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ON SOME NONLINEAR INTEGRAL INEQUALITIES OF PACHPATTE TYPE

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Abstract. In this paper, we establish new integral inequalities which provide an explicit bound on unknown function, and can be used as a tool in the study of certain nonlinear Volterra integral equations.

Keywords: Integral inequality, Volterra integral equation, bound and asymptotic of solution of integral equation.

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1. Introduction

Integral inequalities involving functions of one independent variable, which provide explicit bounds on unknown functions play a fundamental role in the development of the theory of linear and nonlinear differential and integral equations. In recent years nonlinear integral inequalities have received considerable attention because of the important applications to a variety of problems in diverse fields of nonlinear differential and integral equations. Basic inequalities are established by Gronwall [5], Bellman [2] and Pachpatte [8, 9] provide explicit bounds on solutions of a class of differential and integral equations which are further studied by many mathematicians see [1, 3, 4, 6, 7, 10, 11].

In this paper, we extend and improve some of the results reported in [1, 8, 9] to obtain a new

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generalization for some inequalities, which can be used as handy tools to study the qualitative as well as the quantitative properties of solutions of some nonlinear integral equations. Some applications are also given to convey the importance of our results.

2. Preliminaries

Before proceeding to the statement of our main result, we state some important integral inequalities that will be used in further discussion.

Theorem 2.1 (**Gronowall** [5]). *If $u(t)$ is a continuous function defined on the interval $J = [\alpha, \alpha + h]$ and*

$$0 \le u(t) \le \int_{\alpha}^{t} [bu(s) + a] ds, \; t \in J \tag{2.1}$$

where a and b are nonnegative constants, then

$$0 \le u(t) \le ahe^{bh}, \; t \in J. \tag{2.2}$$

Theorem 2.2 (**Gronowall-Bellman** [2]). *Let $u(t)$ and $f(t)$ be a nonnegative continuous functions defined on $J = [\alpha, \alpha + h]$ and c be nonegative constant. If*

$$u(t) \le c + \int_{\alpha}^{t} f(s)u(s)ds, \; t \in J \tag{2.3}$$

then

$$u(t) \le c \, \exp\left(\int_{\alpha}^{t} f(s)ds\right), \; t \in J. \tag{2.4}$$

Theorem 2.3 (see [1]). *We assume that $x(t), f(t)$ and $h(t)$ be nonnegative real valued continuous functions defined on $R_+ = [0, \infty)$, and satisfy the inequality*

$$x^p(t) \le x_0 + \int_0^t f(s)x^p(s)ds + \int_0^t h(s)x^q(s)ds, \tag{2.5}$$

for all $t \in R_+$, where $p > q \ge 0$, are constants. Then

$$x(t) \le \exp\left(\frac{1}{p}\int_0^t f(s)ds\right)\left[x_0^{p_1} + p_1\int_0^t h(s)\exp\left(-p_1\int_0^s f(\lambda)d\lambda\right)ds\right]^{\left[\frac{1}{p-q}\right]}, \tag{2.6}$$

for all $t \in R_+$, where $p_1 = \frac{p-q}{p}$.

Theorem 2.4 (see [9]). *Let u, f, g be real valued nonnegative continuous functions on $R_+ = [0,\infty)$ and c_1, c_2 be nonnegative constants. If*

$$u(t) \leq \left(c_1 + \int_0^t f(s)u(s)ds\right)\left(c_2 + \int_0^t g(s)u(s)ds\right), \tag{2.7}$$

and $\left[c_1c_2\int_0^t R(s)Q(s)ds\right] < 1$, *for all* $t \in R_+$, *then*

$$u(t) \leq \frac{c_1c_2Q(t)}{1 - c_1c_2\int_0^t R(s)Q(s)ds}, \text{ for all } t \in R_+, \tag{2.8}$$

where

$$R(t) = \left[g(t)\int_0^t f(s)ds + f(t)\int_0^t g(s)ds\right] \text{ and } Q(t) = \exp\left(\int_0^t [c_1g(s) + c_2f(s)]\,ds\right). \tag{2.9}$$

3. Main results

In this section, we state and prove some new nonlinear integral inequalities of Pachpatte type and we obtain a bound on an unknown function, which can be used in the analysis of various problems in the theory of nonlinear differential and integral equations.

Theorem 3.1. *Let u, f and g be nonnegative real valued continuous functions on $R_+ = [0,\infty)$ and c_1, c_2, p be nonnegative real constants such that $c_1c_2 \geq 1$ and $p \geq 1$. If*

$$u^p(t) \leq \left(c_1 + \int_0^t f(s)u^p(s)ds\right)\left(c_2 + \int_0^t g(s)u(s)ds\right) \tag{3.1}$$

and $\left[\frac{(c_1c_2)^{\frac{1}{p}}}{p}\int_0^t R(s)Q(s)ds\right] < 1$, *for all* $t \in R_+$ *then*

$$u(t) \leq \left[\frac{(c_1c_2)^{\frac{1}{p}}Q(t)}{1 - \frac{(c_1c_2)^{\frac{1}{p}}}{p}\int_0^t R(s)Q(s)ds}\right], \text{ for all } t \in R_+, \tag{3.2}$$

where, $R(t)$ is as same defined in (2.9) and $Q(t) = \exp\left(\frac{1}{p}\int_0^t [c_1g(s) + c_2f(s)]\,ds\right)$.

Proof. Define a function $z(t)$ by

$$z^p(t) = \left(c_1 + \int_0^t f(s)u^p(s)ds\right)\left(c_2 + \int_0^t g(s)u(s)ds\right), \text{ for all } t \in R_+, \tag{3.3}$$

then $u^p(t) \leq z^p(t)$ and $z^p(0) = c_1c_2$. Differentiating (3.3) and using the fact that $u(t) \leq z(t)$ and $z(t)$ is monotone nondecreasing for $t \in R_+$, we obtain

$$\begin{aligned}
pz^{p-1}(t)z'(t) &= c_2f(t)u^p(t) + c_1g(t)u(t) + f(t)u^p(t)\int_0^t g(s)u(s)ds + g(t)u(t)\int_0^t f(s)u^p(s)ds \\
&\leq c_2f(t)z^p(t) + c_1g(t)z^p(t) + f(t)z^p(t)\int_0^t g(s)z(s)ds + g(t)z(t)\int_0^t f(s)z^p(s)ds \\
&\leq (c_2f(t) + c_1g(t))\,z^p(t) + \left(f(t)\int_0^t g(s)ds + g(t)\int_0^t f(s)ds\right) z^{p+1}(t),
\end{aligned}$$

i.e.,

$$z'(t) \leq \frac{1}{p}\left[c_2f(t) + c_1g(t)\right] z(t) + \frac{1}{p}R(t)z^2(t). \tag{3.4}$$

The inequality (3.4) implies the estimation for $z(t)$ such that

$$z(t) \leq \left[\frac{(c_1c_2)^{\frac{1}{p}}Q(t)}{1 - \frac{(c_1c_2)^{\frac{1}{p}}}{p}\int_0^t R(s)Q(s)ds}\right], \text{ for all } t \in R_+.$$

As $u(t) \leq z(t)$, we obtain a desired bound for $u(t)$ given by (3.2). This completes the proof.. □

Theorem 3.2. *Let u, f and g be nonnegative real valued continuous functions on $R_+ = [0, \infty)$ and c_1, c_2, p, q be nonnegative real constants such that $c_1c_2 \geq 1, p > q \geq 1$ and $p - q \geq 1$. If*

$$u^p(t) \leq \left(c_1 + \int_0^t f(s)u^q(s)ds\right)\left(c_2 + \int_0^t g(s)u(s)ds\right), \text{ for all } t \in R_+, \tag{3.5}$$

then

$$u(t) \leq \left[\frac{(c_1c_2)^{\frac{p-q}{p}} + \frac{p-q}{p}\int_0^t \left[c_2f(s) + c_1g(s)\right] Q(s)ds}{Q(t)}\right]^{\frac{1}{p-q}}, \text{ for all } t \in R_+, \tag{3.6}$$

where, $R(t)$ is as same defined in (2.9) and $Q(t) = \exp\left(-\frac{p-q}{p}\int_0^t R(s)ds\right)$.

Proof. We proceed as in the proof of the Theorem 3.1. Define a function $z(t)$ by

$$z^p(t) = \left(c_1 + \int_0^t f(s)u^q(s)ds\right)\left(c_2 + \int_0^t g(s)u(s)ds\right), \text{ for all } t \in R_+, \tag{3.7}$$

then $u^p(t) \leq z^p(t)$ and $z^p(0) = c_1c_2$. Differentiating (3.7) and using the fact that $u(t) \leq z(t)$ and $z(t)$ is monotone nondecreasing for $t \in R_+$, we obtain

$$pz^{p-1}(t)z'(t) = c_2f(t)u^q(t) + c_1g(t)u(t) + f(t)u^q(t)\int_0^t g(s)u(s)ds + g(t)u(t)\int_0^t f(s)u^q(s)ds$$

$$
\begin{aligned}
&\leq c_2 f(t) z^q(t) + c_1 g(t) z(t) + f(t) z^q(t) \int_0^t g(s) z(s) ds + g(t) z(t) \int_0^t f(s) z^q(s) ds \\
&\leq [c_2 f(t) + c_1 g(t)]) z^q(t) + \left(f(t) \int_0^t g(s) ds + g(t) \int_0^t f(s) ds \right) z^{q+1}(t) \\
&\leq [c_2 f(t) + c_1 g(t)] z^q(t) + R(t) z^{q+1}(t).
\end{aligned}
$$

Since $p-q \geq 1$, we have

$$
\begin{aligned}
p z^{p-q-1}(t) z'(t) &\leq [c_2 f(t) + c_1 g(t)] + R(t) z(t), \\
&\leq [c_2 f(t) + c_1 g(t)] + R(t) z^{p-q}(t). \qquad (3.8)
\end{aligned}
$$

Let $z^{p-q}(t) = v(t)$, we have $v(0) = z^{p-q}(0) = (c_1 c_2)^{\frac{p-q}{p}}$ and

$$
\begin{aligned}
(p-q) z^{p-q-1} z'(t) &= v'(t) \\
z^{p-q-1} z'(t) &= \frac{v'(t)}{p-q}.
\end{aligned}
$$

From equations (3.8) , we obtain

$$
\begin{aligned}
\left(\frac{p}{p-q}\right) v'(t) &\leq [c_2 f(t) + c_1 g(t)] + R(t) v(t), \\
v'(t) &\leq \left(\frac{p-q}{p}\right) R(t) v(t) + \left(\frac{p-q}{p}\right) [c_2 f(t) + c_1 g(t)]. \qquad (3.9)
\end{aligned}
$$

The inequality (3.9) implies the estimation for $v(t)$ such that

$$
\begin{aligned}
v(t) &\leq \frac{(c_1 c_2)^{\frac{p-q}{p}} + \frac{p-q}{p} \int_0^t [c_2 f(s) + c_1 g(s)] Q(s) ds}{Q(t)}, \text{ for all } t \in R_+, \\
z^{p-q}(t) &\leq \frac{(c_1 c_2)^{\frac{p-q}{p}} + \frac{p-q}{p} \int_0^t [c_2 f(s) + c_1 g(s)] Q(s) ds}{Q(t)}, \text{ for all } t \in R_+.
\end{aligned}
$$

This implies

$$
z(t) \leq \left[\frac{(c_1 c_2)^{\frac{p-q}{p}} + \frac{p-q}{p} \int_0^t [c_2 f(s) + c_1 g(s)] Q(s) ds}{Q(t)} \right]^{\frac{1}{p-q}}, \text{ for all } t \in R_+,
$$

As $u^p(t) = z(t)$, we obtain a desired bound for $u(t)$ given by (3.6). This completes the proof. $\square$

Theorem 3.3. *Let u, f and g be nonnegative real valued continuous functions on $R_+ = [0,\infty)$ and c_1, c_2 be nonnegative real constants. If*

$$u^p(t) \le \left(c_1 + \int_0^t f(s)u^p(s)ds\right)\left(c_2 + \int_0^t g(s)u^p(s)ds\right), \tag{3.10}$$

$p > 0$ and $\left[c_1c_2\int_0^t R(s)Q(s)ds\right] < 1$, for all $t \in R_+$ then

$$u(t) \le \left[\frac{c_1c_2Q(t)}{1 - c_1c_2\int_0^t R(s)Q(s)ds}\right]^{\frac{1}{p}}, \ \textit{for all } t \in R_+, \tag{3.11}$$

where $R(t)$ and $Q(t)$ are same as defined in (2.9).

Proof. Put $z(t) = u^p(t)$ in (3.10),

$$z(t) \le \left(c_1 + \int_0^t f(s)z(s)ds\right)\left(c_2 + \int_0^t g(s)z(s)ds\right), \ \text{for all } t \in R_+. \tag{3.12}$$

Applying Theorem 2.4 to (3.12), we get

$$z(t) \le \left[\frac{c_1c_2Q(t)}{1 - c_1c_2\int_0^t R(s)Q(s)ds}\right], \ \text{for all } t \in R_+.$$

As $u^p(t) = z(t)$, we obtain a desired bound for $u(t)$ given by (3.11). This completes the proof. □

Theorem 3.4. *Let u, f and g be nonnegative real valued continuous functions on $R_+ = [0,\infty)$ and c_1, c_2 be nonnegative real constants such that $c_1c_2 \ge 1$. If*

$$u^p(t) \le \left(c_1 + \int_0^t f(s)u^p(s)ds\right)\left(c_2 + \int_0^t g(s)u^q(s)ds\right), \tag{3.13}$$

$p \ge q > 0$ and $\left[\frac{q}{p}(c_1c_2)^{\frac{q}{p}}\int_0^t R(s)Q(s)ds\right] < 1$, for all $t \in R_+$ then

$$u(t) \le \left[\frac{(c_1c_2)^{\frac{q}{p}}Q(t)}{1 - \frac{q}{p}(c_1c_2)^{\frac{q}{p}}\int_0^t R(s)Q(s)ds}\right]^{\frac{1}{q}}, \ \textit{for all } t \in R_+, \tag{3.14}$$

where, $R(t)$ is as same defined in (2.9) and $Q(t) = \exp\left(\frac{q}{p}\int_0^t [c_2f(s) + c_1g(s)]\,ds\right)$.

Proof. Define a function $z(t)$ by

$$z^p(t) = \left(c_1 + \int_0^t f(s)u^p(s)ds\right)\left(c_2 + \int_0^t g(s)u^q(s)ds\right), \text{ for all } t \in R_+, \tag{3.15}$$

then $u^p(t) \leq z^p(t)$ and $z^p(0) = c_1c_2$. Differentiating (3.15) and using the fact that $u(t) \leq z(t)$ and $z(t)$ is monotone nondecreasing for $t \in R_+$, we obtain

$$\begin{aligned} pz^{p-1}(t)z'(t) &= c_2 f(t)u^p(t) + c_1 g(t)u^q(t) + f(t)u^p(t)\int_0^t g(s)u^q(s)ds + g(t)u^q(t)\int_0^t f(s)u^p(s)ds \\ &\leq (c_2 f(t) + c_1 g(t))\, z^p(t) + \left(f(t)\int_0^t g(s)ds + g(t)\int_0^t f(s)ds\right) z^{p+q}(t) \\ &\leq (c_2 f(t) + c_1 g(t))\, z^p(t) + \left(f(t)\int_0^t g(s)ds + g(t)\int_0^t f(s)ds\right) z^{p+q}(t), \\ p\frac{z'(t)}{z(t)} &\leq (c_2 f(t) + c_1 g(t)) + R(t)z^q(t). \end{aligned} \tag{3.16}$$

Putting $z^q(t) = v(t)$ and differentiating with respect to t and using the fact that $v(t)$ is monotone nondecreasing for $t \in R_+$, we have

$$\begin{aligned} qz^{q-1}z'(t) &= v'(t) \\ q\frac{z'(t)}{z(t)} &= \frac{v'(t)}{z^q(t)}, \\ \frac{z'(t)}{z(t)} &= \frac{v'(t)}{qv(t)}. \end{aligned} \tag{3.17}$$

From inequality (3.16) and equations (3.17) and using $p \geq q > 0$, we obtain

$$v'(t) \leq \frac{q}{p}\left[c_2 f(t) + c_1 g(t)\right] v(t) + \frac{q}{p}R(t)v^2(t). \tag{3.18}$$

The inequality (3.18) implies the estimation for $v(t)$ such that

$$v(t) \leq \frac{(c_1c_2)^{\frac{q}{p}}Q(t)}{1 - \frac{q}{p}(c_1c_2)^{\frac{q}{p}}\int_0^t R(s)Q(s)ds}, \text{ for all } t \in R_+,$$

$$z^q(t) \leq \frac{(c_1c_2)^{\frac{q}{p}}Q(t)}{1 - \frac{q}{p}(c_1c_2)^{\frac{q}{p}}\int_0^t R(s)Q(s)ds}, \text{ for all } t \in R_+.$$

As $u(t) = z(t)$, we obtain a desired bound for $u(t)$ given by (3.14). This completes the proof. □

Theorem 3.5. *Let u, f, g be nonnegative real valued continuous functions on $R_+ = [0,\infty)$, $p \geq q > 0$, $c_2 \geq 1$ and c_1 be nonnegative real constants. If*

$$u^p(t) \leq \left(c_1 + \int_0^t f(s)u(s)ds\right)^p \left(c_2 + \int_0^t g(s)u(s)ds\right)^q, \text{ for all } t \in R_+, \tag{3.19}$$

then

$$u(t) \leq \left[\frac{c_1c_2Q(t)}{1 - c_1c_2\int_0^t R(s)Q(s)ds}\right] \text{ for all } t \in R_+, \tag{3.20}$$

where $R(t)$ and $Q(t)$ are same as defined in (2.9).

Proof. Since $p \geq q$ and $c_2 \geq 1$, using nonnegative character of u and g, we have

$$\begin{aligned} u^p(t) &\leq \left(c_1 + \int_0^t f(s)u(s)ds\right)^p \left(c_2 + \int_0^t g(s)u(s)ds\right)^q \\ &\leq \left[\left(c_1 + \int_0^t f(s)u(s)ds\right) \left(c_2 + \int_0^t g(s)u(s)ds\right)\right]^p. \end{aligned}$$

Define a function $z(t)$ by

$$z(t) = \left(c_1 + \int_0^t f(s)u(s)ds\right) \left(c_2 + \int_0^t g(s)u(s)ds\right), \tag{3.21}$$

then $u^p(t) \leq z^p(t)$ and $z(0) = c_1c_2$. Using the relation $u(t) \leq z(t)$ and monotonic character of $z(t)$, we have

$$z(t) \leq \left(c_1 + \int_0^t f(s)z(s)ds\right) \left(c_2 + \int_0^t g(s)z(s)ds\right), \text{ for all } t \in R_+, \tag{3.22}$$

which is of the form (2.7). Using Theorem 2.4, we obtain

$$z(t) \leq \left[\frac{c_1c_2Q(t)}{1 - c_1c_2\int_0^t R(s)Q(s)ds}\right], \text{ for all } t \in R_+.$$

Consequently, we get a desired bound for $u(t)$ given by (3.20). This completes the proof. □

4. **Applications**

In this section, we give the applications of our main results to study the boundedness and asymptotic behaviour of the solutions of nonlinear Volterra integral equations.

Example 4.1. We calculate the explicate bound on the solution of the nonlinear integral equation

$$u^2(t) = \left(1 + \int_0^t u^2(s)ds\right)\left(1 + \int_0^t u(s)ds\right), \tag{4.1}$$

where u be a nonnegative real valued continous function and we assume that every solution $u(t)$ of (4.1) exists on R_+.

Applying Theorem 3.1 to the equation (4.1), we have

$$u(t) \leq \left[\frac{Q(t)}{1 - \frac{1}{2}\int_0^t R(s)Q(s)ds}\right], \tag{4.2}$$

provided

$$\left[\frac{1}{2}\int_0^t R(s)Q(s)ds\right] < 1, \tag{4.3}$$

where $R(t)$ and $Q(t)$ are defined as in Theorem 3.1 and their values are

$$R(t) = \int_0^t 1ds + \int_0^t 1ds = t + t = 2t \tag{4.4}$$

and

$$Q(t) = \exp\left(\frac{1}{2}\int_0^t [1+1]\,ds\right) = e^t. \tag{4.5}$$

Using (4.4) and (4.5), we get

$$\left[\frac{1}{2}\int_0^t R(s)Q(s)ds\right] = \frac{1}{2}\int_0^t 2se^s ds = te^t - e^t + 1.$$

Clearly (4.3) holds for $0 \leq t < 1$. Hence the right hand side of (4.2) gives the bound on the solution of (4.1) in terms of the known quantities

$$u(t) \leq \frac{e^t}{1 - \int_0^t se^s ds} \leq \frac{1}{1-t}, \text{ for } 0 \leq t < 1.$$

Example 4.2. Consider the nonlinear integral equation

$$u^2(t) = \left(1 + \int_0^t u(s)ds\right)\left(1 + \int_0^t u(s)ds\right), \tag{4.6}$$

where u be nonnegative real valued continuous functions and assume that every solution $u(t)$ of (4.6) exists on R_+. We estimate the bound on the solution of nonlinear integral equation (4.6). Applying Theorem 3.2 to equation (4.6), we get

$$u(t) \leq \left[\frac{1+\int_0^t Q(s)ds}{Q(t)}\right], \text{ for all } t \in R_+, \tag{4.7}$$

where $R(t)$ and $Q(t)$ are same as defined in Theorem 3.2. In particular

$$R(t) = \int_0^t ds + \int_0^t ds = 2t \text{ and } Q(t) = \exp\left(-\frac{1}{2}\int_0^t 2sds\right) = e^{\frac{-t^2}{2}}. \tag{4.8}$$

Making use of equations (4.8) in (4.7), we have

$$u(t) \leq \left[\frac{1+\int_0^t e^{\frac{-s^2}{2}}ds}{e^{\frac{-t^2}{2}}}\right], \text{ for all } t \in R_+. \tag{4.9}$$

Example 4.3. We discuss the boundedness and asymptotic behaviour of the solution of a nonlinear Volterra integral equation of the form

$$u^p(t) = \left(h_1(t) + \int_0^t B(t-s)u^p(s)ds\right)\left(h_2(t) + \int_0^t C(t-s)u^q(s)ds\right), \tag{4.10}$$

for all $t \in R_+$, where p and q are as defined in Theorem 3.4, $u(t)$ nonnegative real valued continuous function defined on R_+ and h_1, h_2, B, C are real valued continuous functions defined on R_+. Here we assume that every solution $u(t)$ of (4.10) exists on R_+.

We list the following hypotheses on the functions involved in (4.10):

$$|h_1(t)| \leq c_1, |h_2(t)| \leq c_2, |B(t-s)| \leq M_1 f_1(s) \text{ and } |C(t-s)| \leq N_1 g_1(s), \tag{4.11}$$

$$\begin{aligned} |h_1(t)| \leq c_1 e^{-\alpha t}, |h_2(t)| \leq c_2 e^{-\alpha t}, |B(t)| &\leq M_1 f_1(s) e^{-\alpha(t-2s)} \text{ and} \\ |C(t)| &\leq N_1 g_1(s) e^{-\alpha(t-2s)} \end{aligned} \tag{4.12}$$

$$E_1(t) = \left[\frac{(c_1c_2)^{\frac{q}{p}} Q(t)}{1 - \frac{q}{p}(c_1c_2)^{\frac{q}{p}} \int_0^t R(s)Q(s)ds}\right]^{\frac{1}{q}} < \infty, \tag{4.13}$$

for all $0 \leq s \leq t, s, t \in R_+$, where $c_1, c_2, M_1, N_1, \alpha$ are nonnegative real constants and f_1, g_1 are nonnegative real valued continuous functions defined on R_+.

Firs we discuss the boundedness of solution of nonlinear integral equation (4.10). Suppose that the hypotheses (4.11) and (4.13) are satisfied, and let $u(t)$ be a solution of (4.10), we obtain

$$|u^p(t)| \leq \left(c_1 + \int_0^t M_1 f_1(s)\,|u^p(s)|\,ds\right)\left(c_2 + \int_0^t N_1 g_1(s)\,|u^q(s)|\,ds\right), \tag{4.14}$$

for all $t \in R_+$. Applying integral inequality given in Theorem 3.4 to (4.14) yields

$$|u(t)| \leq E_1(t), \ \text{ for all } t \in R_+, \tag{4.15}$$

where $R(t)$ and $Q(t)$ are same as defined in Theorem 3.4 by replacing f by $M_1 f_1$ and g by $M_2 g_1$, then every solution $u(t)$ of (4.10) existing on R_+ is bounded.

Now, we discuss the asymptotic behaviour of solution of nonlinear integral equation (4.10). Assume that the hypotheses (4.12) and (4.13) are satisfied, and let $u(t)$ be a solution of (4.10), one can easily observe that

$$|u^p(t)| \leq e^{-2\alpha t}\left(c_1 + \int_0^t M_1 f_1(s)\,|u^p(s)|\,e^{2\alpha s}ds\right)\left(c_2 + \int_0^t N_1 g_1(s)\,|u^q(s)|\,e^{2\alpha s}ds\right). \tag{4.16}$$

Multiplying on both sides of (4.16) by $e^{2\alpha t}$, we get

$$|u^p(t)|\,e^{2\alpha t} \leq \left(c_1 + \int_0^t M_1 f_1(s)\,|u^p(s)|\,e^{2\alpha s}ds\right)\left(c_2 + \int_0^t N_1 g_1(s)\,|u^q(s)|\,e^{2\alpha s}ds\right). \tag{4.17}$$

Replace $v^r(t)$ by $|u^r(t)|\,e^{2\alpha t}$ for $r > 0$ in (4.17) and applying Theorem 3.4, we obtain $v(t) \leq E_1(t)$ and hence $u(t) \leq E_1(t)e^{-2\alpha t}$ for all $t \in R_+$. This shows that the solution $u(t)$ of (4.10) is asymptotically stable.

Example 4.4. We also discuss the boundedness of the solution of a nonlinear Volterra integral equation of the form

$$u^p(t) = \left(h_1(t) + \int_0^t B(t-s)u(s)ds\right)^p \left(h_2(t) + \int_0^t C(t-s)u(s)ds\right)^q, \tag{4.18}$$

for all $t \in R_+$, where p and q are as defined in Theorem 3.5, $u(t)$ nonnegative real valued continuous function defined on R_+ and h_1, h_2, B, C are real valued continuous functions defined on R_+. Here we assume that every solution $u(t)$ of (4.18) exists on R_+.

Assume that the hypotheses (4.11) and

$$E_2(t) = \left[\frac{c_1 c_2 Q(t)}{1 - c_1 c_2 \int_0^t R(s)Q(s)ds}\right] < \infty$$

holds for functions involved in (4.18), and let $u(t)$ be a solution of (4.18). Applying Theorem 3.5, one can easily obtain an explicit bound $E_2(t)$ for every solution $u(t)$ of (4.18) existing on R_+.

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