x_2)\left(\frac{\partial L}{\partial x_1}-\frac{\partial L}{\partial x_2}\right)$$

$$=(x_1-x_2)\prod_{i=1}^{n}x_i^{\alpha}\left[\alpha\left(\frac{1}{x_1}-\frac{1}{x_2}\right)\sum_{i=1}^{n}x_i^{\beta}+\beta(x_1^{\beta-1}-x_2^{\beta-1})\right] \tag{10}$$

$$\leq(x_1-x_2)\prod_{i=1}^{n}x_i^{\alpha}\left[\alpha\left(\frac{1}{x_1}-\frac{1}{x_2}\right)(x_1^{\beta}+x_2^{\beta})+\beta(x_1^{\beta-1}-x_2^{\beta-1})\right]$$
$$=(x_1-x_2)\prod_{i=1}^{n}x_i^{\alpha}\left[(\alpha+\beta)(x_1^{\beta-1}-x_2^{\beta-1})+\alpha\left(\frac{x_2^{\beta}}{x_1}-\frac{x_1^{\beta}}{x_2}\right)\right]$$
$$=(x_1-x_2)\prod_{i=1}^{n}x_i^{\alpha}\left[(\alpha+\beta)x_2^{\beta-1}\left(\frac{x_1^{\beta-1}}{x_2^{\beta-1}}-1\right)+\alpha x_2^{\beta-1}\left(\frac{x_2}{x_1}-\frac{x_1^{\beta}}{x_2^{\beta}}\right)\right]$$
$$=(x_1-x_2)x_2^{\beta-1}\prod_{i}^{n}x_i^{\alpha}[(\alpha+\beta)(z^{\beta-1}-1)+\alpha(z^{-1}-z^{\beta})]$$
$$=(x_1-x_2)z^{-1}x_2^{\beta-1}\prod_{i}^{n}x_i^{\alpha}f(z)\leq 0.$$

(2)If $\alpha > 0$, $-1 \le \beta \le 1$ and $\alpha+\beta \ge 0$, then

$$\begin{aligned}\Delta_1 &\le (x_1-x_2)\prod_{i=1}^n x_i^\alpha \left[\alpha\left(\frac{1}{x_1}-\frac{1}{x_2}\right)(x_1^\beta+x_2^\beta)+\beta(x_1^{\beta-1}-x_2^{\beta-1})\right]\\ &=(x_1-x_2)\prod_{i=1}^n x_i^\alpha \left[(\alpha+\beta)(x_1^{\beta-1}-x_2^{\beta-1})+\alpha\left(\frac{x_2^\beta}{x_1}-\frac{x_1^\beta}{x_2}\right)\right]\\ &=(x_1-x_2)\prod_{i=1}^n x_i^\alpha \left[(\alpha+\beta)(x_1^{\beta-1}-x_2^{\beta-1})+\alpha\left(\frac{x_2^{\beta+1}-x_1^{\beta+1}}{x_1x_2}\right)\right]\le 0.\end{aligned}$$

By Lemma 1, when $\alpha > 0$, if one of the following conditions is satisfied: (1)$0<\beta<\frac{1}{2}(1+\sqrt{1+8\alpha})$; (2) $-1\le\beta\le 1$ and $\alpha+\beta\ge 0$, then $L(x)$ is Schur concave with $x_1,\dots,x_n$ on $\mathbb{R}^n_{++}$.

$(ii)$ If $\alpha<0$, $\beta\ge 1$, from the inequality (10) we have $\Delta_1 \ge 0$.

If $\alpha<0$, and $-1\le\beta<0$ or $0\le\beta\le\min(-\alpha,1)$, then

$$\begin{aligned}\Delta_1 &\ge (x_1-x_2)\prod_{i=1}^n x_i^\alpha \left[\alpha\left(\frac{1}{x_1}-\frac{1}{x_2}\right)(x_1^\beta+x_2^\beta)+\beta(x_1^{\beta-1}-x_2^{\beta-1})\right]\\ &=(x_1-x_2)\prod_{i=1}^n x_i^\alpha \left[(\alpha+\beta)(x_1^{\beta-1}-x_2^{\beta-1})+\alpha\left(\frac{x_2^{\beta+1}-x_1^{\beta+1}}{x_1x_2}\right)\right]\ge 0.\end{aligned}$$

By Lemma 1, $L(x)$ is Schur convex with $x_1,\dots,x_n$ on $\mathbb{R}^n_{++}$.

The proof of Theorem 1 is complete. □

**Proof of Theorem 2.**

*Proof.*

$$\begin{aligned}x_1\frac{\partial L}{\partial x_1} &= \alpha x_1^\alpha x_2^\alpha\cdots x_n^\alpha\sum_{i=1}^n x_i^\beta+\beta x_1^\beta\prod_{i=1}^n x_i^\alpha\\ &=\prod_{i=1}^n x_i^\alpha\left(\alpha\sum_{i=1}^n x_i^\beta+\beta x_1^\beta\right),\end{aligned}$$

$$x_2\frac{\partial L}{\partial x_2}=\prod_{i=1}^n x_i^\alpha\left(\alpha\sum_{i=1}^n x_i^\beta+\beta x_2^\beta\right).$$

$$\begin{aligned}\Delta_2 :=&(x_1-x_2)\left(x_1\frac{\partial L}{\partial x_1}-x_2\frac{\partial L}{\partial x_2}\right)\\ =&(x_1-x_2)\prod_{i=1}^n x_i^\alpha\beta(x_1^\beta-x_2^\beta).\end{aligned}$$

For any $\alpha$ , $\beta$, we have $\Delta_2 \geq 0$. By Lemma 2 we know $L(x)$ is Schur geometrically convex with $x_1,\dots,x_n$ on $\mathbb{R}^n_{++}$.

The proof of Theorem 2 is complete. □

**Proof of Theorem 3.**

*Proof.*

$$x_1^2\frac{\partial L}{\partial x_1} = \alpha x_1^{\alpha+1}x_2^{\alpha}\cdots x_n^{\alpha}\sum_{i=1}^n x_i^{\beta} + \beta x_1^{\beta+1}\prod_{i=1}^n x_i^{\alpha}$$
$$= \prod_{i=1}^n x_i^{\alpha}\left(\alpha x_1\sum_{i=1}^n x_i^{\beta} + \beta x_1^{\beta+1}\right),$$
$$x_2^2\frac{\partial L}{\partial x_2} = \prod_{i=1}^n x_i^{\alpha}\left(\alpha x_2\sum_{i=1}^n x_i^{\beta} + \beta x_2^{\beta+1}\right).$$

$$\Delta_3 := (x_1-x_2)\left(x_1^2\frac{\partial L}{\partial x_1} - x_2^2\frac{\partial L}{\partial x_2}\right)$$
$$= (x_1-x_2)\prod_{i=1}^n x_i^{\alpha}\left(\alpha\sum_{i=1}^n x_i^{\beta}(x_1-x_2) + \beta(x_1^{\beta+1}-x_2^{\beta+1})\right).$$

It is easy to see that for $\alpha \geq 0$ and $\beta \geq 0$, or $\alpha \geq 0$ and $\beta \leq -1$, we have $\Delta_3 \geq 0$. By Lemma 3 we know $L(x)$ is Schur harmonically convex with $x_1,\dots,x_n$ on $\mathbb{R}^n_{++}$.

For $\alpha \leq 0$ and $-1 \leq \beta \leq 0$, we have $\Delta_3 \leq 0$. By Lemma 3 we know that $L(x)$ is Schur harmonically concave with $x_1,\dots,x_n$ on $\mathbb{R}^n_{++}$.

For $\alpha \geq 0$, $-1 \leq \beta \leq 1$ and $\alpha+\beta \geq 0$, we have

$$\Delta_3 = (x_1-x_2)\left(x_1^2\frac{\partial L}{\partial x_1} - x_2^2\frac{\partial L}{\partial x_2}\right)$$
$$= (x_1-x_2)\prod_{i=1}^n x_i^{\alpha}\left(\alpha\sum_{i=1}^n x_i^{\beta}(x_1-x_2) + \beta(x_1^{\beta+1}-x_2^{\beta+1})\right)$$
$$\geq (x_1-x_2)\prod_{i=1}^n x_i^{\alpha}\left(\alpha\sum_{i=1}^2 x_i^{\beta}(x_1-x_2) + \beta(x_1^{\beta+1}-x_2^{\beta+1})\right)$$
$$= (x_1-x_2)\prod_{i=1}^n x_i^{\alpha}[\alpha(x_1^{\beta}+x_2^{\beta})(x_1-x_2) + \beta(x_1^{\beta+1}-x_2^{\beta+1})]$$
$$= (x_1-x_2)\prod_{i=1}^n x_i^{\alpha}[(\alpha+\beta)(x_1^{\beta+1}-x_2^{\beta+1}) - \alpha x_1x_2(x_1^{\beta-1}-x_2^{\beta-1}) \geq 0.$$

By Lemma 3, it follows that $L(x)$ is Schur harmonically convex with $x_1,\ldots,x_n$ on $\mathbb{R}^n_{++}$.

For $\alpha \le 0$, $-1 \le \beta \le 1$ and $\alpha+\beta \le 0$, we have

$$\begin{aligned}
\Delta_3 &= (x_1-x_2)\left(x_1^2\frac{\partial L}{\partial x_1}-x_2^2\frac{\partial L}{\partial x_2}\right)\\
&= (x_1-x_2)\prod_{i=1}^n x_i^\alpha\left(\alpha\sum_{i=1}^n x_i^\beta(x_1-x_2)+\beta(x_1^{\beta+1}-x_2^{\beta+1})\right)\\
&\le (x_1-x_2)\prod_{i=1}^n x_i^\alpha\left(\alpha\sum_{i=1}^2 x_i^\beta(x_1-x_2)+\beta(x_1^{\beta+1}-x_2^{\beta+1})\right)\\
&= (x_1-x_2)\prod_{i=1}^n x_i^\alpha[(\alpha+\beta)(x_1^{\beta+1}-x_2^{\beta+1})-\alpha x_1x_2(x_1^{\beta-1}-x_2^{\beta-1})\le 0.
\end{aligned}$$

By Lemma 3, it follows that $L(x)$ is Schur harmonically concave with $x_1,\ldots,x_n$ on $\mathbb{R}^n_{++}$.

For $\alpha \le 0$,$\beta \ge 1$ and $\alpha+\beta^2 \le 0$,let $z=\frac{x_1}{x_2}\ge 1$, then

$$\begin{aligned}
\Delta_3 &= (x_1-x_2)\left(x_1^2\frac{\partial L}{\partial x_1}-x_2^2\frac{\partial L}{\partial x_2}\right)\\
&\le (x_1-x_2)\prod_{i=1}^n x_i^\alpha\left(\alpha\sum_{i=1}^2 x_i^\beta(x_1-x_2)+\beta(x_1^{\beta+1}-x_2^{\beta+1})\right)\\
&= (x_1-x_2)x_i^{\beta+1}\prod_{i=1}^n x_i^\alpha[(\alpha+\beta)z^{\beta+1}-\alpha z^\beta-\alpha-\beta]\\
&= (x_1-x_2)x_2^{\beta+1}\prod_{i=1}^n x_i^\alpha g(z).
\end{aligned}$$

From Lemma 5, we have $\Delta_3 \le 0$, and then by Lemma 3, it follows that $L(x)$ is Schur harmonic concave with $x_1,\ldots,x_n$ on $\mathbb{R}^n_{++}$.

The proof of Theorem 3 is complete. □

## 4. Applications

As an applications of Theorem 1, Theorem 2 and Theorem 3, we get the following corollaries.

**Corollary 1.** *Let* $x=(x_1,\ldots,x_n)\in\mathbb{R}^n_{++}$.

$(i)$ *When* $\alpha>0$*, if one of the following conditions is satisfied: (1)* $0<\beta\le\frac{1}{2}(1+\sqrt{1+8\alpha})$*; (2)* $-1\le\beta\le 1$ *and* $\alpha+\beta\ge 0$*, then*

$$\prod_{i=1}^n x_i^\alpha\sum_{i=1}^n x_i^\beta \le \frac{1}{n^{n\alpha+\beta-1}}\left(\sum_{1=1}^n x_i\right)^{n\alpha+\beta}. \tag{11}$$

$(ii)$ *When* $\alpha < 0$, *if one of the following conditions is satisfied: (1)* $-1 \leq \beta < 0$*; (2)* $0 \leq \beta \leq \min(-\alpha, 1)$*; (3)* $\beta \geq 1$, *then inequality (11) is reversed.*

*Proof.* $(i)$ For $\alpha > 0$, if one of the following conditions is satisfied: (1) $0 < \beta \leq \frac{1}{2}(1+\sqrt{1+8\alpha})$; (2) $-1 \leq \beta \leq 1$ and $\alpha + \beta \geq 0$, by

$$\left(\underbrace{A_n(x), A_n(x), \cdots, A_n(x)}_{n}\right) \prec (x_1, x_2, \cdots, x_n).$$

Theorem 1 and Definition 1, we have

$$\begin{aligned}
\prod_{i=1}^{n} x_i^{\alpha} \sum_{i=1}^{n} x_i^{\beta} &\leq [A_n(x)]^{n\alpha} n [A_n(x)]^{\beta} \\
&= n \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{n\alpha} \left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{\beta} \\
&= \frac{1}{n^{n\alpha+\beta-1}} \left(\sum_{i=1}^{n} x_i\right)^{n\alpha+\beta}.
\end{aligned}$$

Similar argument leads to the proof of the proposition (ii).

The proof of Corollary 1 is complete. □

*Remark* 2. (i) Taking $\alpha = 1$, $\beta = -1$, the inequality (11) is reduces to Sierpinski's inequality.

(ii) Taking $\alpha = 1$, $\beta = 1$, the inequality (11) is reduces to $GM - AM$ inequality.

(iii) Let $\gamma = -\alpha > 0$, $\beta \geq 1$, $\sum_{i=1}^{n} x_i = 1$, by Corollary 1(ii) we have

$$\prod_{i=1}^{n} x_i^{\gamma} \leq n^{\beta - n\gamma - 1} \sum_{i=1}^{n} x_i^{\beta}.$$

**Corollary 2.** *Let* $x_i \in \mathbb{R}_{++}$, $i = 1, 2, \ldots, n$. *If* $0 < m \leq 3$, *then*

$$nG_n^{m(n+1)}(x) \leq \prod_{i=1}^{n} x_i^m \sum_{i=1}^{n} x_i^m \leq nA_n^{m(n+1)}(x). \tag{12}$$

*Proof.* When $m>0$, $m\le \frac{1}{2}(1+\sqrt{1+8m}) \Leftrightarrow m>0,(2m-1)^2\le 1+8m \Leftrightarrow 0<m\le 3$. Let $\alpha=\beta=m$. By Corollary 1, we have

$$\begin{aligned}\prod_{i=1}^{n} x_i^m \sum_{i=1}^{n} x_i^m &\le \frac{1}{n^{nm+m-1}}\left(\sum_{i=1}^{n} x_i\right)^{nm+m}\\ &= n\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{mn+m}\\ &= nA_m^{m(n+1)}(x).\end{aligned}$$

Let $L_1(x)=(x_1\cdots x_n)^m(x_1^m+\cdots+x_n^m)$. By Theorem 2 $L(x)$ is Schur geometrically convex with $x_1,\ldots,x_n$ on $\mathbb{R}_{++}^n$.

By

$$\left(\underbrace{\log G_n(x),\cdots,\log G_n(x)}_{n}\right) \prec (\log x_1,\cdots,\log x_n),$$

and Definition 2, we have $L_1(x)\ge L_1(G_n(x))$, that is

$$(x_1\cdots x_n)^m(x_1^m+\cdots+x_n^m)\ge nG_n^{m(n+1)}(x).$$

The proof of Corollary 2 is complete. □

**Corollary 3.** *Faǐzlev's inequality* (*see*[10]) *Let* $x_i\in\mathbb{R}_{++}$, $i=1,2,\ldots,n$. *Then*

$$(x_1\cdots x_n)(x_1+\cdots+x_n)\le x_1^{n+1}+\cdots+x_n^{n+1}. \tag{13}$$

*Proof.* Taking $m=1$, by Corollary 2 and power mean inequality(see[10])

$$\left(\sum_{i=1}^{n} x_i\right)^p \le n^{p-1}\sum_{i=1}^{n} x_i^p, (p\ge 1),$$

it follows that

$$\begin{aligned}(x_1\cdots x_n)(x_1+\cdots+x_n) &\le n\left(\frac{\sum_{i=1}^{n} x_i}{n}\right)^{n+1}\\ &\le \frac{nn^{(n+1)-1}}{n^{n+1}}\sum_{i=1}^{n} x_i^{n+1}\\ &= \sum_{i=1}^{n} x_i^{n+1}.\end{aligned}$$

The proof of Corollary 3 is complete. □

By Corollary 2, it can easily be shown that the following Corollary.

**Corollary 4.** *Let $x_i \in \mathbb{R}_{++}$, $i = 1,2,\dots,n$. If $0 < m \le 3$ and $\sum_{i=1}^n x_i = n$, then*

$$(x_1^m \cdots x_n^m)(x_1^m + \cdots + x_n^m) \le n. \tag{14}$$

*Remark* 3. (i) Suppose $x, y \in \mathbb{R}_{++}$ and $x + y = 2$, proof

$$x^2y^2(x^2+y^2) \le 2.$$

It is an inequality question for the 2002 Irish Mathematical Olympiad(see[11]).

Suppose $x, y \in \mathbb{R}_{++}$ and $x + y = 2$, proof

$$x^3y^3(x^3+y^3) \le 2.$$

It is inequality question of the Indian Mathematical Olympiad (see[11])

Corollary 4 is generalization of this two inequalities questions.

(ii) If $m > 3$, inequality (14) does not necessarily hold. For example, let $x_1 = 1.1$, $x_2 = 0.9$, $m = 4$, though $x_1 + x_2 = 2$, but $x_1^4x_2^4(x_1^4+x_2^4) \approx 2.0367 > 2$.

By Corollary 2, we have the following Corollary.

**Corollary 5.** *Let $x_i \in \mathbb{R}_{++}$, $i = 1,2,\dots,n$ and $0 < m \le 3$, then*

$$G_n(x) \le [(G_n(x))^n M_n^{[m]}(x)]^{\frac{1}{n+1}} \le A_n(x). \tag{15}$$

From corollary 2, we can get the sharpen of Cauchy inequality $n! < \left(\frac{n+1}{2}\right)^n$(see[12]).

**Corollary 6.** *If $n \ge 4$, then*

$$n! \le \frac{1}{\sqrt[3]{2}}\left(1+\frac{1}{n}\right)\left(\frac{n+1}{2}\right)^n < \left(\frac{n+1}{2}\right)^n. \tag{16}$$

*Proof.* Let $m = 3$, $x_k = k(k = 1,\dots,3)$, by Corollary 2 and Lemma 7, we have

$$\prod_{k=1}^n k^3 \sum_{k=1}^n k^3 = (n!)^3 \sum_{k=1}^n k^3 \le n\left(\frac{n+1}{2}\right)^{3(n+1)}$$
$$\Rightarrow n! \le n^{\frac{1}{3}}\left(\frac{n+1}{2}\right)^{n+1} \frac{1}{(\sum_{k=1}^n k^3)^{\frac{1}{3}}} \le n^{\frac{1}{3}}\left(\frac{n+1}{2}\right)^{n+1} \frac{4^{\frac{1}{3}}}{n^{\frac{4}{3}}}$$
$$= \frac{n+1}{\sqrt[3]{2}n}\left(\frac{n+1}{2}\right)^n = \frac{1}{\sqrt[3]{2}}\left(1+\frac{1}{n}\right)\left(\frac{n+1}{2}\right)^n.$$

If $n \geq 4$, then

$$\frac{1}{\sqrt[3]{2}}\left(1+\frac{1}{n}\right) \leq \frac{1}{\sqrt[3]{2}}\left(1+\frac{1}{4}\right) \approx 0.9921 < 1.$$

□

The proof of Corollary 6 is complete.

**Corollary 7.** *Let* $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$.

($i$) *When* $\alpha \geq 0$, *if one of the following conditions is satisfied:* (1) $\beta \geq 0$;(2) $\beta \leq -1$; (3) $-1 \leq \beta \leq 1$ *and* $\alpha + \beta \geq 0$, *then*

$$\prod_{i=1}^{n} x_i^{\alpha} \sum_{i=1}^{n} x_i^{\beta} \geq n[H_n(x)]^{n\alpha+\beta}. \tag{17}$$

($ii$) *When* $\alpha \leq 0$, *if one of the following conditions is satisfied:*(1) $-1 \leq \beta \leq 0$; (2) $-1 \leq \beta \leq 1$ *and* $\alpha + \beta \leq 0$; (3) $\beta \geq 1$ *and* $\alpha + \beta^2 \leq 0$, *then inequality (17) reverse.*

*Proof.* By Theorem 3, Definition 3 and

$$\Bigg(\underbrace{\frac{1}{(H_n(x))}, \frac{1}{(H_n(x))}, \cdots, \frac{1}{(H_n(x))}}_{n}\Bigg) \prec \left(\frac{1}{x_1}, \frac{1}{x_2}, \cdots, \frac{1}{x_n}\right),$$

it is easy to prove inequality (17) holds.

The proof of Corollary 7 is complete. □

*Remark* 4. If let $\alpha = -1, \beta = 1$, by Corollary 7, then inequality

$$\frac{A_n(x)}{H_n(x)} \leq \left(\frac{G_n(x)}{H_n(x)}\right)^n \tag{18}$$

holds. If let $\alpha = -1$, $\beta = 0$, then $HM - GM$ inequality holds.

## 5. Schur Convexity of $L_1(x) = \prod_{i=1}^{n}(a+x_i)^{\alpha}\sum_{i=1}^{n}(b+x_i)^{\beta}$

**Theorem 4.** *Let* $a \geq 0$, $b \geq 0$, $x = (x_1, \cdots, x_n) \in \mathbb{R}^n_{++}$,

$$L_1(x) = \prod_{i=1}^{n}(a+x_i)^{\alpha}\sum_{i=1}^{n}(b+x_i)^{\beta}.$$

($i$) *When* $\alpha \geq 0$, $0 \leq \beta \leq 1$, *then* $L_1(x)$ *is Schur concave with* $x_1, \ldots, x_n$ *on* $\mathbb{R}^n_{++}$.

($ii$) *When* $\alpha \leq 0$, $\beta \leq 0$ *or* $\beta \geq 1$ *then* $L_1(x)$ *is Schur convex with* $x_1, \ldots, x_n$ *on* $\mathbb{R}^n_{++}$.

*Proof.* It is easy to see $L_1(x)$ is symmetry with $x_1,\cdots,x_n$, without loss of generality, we may assume that $x_1 \ge x_2 > 0$, we have

$$\begin{aligned}\frac{\partial L_1}{\partial x_1} &= \alpha(a+x_1)^{\alpha-1}\prod_{i=2}^{n}(a+x_i)^{\alpha}\sum_{i=1}^{n}(b+x_i)^{\beta}+\beta(b+x_1)^{\beta-1}\prod_{i=1}^{n}(a+x_i)^{\alpha}\\ &= \frac{\alpha}{a+x_1}\prod_{i=1}^{n}(a+x_i)^{\alpha}\sum_{i=1}^{n}(b+x_i)^{\beta}+\beta(b+x_1)^{\beta-1}\prod_{i=1}^{n}(a+x_i)^{\alpha},\end{aligned}$$

$$\begin{aligned}\frac{\partial L_1}{\partial x_2} &= \alpha(a+x_2)^{\alpha-1}\prod_{i-1,i\neq 2}^{n}(a+x_i)^{\alpha}\sum_{i=1}^{n}(b+x_i)^{\beta}+\beta(b+x_2)^{\beta-1}\prod_{i=1}^{n}(a+x_i)^{\alpha}\\ &= \frac{\alpha}{a+x_2}\prod_{i=1}^{n}(a+x_i)^{\alpha}\sum_{i=1}^{n}(b+x_i)^{\beta}+\beta(b+x_2)^{\beta-1}\prod_{i=1}^{n}(a+x_i)^{\alpha},\end{aligned}$$

$$\begin{aligned}\triangle_4 :=&(x_1-x_2)\left(\frac{\partial L_1}{\partial x_1}-\frac{\partial L_2}{\partial x_2}\right)\\ =&(x_1-x_2)\{\alpha\prod_{i=1}^{n}(a+x_i)^{\alpha}\sum_{i=1}^{n}(b+x_i)^{\beta}\left(\frac{1}{a+x_1}-\frac{1}{a+x_2}\right)\\ &+\beta\prod_{i=1}^{n}(a+x_i)^{\alpha}\left[(b+x_1)^{\beta-1}-(b+x_2)^{\beta-1}\right]\}.\end{aligned}$$

Easy to see, when $\alpha \ge 0$, $0 \le \beta \le 1$, we have $\triangle_4 \le 0$, by Lemma 1, it follows that $L_1(x)$ is Schur concave with $x_1,\ldots,x_n$ on $\mathbb{R}^n_{++}$. When $\alpha \le 0$, $\beta \le 0$ or $\beta \ge 1$, we have $\triangle_4 \ge 0$, by Lemma 1, it follows that $L_1(x)$ is Schur convex with $x_1,\ldots,x_n$ on $\mathbb{R}^n_{++}$.

The proof of Theorem 4 is complete. □

**Question:** when $\alpha \ge 0$, $\beta > 1$ or $\beta < 0$, what is the Schur convexity of $L_1(x)$?

By Theorem 4, Lemma 6(i) and Definition 1, we get the following conclusion:

**Corollary 8.** *Let $a \ge 0$, $b \ge 0$, $x = (x_1,\cdots,x_n) \in \mathbb{R}^n_{++}$. If $\alpha \ge 0$, $0 \le \beta \le 1$, then*

$$\prod_{i=1}^{n}(a+x_i)^{\alpha}\sum_{i=1}^{n}(b+x_i)^{\beta} \le n^{1-n\alpha-\beta}\left(\sum_{i=1}^{n}(a+x_i)\right)^{n\alpha}\left(\sum_{i=1}^{n}(b+x_i)\right)^{\beta}. \tag{19}$$

Let $n_1,\cdots,n_m$ are any $m$ $(m>1)$ natural numbers. Then

$$\left(\sum_{i=1}^{m}\frac{2}{n_i}\right)\left(\prod_{j=1}^{m}\frac{n_j}{n_j+1}\right) \le 1. \tag{20}$$

is Minc's inequality[26].

It is easy to see that the equivalent form of Minc's inequality is

$$\sum_{i=1}^{m}\frac{1}{n_i}\leq\frac{1}{2}\prod_{i=1}^{m}\left(1+\frac{1}{n_i}\right). \tag{21}$$

In the following, we give reverse Minc's inequality.

**Corollary 9.** *Let $n_1\leq\cdots\leq n_m$ are any $m$ $(m>1)$ natural numbers. Write*

$$A=\min\left(\frac{m}{n_m}\left(\frac{n_m}{1+n_m}\right)^m,\frac{m}{n_1}\left(\frac{n_1}{1+n_1}\right)^m\right),$$

*then*

$$\sum_{i=1}^{m}\frac{1}{n_i}\geq A\prod_{i=1}^{m}\left(1+\frac{1}{n_i}\right). \tag{22}$$

*Proof.* For any $x=(x_1,\cdots,x_m)\in\mathbb{R}^m_{++}$, by Theorem 4(ii), Lemma 6(i) and Definition 1, we have

$$\begin{aligned}2\prod_{i=1}^{m}(1+x_i)^{-1}\sum_{i=1}^{m}x_i\geq&2\left(\frac{\sum_{i=1}^{m}(1+x_i)}{m}\right)^{-m}m\frac{\sum_{i=1}^{m}x_i}{m}\\=&\frac{2m^m\sum_{i=1}^{m}x_i}{(m+\sum_{i=1}^{m}x_i)^m}.\end{aligned}$$

Let $x_i=\frac{1}{n_i}$, $i=1,\cdots,m$, then

$$2\prod_{i=1}^{m}\frac{n_i}{n_i+1}\sum_{1}^{m}\frac{1}{n_i}\geq\frac{2m^m\sum_{i=1}^{m}\frac{1}{n_i}}{\left(m+\sum_{i=1}^{m}\frac{1}{n_i}\right)^m}.$$

Let $f(t)=\frac{2m^mt}{(m+t)^m}$, $t>0$, we have

$$f'(t)=2m^m[m+(1-m)t].$$

Easy to see, when $0<t\leq\frac{m}{m-1}$, $f$ is increasing, hence

$$\begin{aligned}\prod_{i=1}^{m}\frac{n_i}{n_i+1}\sum_{i=1}^{m}\frac{2}{n_i}\geq&\frac{2m^m\sum_{i=1}^{m}\frac{1}{n_i}}{\left(m+\sum_{i=1}^{m}\frac{1}{n_i}\right)^m}\\\geq&\frac{2m^mm\frac{1}{n_m}}{\left(m+m\frac{1}{n_m}\right)^m}\\=&\frac{2m}{n_m}\left(\frac{n_m}{1+n_m}\right)^m.\end{aligned}$$

When $t \geq \frac{m}{m-1}$, $f$ is decreasing, hence

$$\prod_{i=1}^{m} \frac{n_i}{n_i+1} \sum_{i=1}^{m} \frac{2}{n_i} \geq \frac{2m^m \sum_{i=1}^{m} \frac{1}{n_i}}{\left(m+\sum_{i=1}^{m} \frac{1}{n_i}\right)^m} \geq \frac{2m^m m \frac{1}{n_1}}{\left(m+m\frac{1}{n_1}\right)^m} = \frac{2m}{n_1}\left(\frac{n_1}{1+n_1}\right)^m.$$

The proof of Corollary 9 is complete. □

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## CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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