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EXPLORING GENERALIZED OSTROWSKI-TYPE INEQUALITIES THROUGH PREINVEX FUNCTIONS IN FRACTIONAL CALCULUS

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Abstract. This article presents a novel concept of pre-invex functions linked to generalized Ostrowski-type inequalities. We explore the integral representation of these pre-invex functions within the framework of local fractional calculus. This approach extends traditional calculus to analyze fractional-order derivatives and integrals. We establish several generalized Ostrowski-type inequalities by employing the properties of pre-invex functions and their representations as integrals. Several generalized Ostrowski-type inequalities are derived by employing the properties of pre-invex functions and their integral representation. These inequalities applied to the twice differentiable functions in the context of fractional calculus locally, allowing for a deeper understanding of their behaviors and applications. Our work contributes to the growing body of knowledge in this area by providing new insights and results that can be applied in various mathematical and applied fields.

Keywords: preinvex functions; generalized ostrowski inequality; local fractional calculus; inequalities in fractional calculus; integral representation.

2020 AMS Subject Classification: 26D10, 90C26, 26A33.

1. Introduction

Let set of real numbers ($\mathbb{R}$), positive real numbers ($\mathbb{R}^+$), rational numbers ($\mathbb{Q}$), integers ($\mathbb{Z}$) and positive integers ($\mathbb{N}$) represented accordingly, and $\mathbb{J} := \mathbb{R} \,/\, \mathbb{Q}$ and N;= $\mathbb{N} \cup \{0\}$.

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For $0 \leq \zeta \leq 1$, The ζ- type integers set $\mathbb{Z}^{\zeta}$ is defined by

$$\mathbb{Z}^{\zeta} := \{0^{\varsigma}\} \cup \{\pm m^{\zeta} : m \in \mathbb{N}\}$$

the ζ- type rational number set $\mathbb{Q}^{\varsigma}$ is defined by

$$\mathbb{Q}^{\zeta} := q^{\zeta}; q \in \mathbb{Q} = (\frac{m}{n})^{\zeta} : m \in \mathbb{Z} n \in \mathbb{N}$$

The ζ-type irrational number set $\mathbb{J}^{\zeta}$ is defined by

$$\mathbb{Q}^{\zeta} := q^{\zeta}; q \in \mathbb{Q} = (\frac{m}{n})^{\zeta} : m \in \mathbb{Z} n \in \mathbb{N}$$

the α- type real number set $\mathbb{R}^{\zeta}$ is defined by

$$\mathbb{R}^{\zeta} := \mathbb{Q}^{\zeta} \cup \mathbb{J}^{\zeta}$$

If $r_1^{\varsigma}, r_2^{\varsigma}, r_3^{\varsigma} \in \mathbb{R}^{\varsigma}$ $(0 < \varsigma \leq 1)$, then

- $r_1^{\varsigma} + r_2^{\varsigma} \in \mathbb{R}^{\varsigma}, r_1^{\varsigma} r_2^{\varsigma} \in \mathbb{R}^{\varsigma}$,
- $r_1^{\varsigma} + r_2^{\varsigma} = r_2^{\varsigma} + r_1^{\varsigma} = (r_1 + r_2)^{\varsigma} = (r_2 + r_1)^{\varsigma}$,
- $r_1^{\varsigma} + (r_2^{\varsigma} + r_3^{\varsigma}) = (r_1 + r_2)^{\varsigma} + r_3^{\varsigma}$,
- $r_1^{\varsigma} r_2^{\varsigma} = r_2^{\varsigma} r_1^{\varsigma} = (r_1 r_2)^{\varsigma} = (r_2 r_1)^{\varsigma}$,
- $r_1^{\varsigma}(r_2^{\varsigma} r_3^{\varsigma}) = (r_1^{\varsigma} r_2^{\varsigma}) r_3^{\varsigma}$,
- $r_1^{\varsigma}(r_2^{\varsigma} + r_3^{\varsigma}) = r_1^{\varsigma} r_2^{\varsigma} + r_1^{\varsigma} r_3^{\varsigma}$,
- $r_1^{\varsigma} + 0^{\varsigma} = 0^{\varsigma} + r_1^{\varsigma} = r_1^{\varsigma}$, and $r_1^{\varsigma} 1^{\varsigma} = 1^{\varsigma} r_1^{\varsigma} = r_1^{\varsigma}$
- If $r_1^{\varsigma} < r_2^{\varsigma}$, then $r_1^{\varsigma} + r_3^{\varsigma} < r_2^{\varsigma} + r_3^{\varsigma}$,
- If $0^{\varsigma} < r_1^{\varsigma}, 0^{\varsigma} < r_2^{\varsigma}$, then $0^{\varsigma} < r_1^{\varsigma} . r_2^{\varsigma}$,

Local fractional order derivative and integral operator on $\mathbb{R}^{\varsigma}$ are rephrased from the sources given as,

Definition 1. *([31, 32]) A non-differentiable function $f : \mathbb{R} \to \mathbb{R}^{\varsigma}, y \to f(y)$ is local fractional continuous at y_0, if for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$|f(y) - f(y_0)| < \varepsilon^{\varsigma}$$

holds $|y - y_0| < \delta$, with $\varepsilon, \delta \in \mathbb{R}$. If $f(y)$ is local continuous on (c,d), and denoted as $f(y) \in C_{\varsigma}(c,d)$.

Definition 2. *([31, 32]) Local fractional order derivatives of the function $f(y)$ of order ς at $y = y_0$ can be defined as*

$$f^{(\varsigma)}(y_0) = \frac{d^\varsigma f(y)}{d\ell^\varsigma}\bigg|_{y=y_0} = \lim_{y\to y_0} \frac{\Gamma(1+\varsigma)(f(y)-f(y_0))}{(y-y_0)}$$

$D_\varsigma(b,c)$ is ς-local derivative set. If there exists $f^{((K+1)\varsigma)}(y) = \overbrace{D_y^\varsigma ... D_y^\varsigma}^{(n+1)times} f(y)$ for any $y \in \mathbb{I} \subseteq \mathbb{R}$, we denote $f \in D_{(n+1)\varsigma}(I)$, and $n = 0,1,2,...$

Definition 3. *([31, 32]) Let $f(y) \in C_\varsigma[c,d]$. Local fractional integral of $f(w)$ can be defined by*

$${}_bI_c^\varsigma f(y) = \frac{1}{\Gamma(1+\varsigma)} \int_b^c f(\ell)(d\ell)^\varsigma = \frac{1}{\Gamma(1+\varsigma)} \lim_{\Delta\ell\to 0} \sum_{f=0}^{N-1} f(\ell_f)(\Delta\ell_f)^\varsigma,$$

Where $c = \ell_0 < \ell_1 < ... < \ell_{N-1} < \ell_N = d, [\ell_f, \ell_{f+1}]$ is partition of $[c,d]$, $\Delta\ell_f = \Delta\ell_{f+1} - \Delta\ell_f, \Delta\ell = max\{\ell_0, \ell_1 ... \ell_{N-1}\}$.

Note that ${}_cI_c^\varsigma f(y) = 0$ and ${}_cI_d^\varsigma f(y) = -{}_dI_c^\varsigma f(y)$ if $c < d$. We denote $f(y) \in I_y^\varsigma[c,d]$ if there exists${}_bI_y^\varsigma f(y)$ for any $y \in [b,c]$.
Lets define a couple of identities from the source [31, 32]

Definition 4. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}^\zeta$. If the following inequality*

$$f(\lambda x_1 + (1-\lambda)x_2) \le \lambda^\zeta f(x_1) + (1-\lambda)^\zeta f(x_2) \tag{1.1}$$

holds for any $x_1, x_2 \in I$ and $\lambda \in [0,1]$, then f is said to be a generalised convex function on I. If the inequality in is reversed, then f is called a generalised concave function on I.

Lemma 1. *(1) Let $g(y) = f^{(\varsigma)}(y) \in C_\varsigma[c,d]$, then*

$${}_bI_c^\varsigma g(y) = f(d) - f(c)$$

(2) Let $g(y), f(y) \in D_\varsigma[c,d]$ and $g^{(\varsigma)}(y), f^{(\varsigma)}(y) \in C_\varsigma[b,c]$, then

$${}_bI_c^\varsigma g(y) f^{(\varsigma)}(y) = g(y)f(y)|_b^c - {}_bI_c^\varsigma g^{(\varsigma)}(y) f(y).$$

Lemma 2.

$$\frac{d^{\varsigma}y^{s\varsigma}}{du^{\varsigma}}=\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s-1)\varsigma)}y^{(s-1)\varsigma};$$

$$\frac{1}{\Gamma(\varsigma+1)}\int_{b}^{c}y^{s\varsigma}(du)^{\varsigma}=\frac{\Gamma(1+s\varsigma)}{\Gamma(1+(s+1)\varsigma)}(d^{(s+1)\varsigma}-c^{(s+1)\varsigma}),\quad s>0$$

Lemma 3.

$$_{b}I_{c}^{\varsigma}1^{\varsigma}=\frac{(d-c)^{\varsigma}}{\Gamma(1+\varsigma)}$$

Lemma 4. *(Generalized Hölder's inequality) Let* $p,q>1$ *with* $p^{-1}+q^{-1}=1$*, let* $g(w),f(w)\in C_{\varsigma}[c,d]$ *,Then*

$$\frac{1}{\Gamma(\varsigma+1)}\int_{b}^{c}\left|g(y)f(y)\right|(dy)^{\varsigma}\leq\left(\frac{1}{\Gamma(\varsigma+1)}\int_{b}^{c}\left|g(y)\right|^{p}(dy)^{\varsigma}\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(\varsigma+1)}\int_{b}^{c}\left|f(y)\right|^{q}(dy)^{\varsigma}\right)^{\frac{1}{q}}$$

Recall generalized beta function:

$$B_{\varsigma}(y,x)=\frac{1}{\Gamma(1+\varsigma)}\int_{0}^{1}\ell^{(y-1)\varsigma}(1-\ell)^{(x-1)\varsigma}(d\ell)^{\varsigma},\quad y>0,x>0 \tag{1.2}$$

Local fractional theory has solid applications in control theory, communication engineering, random walk process and Physics [33, 34, 35, 36]. Many researchers studied various types of integral inequalities for generalized definitions of convexity on fractal sets (see [37, 38, 39, 40, 41] are references therein). Recently, in [3], Saud and co introduces generalized (h,m)-preinvex functions, extending convexity concepts to fractal sets and deriving novel Hermite-Hadamard-type inequalities. The work significantly broadens prior research, offering new local fractional integral inequalities and practical applications in midpoint, trapezoidal, and Simpson-type inequality generalizations.

Let $f:I\to R$, where $I\subseteq R$ is an interval, be a differentiable mapping in I^{o}(the interior of I^{o}) and let $a,b\in I^{o}$ with $a\leq b$. If $|f^{'}(t)|\leq M$, for all $t\in[a,b]$, then the following inequality holds

$$\left|f(t)-\frac{1}{b-a}\int_{a}^{b}f(\tau)d\tau\right|\leq\frac{M}{b-a}\left[\frac{(t-b)^{2}+(t-a)^{2}}{2}\right]$$

for all $t\in[a,b]$. This inequality is known in the literature as the Ostrowski inequality (see [9]). For the research and extension concerning Owstrowski inequality, we can refer the Reference [1, 6, 17, 23] and cited in.

For the sake of tackling non-differentiable functions known as Cantor sets, Yang[26, 27] presented local fractional calculus on Yang's fractional set and exemplified the theory comprehensively. It can elaborate the aspect of continuous but differentiable functions can't be explained through this method. Thus, such topics have gained much popularity for the ones who are researching in the fields of mathematical physics and applied sciences[28, 29, 30, 25]. Yang's theory of fractional sets and local fractional calculus has opened various new horizons for the researchers who have further expanded the convexity on fractional set. For instance in [13] Mo et al. introduced generalized convex function on Yang's fractional sets. Mo and sui also presented s-convex functions in [12]. Similarly, Sun in [19] established generalized harmonically convex functions and further in [20] explained generalized harmonically s-convex functions. Moreover, Du et al. in [7] proposed generalized m-convexity on fractal sets and and studied related integral inequalities. For more results we refer the readers to [2, 4, 7, 8, 10, 11, 14, 15, 16, 21, 22, 24] and further reference therein. Ostrowski [6] established an integral inequality which is now classical and given in Theorem 1.

Lemma 5. [26, 31] *The subsequent equations are valid:*

- *(Local fractional derivative of order ζ)* $\frac{d^{\zeta}x^{k\zeta}}{dx^{\zeta}} = \frac{\Gamma(1+k\zeta)}{\Gamma^(1+(k-1)\zeta)x^{(k-1)\zeta}}(k \in \mathbb{R})$;
- *(Local fractional integration corresponds to anti-differentiation) Suppose* $f = g^{(\zeta)} \in C_{\zeta}[\mu_1, \mu_2]$. *Consequently, we have* ${}_{\mu_1}I_{\mu_2}^{(\zeta)} f = g(\mu_2) - g(\mu_1)$.
- *(Local fractional integration by parts) Assuming* $f, g \in D_{\zeta}[\mu_1, \mu_2]$ *as well as* $f^{(\mu_1)}, g^{(\mu_1)} \in C_{\zeta}[\mu_1, \mu_2]$.*The expression is given by* ${}_{\mu_1}I_{\mu_2}^{(\zeta)}\left(fg^{(\zeta)}\right) g|_{\mu_1}^{\mu_2} - {}_{\mu_1}I_{\mu_2}^{(\zeta)}\left(f^{(\zeta)}g\right)$;
- *(Definite integrals of $x^{k\zeta}$ using local fractional calculus)* $\frac{1}{\Gamma(1+\zeta)}\int_{\mu_1}^{\mu_2} x^{k\zeta}(dx)^{\zeta} = \frac{\Gamma(1+k\zeta)}{\Gamma(1+(k+1)\zeta)}\left(\mu_2^{(k+1)\zeta} - \mu_1^{(k+1)\mu_1}\right)(k \in \mathbb{R})$.

Ostrowski [6] *Introduced μ_1 classical integral inequality, as presented and established in Theorem 1.*

Theorem 1. *Let $f : [a,b] \to \mathbb{R}$ be a differentiable function whose derivative $f' : [a,b] \longrightarrow \mathbb{R}$ is bounded on (a,b),i.e., $\|f'\|_{\infty}=sup_{t\in(a,b)}|f'(t)| \leq \infty$.Then the following inequality holds true:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{(x - \frac{a+b}{2})^2}{(b-a)^2} \right](b-a)\|f'\|_{\infty} \tag{1.3}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is best possible.

Inequality 1.3 has attracted significant attention from mathematicians and other researchers, primarily due to its broad and diverse applications in fields such as numerical analysis and the theory of certain special means [9]. Recently, Sarikaya and Budak [18] derived a generalised Ostrowski inequality for local fractional integrals, which is presented in Theorem 2

Theorem 2. *Let $I \subseteq \mathbb{R}$ be 1n interval, $f : I^o \subseteq \mathbb{R} \to \mathbb{R}^{\zeta}$ (I^o is the interior of I) such that $f^{(\zeta)} \in D_{\zeta} f^{(2\zeta)} \in C_a[a,b]$ for $a,b \in I^o$ with $a \leq b$. Also, assume that. $\|f^{(\zeta)}\|_{\infty} := sup_{t\in[a,b]}|f^{(\zeta)}(t)| \leq \infty$. Then, for all $x \in [a,b]$.*

$$\left| f(x) - \frac{\Gamma(1+\zeta)}{(b-a)^{\zeta}} {}_aI_b^{(\zeta)} f \right| \leq 2^{\zeta} \frac{\Gamma(1+\zeta)}{\Gamma(1+2\zeta)} \left[\frac{1}{4^{\zeta}} + \left(\frac{x - \frac{a+b}{2}}{b-a} \right)^{2\zeta} \right] (b-a)^{\zeta} \|f^{(\zeta)}\|_{\infty} \tag{1.4}$$

2. The Results

To derive additional inequalities of the generalised Ostrowski type for twice locally fractional differentiable functions, we first introduce a function along with its integral representation for such functions, as stated in the following lemma.

Lemma 6. *Let $I \subseteq \mathbb{R}$ be an interval $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}^{\zeta}$ (where I° is the interior of I), such that $f, f^{(\zeta)} \in D_{\zeta}(I^{\circ})$ and $f^{(2\zeta)} \in C_{\zeta}[a,b]$ for $a,b \in I^{\circ}$ with $a < b$. Then the following equality holds true: For any $x \in \left[\frac{a+b}{2}, b\right]$,*

$$L(\zeta;a,b;x) = \frac{\eta(b,a)^{2\zeta}}{\Gamma(1+\zeta)\Gamma(1+2\zeta)} \int_0^1 k(t) f^{(2\zeta)}(a+t\eta(b,a))(dt)^{\zeta}, \tag{2.5}$$

The function $k(t)$ is defined as follows:

$$k(t) = \begin{cases} t^{2\zeta} & if\, 0 \leq t \leq \frac{x-a}{\eta(b,a)}, \\ \left(t-\frac{1}{2}\right)^{2\zeta} & if\, \frac{x-a}{\eta(b,a)} < t < \frac{b-x}{\eta(b,a)}, \\ (t-1)^{2\zeta} & if\, \frac{b-x}{\eta(b,a)} \leq t \leq 1. \end{cases} \tag{2.6}$$

Proof. Let I represent the following integral:

$$I := \frac{1}{\Gamma(1+\zeta)} \int_0^1 k(t) f^{(2\zeta)}(a+t\eta(b,a))(dt)^{\zeta}.$$

Therefore, considering $k(t)$, we have

$$I = I_1 + I_2 + I_3, \tag{2.7}$$

where

$$I_1 := \frac{1}{\Gamma(1+\zeta)} \int_0^{\frac{x-a}{\eta(b,a)}} t^{2\zeta} f^{(2\zeta)}(a+t\eta(b,a))(dt)^{\zeta}.$$

and

$$I_2 := \frac{1}{\Gamma(1+\zeta)} \int_{\frac{b-x}{\eta(b,a)}}^{\frac{x-a}{\eta(b,a)}} \left(t-\frac{1}{2}\right)^{2\zeta} f^{(2\zeta)}(a+t\eta(b,a))(dt)^{\zeta}.$$

$$I_3 := \frac{1}{\Gamma(1+\zeta)} \int_{\frac{b-x}{\eta(b,a)}}^{1} (t-1)^{2\zeta} f^{(2\zeta)}(a+t\eta(b,a))(dt)^{\zeta}.$$

Using local fractional integration by parts, we get

$$I_1 := \frac{1}{\Gamma(1+\zeta)} \int_0^{\frac{x-a}{\eta(b,a)}} t^{2\zeta} f^{(2\zeta)}(a+t\eta(b,a))(dt)^{\zeta}.$$

Let's denote $u(t) = t^{(2\zeta)}$ and $v(t) = f(a+t\eta(b,a))$. We need to compute their local fractional derivatives and then apply the integration by parts formula. First, compute the local fractional derivative of $u(t)$:

$$u^{(\zeta)}(t) = (t^{2\zeta})^{\zeta}$$

Applying the rule for functional derivatives of power functions, we have:

$$(t^{2\zeta})^{\zeta} = \frac{\Gamma(2\zeta+1)}{\Gamma(2\zeta+1-\zeta)} t^{(2\zeta+1)} = \frac{\Gamma(2\zeta+1)}{\Gamma(\zeta+1)} t^{(\zeta)}$$

Now the local derivative of $v(t)$.

$$v^{(\zeta)}(t) = (f(a+t\eta(b,a)))^{(d)} = \eta(b,a)^d f^{(\zeta)}(a+t\eta(b,a)).$$

Now apply the local fractional integration by parts

$$I = u(t)v(t)\Big|_0^{\frac{x-a}{\eta(b,a)}} - \int_0^{\frac{x-a}{\eta(b,a)}} u(t)v(\alpha)(dt)^{\alpha}$$

substitute $u(t)$ and $v(t)$.

$$I_1 = t^{2\zeta} f(a+t\eta(b,a))\Big|_0^{\frac{x-a}{\eta(b,a)}} - \int_0^{\frac{x-a}{\eta(b,a)}} t^{2\zeta} (\eta(b,a)^{\zeta} f^{(\zeta)}(a+t\eta(b,a)))(dt)^{\zeta}.$$

Evaluating the boundary term.

$$t^{2\zeta} f(a+t\eta(b,a))\Big|_0^{\frac{x-a}{\eta(b,a)}} = \big(\frac{x-a}{\eta(b,a)}\big)^{2\zeta} f(x) - 0 = \big(\frac{x-a}{\eta(b,a)}\big)^{2\zeta} f(x).$$

The integral term becomes

$$-\eta(b,a)^{\zeta} \int_0^{\frac{x-a}{\eta(b,a)}} t^{2\zeta} f^{(\zeta)}(a+t\eta(b,a))(dt)^{\zeta}.$$

Combining the results we get:

$$I_1 = \big(\frac{x-a}{\eta(b,a)}\big)^{2\zeta} f(x) - \eta(b,a)^{\zeta} \int_0^{\frac{x-a}{\eta(b,a)}} t^{2\zeta} f^{(\zeta)}(a+t\eta(b,a))(dt)^{\zeta}. \tag{2.8}$$

Similarly, we have

(2.9)

$$I_2 = \big(\frac{x-a}{\eta(b,a)} - \frac{1}{2}\big)^{2\zeta} f(b) - \big(\frac{b-x}{\eta(b,a)} - \frac{1}{2}\big)^{2\zeta} f(a) - 2\zeta \int_{\frac{b-x}{\eta(b,a)}}^{\frac{x-a}{\eta(b,a)}} f(a+t\eta(b,a))\big(t-\frac{1}{2}\big)^{2\zeta-1}(dt)^{\zeta}.$$

$$I_3 = \frac{1}{\Gamma(1+\zeta)}\big[f(a+\eta(b,a)) - \big(\frac{b-x}{\eta(b,a)} - 1\big)^{2\zeta} f(a+\frac{b-x}{\eta(b,a)}\eta(b,a))\big] -$$

$$\frac{\eta(b,a)^{\zeta}}{\Gamma(1+\zeta)} \int_{\frac{b-x}{\eta(b,a)}}^{1} (t-1)^{2\zeta} f^{(\zeta)}(a+t\eta(b,a))(dt)^{\zeta}. \tag{2.10}$$

From (2.8), (2.9) and (2.10) we observe that

$$I = \left(\frac{1}{\eta(b,a)}\right)^{2\zeta}\left[(x-a)^{2\zeta} f(x) + \left((x-a) - \frac{\eta(b,a)}{2}\right)^{2\zeta} f(b) - \left((b-x) - \frac{\eta(b,a)}{2}\right)^{2\zeta} f(a)\right]$$

$$+\frac{1}{\Gamma(1+\zeta)}\left[f(a+\eta(b,a)) - \left(\frac{b-x}{\eta(b,a)} - 1\right)^{2\zeta} f(a+(b-x))\right]$$

$$-\eta(b,a)^{\zeta}\left[\int_0^{\frac{x-a}{\eta(b,a)}} t^{2\zeta} f^{(\alpha)}(a+t\eta(b,a))\,(dt)^{\nu} + 2\zeta\int_{\frac{b-x}{\eta(b,a)}}^{\frac{x-a}{\eta(b,a)}} f(a+t\eta(b,a))\left(t-\frac{1}{2}\right)^{2\zeta-1}(dt)^{\zeta}\right.$$

(2.11)

$$\left.+\frac{1}{\Gamma(1+\zeta)}\int_{\frac{b-x}{\eta(b,a)}}^{1} (t-1)^{2\zeta} f^{(\zeta)}(a+t\eta(b,a))\,(dt)^{\zeta}\right].$$

Finally, by substituting the variable $u = a+t\eta(b,a)$ ($t\in[0,1]$) into Equation (2.11) and multiplying both sides of the resulting identity by $\frac{\eta(b,a)^{2\alpha}}{\Gamma(1+2\zeta)}$, we can get the desired equality (2.5).

Theorem 3. *Let $I \subseteq \mathbb{R}$ be an interval $f : I^\circ \subseteq \mathbb{R} \to \mathbb{R}^\zeta$ (where I° is the interior of I) such that $f, f^{(\zeta)} \in D_\zeta(I^\circ)$, and $f^{(2\zeta)} \in C_\zeta[a,b]$ for $a,b \in I^\circ$ with $a<b$. Also, assume that*

$$\|f^{(2\zeta)}\|_\infty := \sup_{t\in[a,b]} |f^{(2\zeta)}(t)| < \infty.$$

in that case, the following inequality holds: For any $x \in \left[\frac{a+b}{2}, b\right]$,

$$|L(\zeta;a,b;x)| \leq \frac{\|f^{(2\zeta)}\|_\infty}{\Gamma(1+3\alpha)\eta(b,a)^\zeta}\left[2^\zeta(x-a)^{3\zeta} + (2x-a-b)^{3\zeta}\frac{1}{4^\zeta}\right], \tag{2.12}$$

Proof. Let $L := L(\zeta;a,b;x)$. As a result, we obtain:

$$\begin{aligned} |L| &\leq \frac{\eta(b,a)^{2\zeta}}{\Gamma(1+\zeta)\Gamma(1+2\zeta)}\int_0^1 |k(t)|\left|f^{(2\zeta)}(a+t\eta(b,a))\right|(dt)^\zeta \\ &\leq \frac{\eta(b,a)^{2\zeta}\|f(2\zeta)\|_\infty}{\Gamma(1+\zeta)\Gamma(1+2\zeta)}\int_0^1 |k(x,t)|(dt)^\zeta, \end{aligned}$$

Using (2.6), we get

$$\begin{aligned} |L| \leq & \frac{\eta(b,a)^{2\zeta}\|f^{(2\zeta)}\|_\infty}{\Gamma(1+2\zeta)} \times \left\{ \frac{1}{\Gamma(1+\alpha)}\int_0^{\frac{b-x}{b-a}} t^{2\zeta}(dt)^\zeta + \frac{1}{\Gamma(1+\zeta)}\int_{\frac{b-x}{\eta(b,a)}}^{\frac{x-a}{\eta(b,a)}} (t-\tfrac{1}{2})^{2\zeta}(dt)^\zeta \right. \\ & \left. + \frac{1}{\Gamma(1+\zeta)}\int_{\frac{b-x}{\eta(b,a)}}^1 (t-1)^{2\zeta}(dt)^\zeta \right\}. \end{aligned} \tag{2.13}$$

Using Lemma 5, we have

$$\frac{1}{\Gamma(1+\zeta)}\int_0^{\frac{x-a}{\eta(b,a)}} t^{2\zeta}(dt)^\zeta = \frac{\Gamma(1+2\zeta)}{\Gamma(1+3\zeta)}\left(\frac{x-a}{\eta(b,a)}\right)^{3\zeta}. \tag{2.14}$$

$$\frac{1}{\Gamma(1+\zeta)}\int_{\frac{b-x}{\eta(b,a)}}^{\frac{x-a}{\eta(b,a)}} (t-\tfrac{1}{2})^{2\zeta}(dt)^\zeta = \frac{\Gamma(1+2\zeta)}{4^\alpha\Gamma(1+3\zeta)}\left(\frac{2x-a-b}{\eta(b,a)}\right)^{3\zeta}. \tag{2.15}$$

$$\begin{aligned} \frac{1}{\Gamma(1+\zeta)}\int_{\frac{b-x}{\eta(b,a)}}^1 (t-1)^{2\zeta}(dt)^\zeta &= \frac{1}{\Gamma(1+\zeta)}\int_0^{\frac{x-a}{\eta(b,a)}} u^{2\zeta}(du)^\zeta \\ &= \frac{\Gamma(1+2\zeta)}{\Gamma(1+3\zeta)}\left(\frac{x-a}{\eta(b,a)}\right)^{3\zeta}. \end{aligned} \tag{2.16}$$

Finally, it is evident that substituting (2.14), (2.15), and (2.16) into (2.13) directly leads to the desired inequality (2.12).

Substituting $x = (a+b)/2$ into Theorem 3 results in an intriguing inequality related to the local fractional integral, as stated in the following corollary.

Corollary 1. *Under the conditions stated in Theorem 3, the following inequality is true:*

$$\left|\frac{1}{\eta(b,a)^{\zeta}}aI_b^{\zeta}f-\frac{2^{\zeta}}{\Gamma(1+\zeta)\Gamma(1+2\zeta)}f\left(\frac{a+b}{2}\right)\right|\leq\frac{\|f^{(2\zeta)}\|_{\infty}\eta(b,a)^{2\zeta}}{4^{\zeta}\Gamma(1+3\zeta)}.$$

Theorem 4. *Let $I\subseteq\mathbb{R}$ be an interval, and let $f:I^{\circ}\subseteq\mathbb{R}\to\mathbb{R}^{\zeta}$ (where I° represents the interior of I) such that f and $f^{(\zeta)}\in D_{\zeta}(I^{\circ})$, and $f^{(2\zeta)}\in C_{\zeta}[a,b]$ for $a,b\in I^{\circ}$ with $a<b$. If $|f^{(2\zeta)}|$ is generalized convex, the ensuing inequality is true for all $x\in\left[\frac{a+b}{2},b\right]$:*

$$|L(\zeta;a,b;x)|\leq\frac{\eta(b,a)^{2\zeta}}{\Gamma(1+2\zeta)}[K_{\zeta}(x;a,b)+L_{\zeta}(x;a,b)+M_{\zeta}(x;a,b)], \tag{2.17}$$

$$K_{\zeta}(x;a,b):=\frac{\Gamma(1+3\zeta)}{\Gamma(1+4\zeta)}\left(\frac{x-a}{\eta(b,a)}\right)^{4\zeta}\left|f^{(2\zeta)}(a)\right|+$$
$$\left[\frac{\Gamma(1+2\zeta)}{\Gamma(1+3\zeta)}\left(\frac{x-a}{\eta(b,a)}\right)^{3\zeta}-\frac{\Gamma(1+3\zeta)}{\Gamma(1+4\zeta)}\left(\frac{x-a}{\eta(b,a)}\right)^{4\zeta}\right]\left|f^{(2\zeta)}(b)\right|. \tag{2.18}$$

$$L_{\zeta}(x;a,b):=C_{\zeta}(x;a,b)\left|\left|f^{(2\zeta)}(a)\right|-D_{\zeta}(x;a,b)\left|f^{(2\zeta)}(b)\right|\right|. \tag{2.19}$$

$$C_{\zeta}(x;a,b):=\frac{\Gamma(1+3\zeta)}{\Gamma(1+4\zeta)}\left(\frac{(x-a)^{4\zeta}-(b-x)^{4\zeta}}{\eta(b,a)^{4\zeta}}\right)-$$
$$\frac{\Gamma(1+2\zeta)}{\Gamma(1+3\zeta)}\left(\frac{(x-a)^{3\zeta}-(b-x)^{3\zeta}}{\eta(b,a)^{3\zeta}}\right)+\frac{\Gamma(1+\zeta)}{4^{\zeta}\Gamma(1+2\zeta)}\left(\frac{(x-a)^{2\zeta}-(b-x)^{2\zeta}}{\eta(b,a)^{2\zeta}}\right).$$

and

$$D_{\zeta}(x;a,b):=\frac{\Gamma(1+3\zeta)}{\Gamma(1+4\zeta)}\left(\frac{(x-a)^{4\zeta}-(b-x)^{4\zeta}}{\eta(b,a)^{4\zeta}}\right)-2^{\zeta}\frac{\Gamma(1+2\zeta)}{\Gamma(1+3\zeta)}\left(\frac{(x-a)^{3\zeta}-(b-x)^{3\zeta}}{\eta(b,a)^{3\zeta}}\right)+$$
$$\left(\frac{5}{4}\right)^{v}\frac{\Gamma(1+\zeta)}{\Gamma(1+2\zeta)}\left(\frac{(x-a)^{2\zeta}-(b-x)^{2\zeta}}{\eta(b,a)^{2\zeta}}\right)-\frac{1}{4^{\zeta}\Gamma(1+\zeta)}\frac{(2x-b-a)^{\zeta}}{\eta(b,a)^{\zeta}}.$$

and

$$M_{\zeta}(x;a,b):=\left[\frac{\Gamma(1+3\zeta)}{\Gamma(1+4\zeta)}\left(\frac{\eta(b,a)^{4\zeta}-(b-x)^{4\zeta}}{\eta(b,a)^{4\zeta}}\right)-\frac{2^{\zeta}\Gamma(1+2\zeta)}{\Gamma(1+3\zeta)}\left(\frac{\eta(b,a)^{3\zeta}-(b-x)^{3\zeta}}{\eta(b,a)^{3\zeta}}\right)\right.$$
$$\left.+\frac{\Gamma(1+\zeta)}{\Gamma(1+2\zeta)}\left(\frac{b+a-2x}{\eta(b,a)}\right)^{\zeta}\right]\left|f^{(2\zeta)}(a)\right|+\frac{\Gamma(1+3\zeta)}{\Gamma(1+4\zeta)}\left(\frac{x-a}{\eta(b,a)}\right)^{4\zeta}\left|f^{(2\zeta)}(b)\right|. \tag{2.20}$$

Proof. As demonstrated in the proof of Theorem 4, let $L := L(\zeta;a,b;x)$ in (2.5).Then, by examining $k(t)$, we have

$$|L| \leq \frac{\eta(b,a)^{2\zeta}}{\Gamma(1+2\zeta)} \frac{1}{\Gamma(1+\zeta)} \int_0^1 |k(t)| \left| f^{(2\zeta)}(a+t\eta(b,a)) \right| (dt)^{\zeta}. \tag{2.21}$$

$$\leq \frac{\eta(b,a)^{2\zeta}}{\Gamma(1+2\zeta)} (H_1 + H_2 + H_3),$$

where

$$H_1 := \frac{1}{\Gamma(1+\zeta)} \int_0^{\frac{x-a}{\eta(b,a)}} t^{2\zeta} \left| f^{(2\zeta)}(a+t\eta(b,a)) \right| (dt)^{\zeta},$$

$$H_2 := \frac{1}{\Gamma(1+\zeta)} \int_{\frac{b-x}{\eta(b,a)}}^{\frac{x-a}{\eta(b,a)}} \left(t-\frac{1}{2}\right)^{2\zeta} \left| f^{(2\zeta)}(a+t\eta(b,a)) \right| (dt)^{\zeta},$$

$$H_3 := \frac{1}{\Gamma(1+\zeta)} \int_{\frac{b-x}{\eta(b,a)}}^{1} (t-1)^{2\zeta} \left| f^{(2\zeta)}(a+t\eta(b,a)) \right| (dt)^{\zeta}.$$

By utilizing the generalized convexity of $|f^{(2\zeta)}|$ (as defined in Definition 4 and applying Lemma 5 to evaluate the local fractional integrals of the relevant powers, we obtain:

$$H1 \leq \frac{1}{\Gamma(1+\zeta)} \int_0^{\frac{x-a}{\eta(b,a)}} \left[\left| f^{(2\zeta)}(a) \right| + t^{2\zeta} \eta(b,a)^{\zeta} \left| f^{(2\zeta)} \right| \right] (dt)^{\zeta} = K_{\zeta}(x;a,b) \tag{2.22}$$

(2.23)

$$H2 \leq \frac{1}{\Gamma(1+\zeta)} \int_{\frac{b-x}{\eta(b,a)}}^{\frac{x-a}{\eta(b,a)}} \left[(t-\frac{1}{2})^{2\zeta} \left| f^{(2\zeta)}(a) \right| + \eta(b,a)^{\zeta} (t-\frac{1}{2})^{2\zeta} \left| f^{(2\zeta)} \right| \right] (dt)^{\zeta} = L_{\zeta}(x;a,b),$$

and

(2.24)

$$H3 \leq \frac{1}{\Gamma(1+\zeta)} \int_{\frac{b-x}{\eta(b,a)}}^{1} (t-1)^{2\zeta} \left| f^{(2\zeta)}(a) \right| + \eta(b,a)^{\zeta} (t-1)^{2\zeta} \left| f^{(2\zeta)} \right| (dt)^{\zeta} = M_{\zeta}(x;a,b),$$

At last, by plugging (2.22), (2.23), and (2.24) into (2.21), we acquire the inequality in question.

Theorem 5. *Let $I \subseteq \mathbb{R}$ be an interval $f : I^{\circ} \subseteq \mathbb{R} \to \mathbb{R}^{\zeta}$ (I° is the interior of I) such that $f, f^{(\zeta)} \in D_{\zeta}(I^{\circ})$, and $f^{(2\zeta)} \in C_{\zeta}[a,b]$ for $a,b \in I^{\circ}$ with $a < b$. Also let $p,q \in \mathbb{R}$ with $p,q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$. If $|f^{(2\zeta)}|^q$ is generalized convex, then the subsequent inequality is valid for any $x \in \left[\frac{a+b}{2}, b\right]$:*

$$|L(\zeta;a,b;x)| \leq \frac{\eta(b,a)^{2\zeta}}{\Gamma(1+2\zeta)} \frac{1}{\left(2^{\zeta}\Gamma(1+\zeta)\right)^{\frac{1}{q}}} \left(\frac{\Gamma(1+2p\zeta)}{\Gamma(1+(2p+1)\zeta)} \right)^{\frac{1}{p}} J_{\zeta}(x;a,b;p,q), \tag{2.25}$$

Proof. We find from (2.21) that

(2.26)
$$|L| \leq \frac{\eta(b,a)^{2\zeta}}{\Gamma(1+2\zeta)} \frac{1}{\Gamma(1+\zeta)} \int_0^1 |k(t)| \left| f^{(2\zeta)}(a+t\eta(b,a)) \right| (dt)^\zeta \leq \frac{\eta(b,a)^{2\zeta}}{\Gamma(1+2\zeta)} (H_1+H_2+H_3),$$

Where H_i $(i=1,2,3)$ are described according to (2.21).

By employing H"older's inequality for the local fractional integral, as described in Lemma 5 on H_i $(i=1,2,3)$, we get:

(2.27)
$$H_1 \leq \left(\frac{1}{\Gamma(1+\zeta)} \int_0^{\frac{x-a}{\eta(b,a)}} t^{2p\zeta} (dt)^\zeta \right)^{\frac{1}{p}} \times \left(\frac{1}{\Gamma(1+\zeta)} \int_0^{\frac{x-a}{\eta(b,a)}} \left| f^{(2\zeta)}(a+t\eta(b,a)) \right|^q (dt)^\zeta \right)^{\frac{1}{q}}$$

(2.28)
$$H_2 \leq \left(\frac{1}{\Gamma(1+\zeta)} \int_{\frac{b-x}{\eta(b,a)}}^{\frac{x-a}{\eta(b,a)}} \left(t-\frac{1}{2}\right)^{2p\zeta} (dt)^\zeta \right)^{\frac{1}{p}} \times \left(\frac{1}{\Gamma(1+\zeta)} \int_{\frac{b-x}{\eta(b,a)}}^{\frac{x-a}{\eta(b,a)}} \left| f^{(2\zeta)}(a+t\eta(b,a)) \right|^q (dt)^\zeta \right)^{\frac{1}{q}}$$

and

(2.29)
$$H_3 \leq \left(\frac{1}{\Gamma(1+\zeta)} \int_{\frac{b-x}{\eta(b,a)}}^1 (t-1)^{2p\zeta} (dt)^\zeta \right)^{\frac{1}{p}} \times \left(\frac{1}{\Gamma(1+\zeta)} \int_{\frac{b-x}{\eta(b,a)}}^1 \left| f^{(2\zeta)}(a+t\eta(b,a)) \right|^q (dt)^\zeta \right)^{\frac{1}{q}}$$

Here, by applying Lemma 5, we derive:

$$\frac{1}{\Gamma(1+\zeta)} \int_0^{\frac{x-a}{\eta(b,a)}} t^{2p\zeta} (dt)^\zeta = \frac{\Gamma(1+2p\zeta)}{\Gamma(1+(2p+1)\zeta)} \left(\frac{x-a}{\eta(b,a)} \right)^{(2p+1)\zeta} \tag{2.30}$$

(2.31)
$$\frac{1}{\Gamma(1+\zeta)} \int_{\frac{b-x}{\eta(b,a)}}^{\frac{x-a}{\eta(b,a)}} \left(t-\frac{1}{2}\right)^{2p\zeta} dt^\zeta = \frac{\Gamma(1+2p\zeta)}{\Gamma(1+(2p+1)\zeta)} \left[\left(\frac{2x-b-a}{2\eta(b,a)} \right)^{(2p+1)\zeta} - \left(\frac{a+b-2x}{2\eta(b,a)} \right)^{(2p+1)\zeta} \right]$$

and

$$\frac{1}{\Gamma(1+\alpha)} \int_{\frac{b-x}{\eta(b,a)}}^1 (t-1)^{2p\zeta} dt^\zeta = \frac{1}{\Gamma(1+\zeta)} \int_{\frac{b-x}{\eta(b,a)}}^1 (1-t)^{2p\zeta} (dt)^\zeta$$

$$= \frac{1}{\Gamma(1+\zeta)} \int_0^{\frac{x-a}{\eta(b,a)}} u^{2p\zeta} (du)^\zeta \tag{2.32}$$

$$= \frac{\Gamma(1+2p\zeta)}{\Gamma(1+(2p+1)\zeta)} \left(\frac{x-a}{\eta(b,a)} \right)^{(2p+1)\zeta}$$

Moreover, as $\left|f^{(2\zeta)}\right|^q$ exhibits generalized convexity on $[a,b]$, we can apply the generalized Hermite-Hadamard inequality from (1.10) to obtain:

$$\begin{aligned}\int_0^{\frac{x-a}{\eta(b,a)}}\left|f^{(2\zeta)}(a+t\eta(b,a))\right|^q(dt)^\zeta &= \frac{1}{(\eta(b,a))^\zeta}\int_x^b\left|f^{(2\zeta)}(u)\right|^q(du)^\zeta\\ &\leq \frac{\left|f^{(2\zeta)}(b)\right|^q+\left|f^{(2\zeta)}(x)\right|^q}{2^\zeta}\end{aligned}\tag{2.33}$$

$$\int_{\frac{b-x}{\eta(b,a)}}^{\frac{x-a}{\eta(b,a)}}\left|f^{(2\zeta)}(a+t\eta(b,a))\right|^q(dt)^\zeta \leq \frac{\left|f^{(2\zeta)}(x)\right|^q+\left|f^{(2\zeta)}(a+b-x)\right|^q}{2^\zeta}\tag{2.34}$$

and

$$\int_{\frac{x-a}{\eta(b,a)}}^{1}\left|f^{(2\zeta)}(a+t\eta(b,a))\right|^q(dt)^\zeta \leq \frac{\left|f^{(2\zeta)}(a+b-x)\right|^q+\left|f^{(2\zeta)}(a)\right|^q}{2^\zeta}\tag{2.35}$$

By integrating the equalities (2.30)–(2.32) and the inequalities (2.33)–(2.35) into the inequalities (2.27)–(2.29), and subsequently incorporating the obtained inequalities into (2.26), we derive the final inequality (2.25).

Substituting $x=b$ into the aforementioned theorem results in an inequality that includes a local fractional integral, as stated in Corollary 2. To proceed, we must first recall the following inequality:

$$\sum_{k=1}^n(u_k+v_k)^s\leq\sum_{k=1}^n(u_k)^s+\sum_{k=1}^n(v_k)^s \quad (n\in\mathbb{N};\ 0\leq s\leq 1;\ u_k,v_k\geq 0,\ 1\leq k\leq n)\tag{2.36}$$

Corollary 2. *Based on the assumption that the hypothesis holds true of Theorem 5, the following inequality is true:*

$$\begin{aligned}&\left|\frac{1}{(\eta(b,a))^\zeta}{}_aI_b^\zeta f-\frac{f(a)+f(b)}{\Gamma(1+\zeta)\Gamma(1+2\zeta)}+\frac{(\eta(b,a))^\zeta}{8}\frac{f^{(\zeta)}(b)-f^{(\zeta)}(a)}{\Gamma(1+\zeta)}\right|\\ &\leq\frac{(\eta(b,a))^{2\zeta}}{4^\zeta\Gamma(1+2\zeta)}\left\{\frac{\Gamma(1+2p\zeta)}{\Gamma(1+(2p+1)\zeta)}\right\}^{\frac{1}{p}}\left\{\frac{|f^{(2\zeta)}(a)|+|f^{(2\zeta)}(b)|}{(2^\zeta\Gamma(1+\zeta))^{\frac{1}{q}}}\right\}\end{aligned}\tag{2.37}$$

Proof. By considering $x=b$ in the aforementioned Theorem, we assume that L_1 is associated with the left-hand side of the inequality in (2.37)

$$L_1\leq\frac{(\eta(b,a))^{2\zeta}}{\Gamma(1+2\zeta)}\left\{\frac{\Gamma(1+2p\alpha)}{\Gamma(1+(2p+1)\zeta)}\right\}^{\frac{1}{p}}\left(\frac{1^\zeta-(-1)^\zeta}{2^{(2+\frac{1}{p})\zeta}}\right)\times\left\{\frac{|f^{(2\zeta)}(a)|+|f^{(2\zeta)}(b)|}{(2^\zeta\Gamma(1+\zeta))^{\frac{1}{q}}}\right\}\tag{2.38}$$

It is now clear that the expected inequality (2.37) is obtained by fitting the inequality (2.36) to every single of the final two terms in (2.38).

$$f(\frac{a+b}{2}) \leq \frac{\Gamma(1+\alpha)}{\eta(b,a)^{\alpha}} aI_{b}^{(\alpha)} f \leq \frac{f(a)+f(b)}{2^{\alpha}} \tag{2.39}$$

3. Application of Results: Modeling the Spread of Infectious Diseases in Biomedical Engineering

The spread of infectious diseases can be modeled using differential equations to predict the dynamics of infection over time. Traditional models often use integer-order derivatives, but these models may not adequately capture the memory effects and long-range dependencies present in real-world disease spread. Fractional calculus provides a more accurate framework by incorporating these effects. Generalized Ostrowski-type inequalities for fractional integrals can be used to derive bounds on the errors in predicting the spread of infectious diseases, leading to more reliable and accurate models.

Assume that the susceptible, infected, and recovered populations at time t are denoted by $S(t)$, $I(t)$, and $R(t)$ respectively. The following system of fractional differential equations can be used to characterize the fractional SIR (Susceptible-Infected-Recovered) model:

$$D^{\zeta} S(t) = -\beta S(t) I(t)$$

$$D^{\zeta} I(t) = \beta S(t) I(t) - \gamma I(t)$$

$$D^{\zeta} R(t) = \gamma I(t)$$

in which β represents the transfer level, γ represents the recuperation level, and D^{ζ} represents a fractional derivative with order ζ.

Define error functions for each population group as the difference between the actual and predicted values:

$$E_S(t) = S_{actual}(t) - S_{pred}(t)$$

$$E_I(t) = I_{actual}(t) - I_{pred}(t)$$

$$E_R(t) = R_{actual}(t) - R_{pred}(t)$$

Apply the generalized Ostrowski-type inequalities to derive bounds for these error functions. For example:

$$\begin{aligned}
\left|I^{\zeta}\left[E_S(t)\right]\right| &\leq K_S\left|I^{\zeta}\left[S_{actual}(t)-S_{pred}(t)\right]\right| \\
\left|I^{\zeta}\left[E_I(t)\right]\right| &\leq K_I\left|I^{\zeta}\left[I_{actual}(t)-I_{pred}(t)\right]\right| \\
\left|I^{\zeta}\left[E_R(t)\right]\right| &\leq K_R\left|I^{\zeta}\left[R_{actual}(t)-R_{pred}(t)\right]\right|
\end{aligned}$$

where K_S, K_I and K_R are constants dependent on the model parameters.

To minimize the prediction errors $E_S(t), E_I(t)$ and $E_R(t)$, optimize the model parameters (ζ, β and γ). The goal is to choose these parameters such that the bounds on $\left|I^{\zeta}\left[E_S(t)\right]\right|$, $\left|I^{\zeta}\left[E_I(t)\right]\right|$ and $\left|I^{\zeta}\left[E_R(t)\right]\right|$ are minimized.

Perform simulations to validate the theoretical bounds. Use historical data on infectious disease outbreaks to fit the fractional SIR model. Compare the model's predictions with the actual data to evaluate the performance of the model. Measure the prediction error for each population group before and after optimizing the parameters.

Example Simulation: Assume an outbreak of a disease with known transmission and recovery rates. Fit the fractional SIR model to the data and apply the generalized Ostrowski-type inequalities to derive error bounds. Optimize the parameters α, β, and γ to minimize the prediction error.

Calculate the *MAE* and *RMSE* for each population group before and after parameter optimization:

$$\begin{aligned}
MAE_S &= \frac{1}{N}\sum_{i=1}^{N}\left|S_{actual}(t_i)-S_{pred}(t_i)\right| \\
RMSE_I &= \sqrt{\frac{1}{N}\sum_{i=1}^{N}\left(I_{actual}(t_i)-I_{pred}(t_i)\right)^2} \\
MAE_R &= \frac{1}{N}\sum_{i=1}^{N}\left|R_{actual}(t_i)-R_{pred}(t_i)\right|
\end{aligned}$$

By applying generalized Ostrowski-type inequalities in the design and optimization of fractional-order models, one can effectively predict the spread of infectious diseases with higher accuracy. This detailed application demonstrates the practical utility of fractional calculus and generalized inequalities in real-world biomedical engineering tasks, providing a framework for developing more accurate and robust disease spread models.

4. CONCLUSION

In our research, we delve into the concept of a preinvex function in the realm of local fractional calculus. We start by presenting a preinvex function that is closely tied to a generalized Ostrowski-type inequality. Ostrowski-type inequalities are important in mathematical modeling as they have numerous uses, including error estimates for numerical integration and approximation theory, which makes this association essential. By developing an integral representation for this preinvex function, we create a robust mathematical framework that extends the classical Ostrowski inequality to the domain of fractional calculus locally.

Local fractional calculus, a generalization of traditional calculus, is particularly useful for analyzing functions that exhibit fractal or nondifferentiable behavior at certain points. It provides tools for dealing with problems that cannot be addressed adequately by standard calculus. In this context, our introduction of the preinvex function and its integral representation offers new insights and methods for handling such complex functions.

Building upon this foundation, we utilize the preinvex function and its integral representation to derive a series of generalized Ostrowski-type inequalities. These inequalities are tailored for functions that are twice locally fractionally differentiable. This means that the functions possess a fractional order of differentiation, which adds a layer of complexity and precision to the analysis. By focusing on twice-local fractionally differentiable functions, we ensure that our results apply to a wide range of practical problems where such differentiability conditions are met.

The inequalities we establish provide bounds and estimates that are more refined than those available through traditional methods. This is particularly valuable in fields such as signal processing, image analysis, and other areas where fractal and irregular patterns are common. By providing tangible methods for error estimates and function approximation, our study not exclusively broadens the mathematical foundation of local fractional calculus yet additionally improves its practicality.

Our research significantly contributes to the theoretical and practical aspects of local fractional calculus, offering novel approaches and perspectives applicable to a range of complex mathematical challenges.

Accessibility of Data and Resources

Non-Applicable

Conflict of Interests

The authors declare that there is no conflict of interests.

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