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INVESTIGATING SYNCHRONOUS FUNCTION INEQUALITIES AND EXPONENTIAL KERNEL-BASED FRACTIONAL CALCULUS IN POPULATION DYNAMICS

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Abstract. The present paper is focused on developing Chebyshev-type inequalities using the fractional-order integrals with exponential kernels. We obtain new comparison results for synchronous functions and by the method of mathematical induction we will use these inequalities for a family of non-negative increasing functions. Also we prove a related Chebyshev-type inequality with fractional integral operators under conditions of monotonicity of the functions under consideration. These findings enrich the theory of inequalities in the fractional calculus that supplies methods used in math analysis, engineering, and other areas where memory impacts or time delays exist.

Keywords: Chebyshev-type inequalities; fractional-order integrals; monotonic Function; synchronous function; exponential kernel.

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1. Introduction

As for the presented work, it can be associated with the Chebyshev-type inequality that has been developed within the context of fractional-order integrals with exponential kernels, as well as the barely mentioned Hermite–Hadamard inequality that defines bounds for convex functions. [1, 2, 3, 4, 5].

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$$\frac{f(s)+f(t)}{2} \leq \frac{1}{t-s}\int_s^t f(x)\,dx \leq f\left(\frac{s+t}{2}\right).$$

For the consideration of fractional calculus, this relation can be used to derive new constraints for differential inequalities of fractional order of convex functions with exponential kernel [6, 7, 8], thus expanding the mathematical basis of these types of programs and improving the ability to address inequalities in this generalized context.

They are practical in mathematics, especially in analysis, probability theory, and mathematical inequalities [9, 10]. All these inequalities give estimates for integrals or sums of functions under some conditions of monotonicity and convexity [11, 12, 13, 14]. In the classical context, such inequalities help in controlling the difference of functions or variables and assessing the degree of dependency, thereby knowing their value in various applications. Nevertheless, with the help of fractional calculus, the application field of the Chebyshev-type inequality has risen, and that is why it requires new investigations regarding its application for the given settings containing fractional integrals in view of the regulated exponent, particularly when the exponential kernel is used.

Thus, this paper examines the relationship between these two fields by concentrating on Chebyshev-type inequalities [15, 16], as applied to fractional-order integrals with exponential kernel functions. To this end, it is our desire to arrive at improved inequalities that would lead to more accurate estimates of the various processes described by such integrals. The subject has applications in the analysis of engineering systems that involve memory effects and in applied physics and economics where time delay is an issue. These results can be used for the description of systems in which fractional behavior and exponential growth or decay are essential characteristics; thus, the obtained results extend the applicability of both Chebyshev inequalities and fractional calculus in scientific practice.

2. PRELIMINARIES

Within this part , we remind some important preliminaries.

Lemma 1. ***(A)*** *If there exist synchronous functions β and γ, subsequently* [17, 18]

$$[\gamma(\psi)-\gamma(\psi_0)][\beta(\psi)-\beta(\psi_0)] \geq 0, \tag{2.1}$$

And also, $\psi, \psi_0 \in [a,b]$.

(B) *If there exist asynchronous functions β and γ, after that* [19, 20]

$$[\gamma(\psi)-\gamma(\psi_0)][\beta(\psi)-\beta(\psi_0)] \leq 0, \tag{2.2}$$

In addition, $\psi, \psi_0 \in [a,b]$.

3. Main Results

Within light of Lemma 1, in this section With the exponential kernals, we propose four Inequalities of Chebyshev type for integrals of the general fractional-order.

Theorem 1. *If two synchronous functions are β and γ, on $[a,b]$ for $\alpha = 1$, the inequality having exponential*

$$_aI_t^{\alpha}[\gamma(t)\beta(t)] \geq \frac{_aI_t^{\alpha}\gamma(t) \cdot {_aI_t^{\alpha}}\beta(t)}{\int_t^a \Omega_\alpha \mathscr{K}^{(s,y)\alpha}} \tag{3.3}$$

Proof. By using (2.1), we obtain

$$\gamma(\tau)\beta(\tau)+\gamma(\lambda)\beta(\lambda) \geq \gamma(\tau)\beta(\lambda)+\gamma(\lambda)\beta(\tau) \tag{3.4}$$

where $\lambda, \tau \in [a, j]$.

Multiplying inequality (3.4) by $\Omega_\alpha e^{y\alpha}$

$$\gamma(\tau)\beta(\tau)\Omega_\alpha e^{y\alpha}+\gamma(\lambda)\beta(\lambda)\Omega_\alpha e^{y\alpha} \geq \gamma(\tau)\beta(\lambda)\Omega_\alpha e^{y\alpha}+\gamma(\lambda)\beta(\tau)\Omega_\alpha e^{y\alpha} \tag{3.5}$$

Integrating, we get

$$\begin{aligned} &\int_a^t \gamma(\tau)\beta(\tau)\Omega_\alpha e^{y\alpha}dy+\gamma(\lambda)\beta(\lambda)\int_a^t \Omega_\alpha e^{y\alpha} \\ &\geq \beta(\lambda)\int_a^t \gamma(\tau)\Omega_\alpha e^{y\alpha}dy+\gamma(\lambda)\int_a^t \beta(\tau)\Omega_\alpha e^{y\alpha}dy \end{aligned} \tag{3.6}$$

Now (3.6) can be expressed as

$$_aI_t^{\alpha}[\gamma(t)\beta(t)]+[\gamma(\lambda)\beta(\lambda)]\int_a^t \Omega_\alpha e^{y\alpha} \geq \beta(\lambda)_aI_t^{\alpha}\gamma(t)+\gamma(\lambda)_aI_t^{\alpha}\beta(\lambda) \tag{3.7}$$

Multiplying (3.7) by $\Omega_\alpha e^{s\alpha}$ gives the inequality

$$\begin{aligned}
&{}_aI_t^\alpha[\gamma(t)\beta(t)]\Omega_\alpha e^{s\alpha}+[\gamma(\lambda)\beta(\lambda)]\Omega_\alpha e^{s\alpha}\int_a \Omega_\alpha e^{y\alpha}\\
&\geq \beta(\lambda)\Omega_\alpha e^{s\alpha}{}_aI_t^\alpha\gamma(t)+\gamma(\lambda)\Omega_\alpha e^{s\alpha}{}_aI_t^\alpha\beta(t) \qquad (3.8)
\end{aligned}$$

By integrating the inequality (3.8), we get

$$\begin{aligned}
&{}_aI_t^\alpha[\gamma(t)\beta(t)]\int_a^t \Omega_\alpha e^{s\alpha}+\int_a^t[\gamma(\lambda)\beta(\lambda)]\Omega_\alpha e^{s\alpha}ds\int_a^t \Omega_\alpha e^{y\alpha}\\
&\geq {}_aI_t^\alpha\gamma(t)\int_a^t \beta(\lambda)\Omega_\alpha e^{s\alpha}ds+{}_aI_t^\alpha\beta(t)\int_a^t \gamma(\lambda)\Omega_\alpha e^{s\alpha}ds \qquad (3.9)
\end{aligned}$$

$$[{}_aI_t^\alpha\gamma(t)\beta(t)]\int_a \Omega_\alpha e^{s\alpha}+{}_aI_t^\alpha[\gamma(t)\beta(t)]\int_a \Omega_\alpha e^{y\alpha}\geq 2{}_aI_t^\alpha\beta(t){}_aI_t^\alpha\gamma(t) \qquad (3.10)$$

Hence (3.3) is true.

Theorem 2. *If two synchronous functions are γ and β on $[a,j]$ and $\alpha,\eta=1$ and $t>0$.*

$$\begin{aligned}
&[{}_aI_t^\alpha\gamma(t)\beta(t)]\int_a^t \Omega_\eta \mathscr{K}^{s\eta}+{}_aI_t^\eta[\gamma(\tau)\beta(t)]\int_a^t \Omega_\alpha \mathscr{K}^{y\alpha}\\
&\geq {}_aI_t^\eta\beta(t){}_aI_t^\alpha\gamma(t)+{}_aI_t^\eta\gamma(t){}_aI_t^\alpha\beta(t) \qquad (3.11)
\end{aligned}$$

Proof. Multiplying (3.11) by $\Omega_\eta e^{s\eta}$, we get

$$\begin{aligned}
&{}_aI_t^\alpha[\gamma(t)\beta(t)]\Omega_\eta e^{s\eta}+[\gamma(\lambda)\beta(\lambda)]\Omega_\eta e^{s\eta}\int_a^t e^{y\alpha}\\
&\geq \Omega_\eta e^{s\eta}\beta(\lambda){}_aI_t^\alpha\gamma(t)+\Omega_\eta e^{s\eta}\gamma(\lambda){}_aI_t^\alpha\beta(t) \qquad (3.12)
\end{aligned}$$

After integrating, we get (3.12).

$$\begin{aligned}
&{}_aI_t^\alpha[\gamma(t)\beta(t)]\int_a^t \Omega_\eta e^{s\eta}+\int_a^t \Omega_\eta e^{s\eta}[\gamma(\lambda)\beta(\lambda)]d\rho\int_a^t \Omega_\alpha e^{y\alpha}\\
&\geq \int_a^t \Omega_\eta e^{s\eta}\beta(\lambda)d\rho{}_aI_t^\alpha\gamma(t)+\int_a^t \Omega_\eta e^{s\eta}\gamma(\lambda)d\rho{}_aI_t^\alpha\beta(t) \qquad (3.13)
\end{aligned}$$

By using (3.13), we get:

$$\begin{aligned}
&{}_aI_t^\alpha[\gamma(t)\beta(t)]\int_a^t \Omega_\eta e^{s\eta}+{}_aI_t^\eta[\gamma(t)\beta(t)]\int_a^t \Omega_\alpha e^{y\alpha}\\
&\geq {}_aI_t^\eta\beta(t){}_aI_t^\alpha\gamma(t)+{}_aI_t^\eta\gamma(t){}_aI_t^\alpha\beta(t) \qquad (3.14)
\end{aligned}$$

Lastly, (3.14) can be stated as:

$$\begin{aligned} & {}_aI_t^{\alpha}[\gamma(t)\beta(t)]\int_a^t \Omega_\eta \mathscr{K}^{y\alpha} + {}_aI_t^{\eta}[\gamma(t)\beta(t)]\int_a^t \Omega_\alpha \mathscr{K}^{y\alpha} \\ & \geq {}_aI_t^{\eta}\beta(t){}_aI_t^{\alpha}\gamma() + {}_aI_t^{\eta}\gamma(t){}_aI_t^{\alpha}\beta(t) \end{aligned} \tag{3.15}$$

Finally, the proof of (3.11) is complete.

Remark 1. *Assume β and γ are functions that are asynchronous on $[a, j]$, then the inequalities (3.3) and (3.11) are inverted where $[\gamma(\lambda)-\beta(\tau)][\gamma(\tau)-\beta(\lambda)] \leq 0$.*

Remark 2. *For $\alpha = \eta$, Theorem 2 overlaps with Theorem 1.*

Theorem 3. *If $(\gamma_i)_{i=1,\ldots,n}$ are n functions that increase positively on $[a, j]$ and $\alpha = 1$, then*

$$ {}_aI_t^{\alpha}[\prod_{i=1}^{n}\gamma_i(t)] \geq \left[\int_a^t \mathscr{K}^{(s,y)\alpha}\right]^{1-n} [\prod_{i=1}^{n}\gamma_i(t)]_aI_t^{\alpha} \tag{3.16}$$

Proof. Using the inductive method of mathematics, let $n = 1$ in (3.16), we have

$$ {}_aI_t^{\alpha}\gamma_1(t) \geq {}_aI_t^{\alpha}\gamma_2(t) \tag{3.17}$$

$$ {}_aI_t^{\alpha}[\gamma_1(t)\gamma_1(t)] \geq \frac{{}_aI_t^{\alpha}[\gamma_1(t)] \cdot {}_aI_t^{\alpha}[\gamma_2(t)]}{\int_t^a \mathscr{K}^{(s,y)\alpha}} \tag{3.18}$$

which is due to (3.3) of Theorem 1. Based on the principle of induction, we assume

$$ {}_aI_t^{\alpha}[\prod_{i=1}^{n-1}\gamma_i(t)] \geq \left[\int_a^t t^{(s,y)k}\right]^{2-n} [\prod_{i=1}^{n-1}\gamma_i(t)]_aI_t^{\alpha} \tag{3.19}$$

Now $(\gamma_i)_{i=1,\ldots,n}$ are increasing functions, indicating that the function $[\prod_{i=1}^{n-1}\gamma_i(t)]$. Thus, we apply (3.3) to the functions $[\prod_{i=1}^{n-1}\gamma_i(t)]$=$\gamma$ and $\gamma_n = \gamma$, to get

$$ {}_aI_t^{\alpha}[\prod_{i=1}^{n}\gamma_i(t)] \geq \left[\frac{1}{\int_a^\tau \mathscr{K}^{(s,y)\alpha}}\right] \times [{}_aI_t^{\alpha}[\prod_{i=1}^{n-1}\gamma_i(t)]_aI_t^{\alpha}\gamma_{n(t)}] \tag{3.20}$$

From (3.19) and (3.20), we can observe that (3.16) is true.

Theorem 4. *Assume ω, β, and γ functions that are monotone on $[a, j]$ for $\alpha, \eta = 1$ and $t > 0$ then*

$$ {}_aI_t^{\alpha}[\gamma(t)\beta(t)\omega(t)]\int_a^t \Omega_\eta \mathscr{K}^{s\eta} - {}_aI_t^{\eta}[\gamma(t)\beta(t)\omega(t)]\int_a^t \Omega_\alpha \mathscr{K}^{y\alpha}$$

$$\geq {}_aI_t^{\eta}\beta(t)_aI_t^{\alpha}[\gamma(t)\omega(t)] + {}_aI_t^{\eta}\gamma(t)_aI_t^{\alpha}[\beta(t)\omega(t)] - {}_aI_t^{\eta}[\gamma(t)\beta(t)]_aI_t^{\alpha}\omega(t)$$
$$+{}_aI_t^{\eta}\omega(t)_aI_t^{\alpha}[\gamma(t)\beta(t)] + {}_aI_t^{\eta}[\beta(t)\omega(t)]_aI_t^{\alpha}\gamma(t) - {}_aI_t^{\eta}[\gamma(t)\omega(t)]\beta(t) \tag{3.21}$$

Proof. By using

$$[\gamma(\tau)-\gamma(\lambda)][\beta(\tau)-\beta(\lambda)][\omega(\tau)-\omega(\lambda)] \geq 0 \tag{3.22}$$

$$[\gamma(\tau)\beta(\tau)\omega(\tau)] - [\gamma(\lambda)\beta(\lambda)\omega(\lambda)] - [\gamma(\tau)\beta(\lambda)\omega(\tau)] - [\gamma(\lambda)\beta(\tau)\omega(\tau)]$$
$$+[\gamma(\lambda)\beta(\lambda)\omega(\tau)] - [\gamma(\tau)\beta(\tau)\omega(\lambda)] - [\gamma(\tau)\beta(\lambda)\omega(\lambda)] + [\gamma(\lambda)\beta(\tau)\omega(\lambda)] \geq 0 \tag{3.23}$$

$$\int_a^t \Omega_\alpha e^{y\alpha}[\gamma(\tau)\beta(\tau)\omega(\tau)]dy - [\gamma(\lambda)\beta(\lambda)\omega(\lambda)]\int_a^t \Omega_\alpha e^{y\alpha} \geq \beta(\lambda)\int_a^t \Omega_\alpha e^{y\alpha}[\gamma(\tau)\omega(\tau)]dy$$
$$+\gamma(\lambda)\int_a^t \Omega_\alpha e^{y\alpha}[\beta(\tau)\omega(\tau)]dy - [\gamma(\lambda)\beta(\lambda)]\int_a^t \Omega_\alpha e^{y\alpha}\omega(\tau)dy + \omega(\lambda)\int_a^t \Omega_\alpha e^{y\alpha}[\gamma(\tau)\beta(\tau)]dy$$
$$+[\beta(\lambda)\omega(\lambda)]\int_a^t \Omega_\alpha e^{y\alpha}\gamma(\tau)dy - [\gamma(\lambda)\omega(\lambda)]\int_a^t \Omega_\alpha e^{y\alpha}\beta(\tau)dy \tag{3.24}$$

Immediately, (3.24) can be displayed as.

$${}_aI_t^{\alpha}[\gamma(t)\beta(t)\omega(t)] - [\gamma(\lambda)\beta(\lambda)\omega(\lambda)]\int_a^t \Omega_\alpha e^{y\alpha} \geq \beta(\lambda)_aI_t^{\alpha}[\gamma(t)\omega(t)]$$
$$+\gamma(\lambda)_aI_t^{\alpha}[\beta(t)\omega(t)] - [\gamma(\lambda)\beta(\lambda)]_aI_t^{\alpha}\omega(t) + \omega(\lambda)_aI_t^{\alpha}[\gamma(t)\beta(t)]$$
$$+[\beta(\lambda)\omega(\lambda)]_aI_t^{\alpha}\gamma(t) - [\gamma(\lambda)\omega(\lambda)]_aI_t^{\alpha}\beta(t) \tag{3.25}$$

Multiplying (3.25) by $\Omega_\eta e^{s\eta}$ and integrating we get

$${}_aI_t^{\alpha}[\gamma(t)\beta(t)\omega(t)]\int_a^t \Omega_\eta e^{s\eta} - {}_aI_t^{\eta}[\gamma(t)\beta(t)\omega(t)]\int_a^t \Omega_\alpha e^{y\alpha}$$
$$\geq {}_aI_t^{\eta}\beta(t)_aI_t^{\alpha}[\gamma(t)\omega(t)] + {}_aI_t^{\eta}\gamma(t)_aI_t^{\alpha}[\beta(t)\omega(t)] - {}_aI_t^{\eta}[\gamma(t)\beta(t)]_aI_t^{\alpha}\omega(t)$$
$$+{}_aI_t^{\eta}\omega(t)_aI_t^{\alpha}[\gamma(t)\beta(t)] + {}_aI_t^{\rho}[\beta(t)\omega(t)]_aI_t^{\alpha}\gamma(t) - {}_aI_t^{\eta}[\gamma(t)\omega(t)]_aI_t^{\alpha}\beta(t) \tag{3.26}$$

$${}_aI_t^{\alpha}[\gamma(t)\beta(t)\omega(t)]\int_a^t \Omega_\eta e^{s\eta} - {}_aI_t^{\eta}[\gamma(t)\beta(t)\omega(t)]\int_a^t \Omega_\alpha e^{y\alpha}$$
$$\geq {}_aI_t^{\eta}\beta(t)_aI_t^{\alpha}[\gamma(t)\omega(t)] + {}_aI_t^{\eta}\gamma(t)_aI_t^{\alpha}[\beta(t)\omega(t)] - {}_aI_t^{\eta}[\gamma(t)\beta(t)]_aI_t^{\alpha}\omega(t)$$
$$+{}_aI_t^{\eta}\omega(t)_aI_t^{\alpha}[\gamma(t)\beta(t)] + {}_aI_t^{\rho}[\beta(t)\omega(t)]_aI_t^{\alpha}\gamma(t) - {}_aI_t^{\eta}[\gamma(t)\omega(t)]_aI_t^{\alpha}\beta(t) \tag{3.27}$$

Thus, (3.27) can be demonstrated.

4. Application of Chebyshev-type Inequalities in Modeling Population Dynamics

Description of the Issue In the ecosystem context a population has interaction effects over time, delayed response to change, and competition for resources by predation. According to Patteman and Meirack, the effects of memory in such dynamics have to be taken into account when modelling such phenomenon especially in biological species that exhibit cyclical or oscillatory behavior. The processes governing such systems may not be easily discerned and in general, precise description and forecasting with the help of the dynamic models of the integer order can cause major challenges. From the perspective of the correction of the using fractional-order, to describe the dynamics of the memory effects and, in this way, the phenomena under consideration, the proposed approach provides better models, and, therefore, better predictions.

Applying Chebyshev-type Inequalities Chebyshev-type inequalities provide us with a lever by which we can bound responses of population models to such delayed or memory dependent interactions. These bounds help maintain the population within reasonable values so that it does not go all over the place full of oscillations and deviation from agencies' growth rates projecting more useful in models sensitive to fluctuations in ecological factors.

Application Example Let us now discuss a population dynamics model for a competing species system The two competing species are denoted by the populations $Q_1(t)$ and $Q_2(t)$. The objective is to reduce the discrepancy between the dynamics of the actual population and those of the formal ideal population where resources are abundant and interactivities optimal.

Dynamics Model The fractional-order dynamics of species Q_1 can be expressed using a fractional differential equation:

$$D^{\alpha}Q_1(t) = r_1Q_1(t)\left(1-\frac{Q_1(t)}{K_1}\right) - cQ_1(t)Q_2(t),$$

where D^{α} is the fractional derivative of order α, r_1 is the intrinsic growth rate, K_1 is the carrying capacity, and c represents the competition coefficient with species Q_2.

Error Function Definition Define the error function $E_1(t)$ to measure the difference between the observed population $Q_1(t)$ and an ideal population $Q_{1,\text{ideal}}(t)$:

$$E_1(t) = Q_1(t) - Q_{1,\text{ideal}}(t).$$

This function measures fluctuations as caused by environmental or competition forces that shift the rates of population growth.

Apply Chebyshev-type Inequalities Using inequalities for fractional order integral, Chebyshev-type inequalities are used to bound the error function $E_1(t)$ for the realism of the estimated population. If it is assumed that the noise processes of the two populations $Q_1(t)$ and $Q_{1,\text{ideal}}(t)$/are in phase (that is both populations are growing or decreasing), then the inequality is:

$$I_\alpha[E_1(t)] \geq I_\alpha[Q_1(t)] - I_\alpha[Q_{1,\text{ideal}}(t)],$$

where I_α stand for the fractional integral operator. This inequality offers a control of the error in terms of the fractional integrals of the actual and ideal populations, it does not allow the error to grow unbounded in oscillating conditions.

Optimization of Model Parameters The derived bounds on $E_1(t)$ allows for tuning and optimization of other parameters in the model which include α, r_1, K_1 and c to ensure that the dynamics of population are within acceptable levels. Therefore, adjusting most of these parameters can help to capture the simulation closer to the data and bring a reasonable level of competitiveness and environmental factors into the model.

Validation and Simulation Subsequent simulation is then performed to compare the bounds yielded with those that can be derived using Chebyshev-type of inequalities for the purpose of ascertaining whether the optimized model fits the empirical population. This validation also demonstrates that the chosen model is stable and that the use of the fractional order of the parameters accurately reflects the interactions within a population over time.

The image also gives an overview of what goes into population dynamics modeling. The first plot at the top of the chart illustrates the actual population. Labor $Q_1(t)$ (blue line) grows with an increase in the ideal population $Q_{1,\text{ideal}}(t)$, but the curves differ (blue line breaks away from the red dashed line). first a boom and then a bust. The middle plot describes the error function $E_1(t)$ that estimates the real distance between the zones. the disparity between actual and identified population, which declines gradually year by year. The bottom plot highlights The error function constraints are derived, as well as showing how elements of Chebyshev-type inequalities can be used to constrain population fluctuations. These plots check the stability and

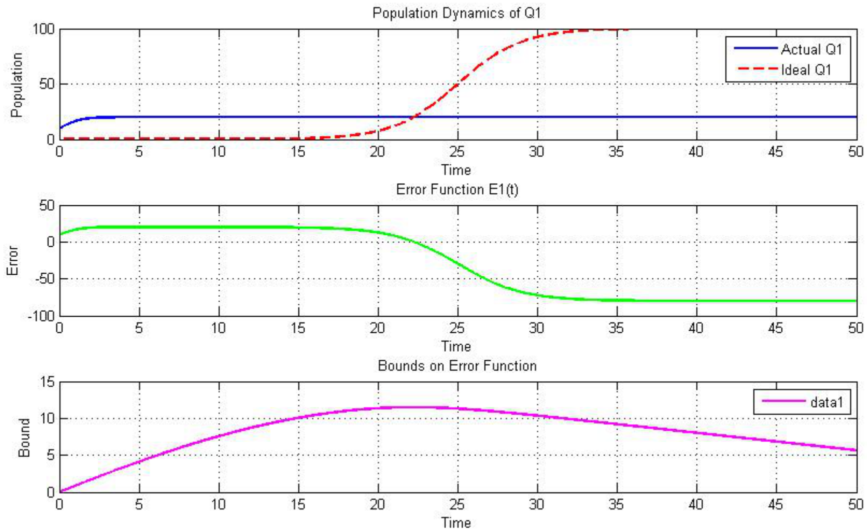

FIGURE 1. Graphical Analysis

proved the theoretical fractional-order model proposed. the synchrony of the two population curves. In total, they strengthen the idea of the model as an efficient tool to regulate competition and memory effects.

5. Conclusion

In this work, we notice that Chebyshev-type inequalities play a significant role with regard to fractional-order integrals based on exponential kernels. The stated results confirm not only that these inequalities can give accurate estimates, but also that they can be applied to solve problems arising in various fields of mathematics in one form or another. From the interconnectedness of classical inequalities and fractional calculus in this study, there is great potential for future analysis in understanding functional behavior and approximation. In future work, efforts should be made to generalize these inequalities and identify further situations where they may be effective. Thus, the findings of this study facilitate the enrichment of theoretical knowledge and the development of practical applications in the field of mathematical analysis.

Conflict of Interests

The authors declare that there is no conflict of interests.

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