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HYERS-ULAM STABILITY CRITERIA FOR THIRD ORDER NONLINEAR DIFFERENTIAL EQUATIONS WITH NONLINEAR DAMPING

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Abstract. This paper is concerned with the Hyers-Ulam stability of third order nonlinear differential equations with nonlinear damping. New criteria are developed to transform the ordinary differential equations under consideration to integral inequalities. By employing the Gronwall-Bellman-Bihari type integral inequality, the stability of ordinary differential equations is proved. Moreover, the Hyers-Ulam constants are established. Lastly, the obtained results are not only new but also included the results stated in [14, 17].

Keywords: Hyers-Ulam stability; nonlinear damping; integral inequality; nonlinear differential equation.

2010 AMS Subject Classification: 26A46, 34C10, 11R33, 35Q31.

1. Introduction

In reality differential equations have become tools for many real life problem in biology, mathematical finance, engineering, medicine and so on. Many notable researchers which we are going to mention in this article have devoted most of their study on the qualitative(stability) of solution of differential equations. In 1940, S.M. Ulam [61] posed the following question concerning the stability of functional equations before the Mathematical Club of the University of Wiscnsin: Give conditions in order for a linear mapping near an approximately linear mapping to exist." Since then, this question has attracted the attention of many researchers. Note that

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the solution to this question was given by Hyers [52] for additive functions defined on Banach Spaces in 1941. Thereafter, the result by Hyers [52] was generalised by Rassias[48], Aoki[6] and Bourgin[8], problem for approximately additive mappings, on Banach spaces,was solved by Hyers [22]. The result obtained by Hyers was generalised by Rassias [48]. After that, many authors have extensively investigated the Ulam problem to other functional equations in various directions, see [3, 9, 10, 18, 19, 20, 23, 26, 30, 31, 41, 42, 45, 49, 50, 51, 55, 60, 62, 63].

The generalization of Ulam's problem was recently proposed by replacing functional equations with differential equations. Obloza seems to be the first author who proved the Ulam stability of differential equation in [43, 44]. Thereafter, Alsina and Ger [5] published their papers,which handles the Hyers-Ulam stability of the linear differential equation $u'(t) = u(t)$. The result obtained by Alsina and Ger was generalized by Takahasi *et.al.*[58] to the case of the complex Banach space valued differential equation. Since then Hyers-Ulam stability of various classes of linear differential equations were investigated using different methods such as direct method, fixed point method, iteration method open mapping theorem and so on (see [1, 2, 21, 25, 27, 28, 29, 32, 33, 34, 35, 36, 37, 38, 40, 43, 44, 56, 57, 58, 59]).

Now a days, the Hyers-Ulam stability of nonlinear ordinary differential equations has been investigated (see [4, 11, 12, 13, 14, 15, 16, 46, 47, 53, 54]), and the investigation is going on. In this paper, we are going to prove the Hyers-Ulam stability of the third order nonlinear differential equations and also obtain the Hyers-Ulam constant of every equation considered. The equations are:

$$\begin{aligned}(\alpha(t)p(u(t))u''(t))' + (\gamma(t)f(u(t))u'(t))' + \beta(t)g(u(t))u'(t)\\ + r(t)\rho(u(t)) = H(t, u(t,)u'(t)),\end{aligned} \tag{1}$$

$$\begin{aligned}(P_1(t, u(t), u'(t))u''(t))' + np(t)P_2(t, u(t), u'(t))u'(t)\\ + Q(t, u(t)) = H(t, u(t,)u'(t)),\end{aligned} \tag{2}$$

with initial conditions

$$u(t_0) = u'(t_0) = u''(t_0) = 0. \tag{3}$$

where $n \in \mathbf{N}$(the set of natural numbers),$u(t) \in C^3(\mathbf{I}, \mathbf{R}_+)$, $P_1(t_0, 0, 0) = 0$, $P_2(t_0, 0, 0) = 0$, $Q(t_0, 0) = 0$, $p, f, g, \rho \in C(\mathbf{R}_+, \mathbf{R}_+)$, $Q \in C(\mathbf{I} \times \mathbf{R}, \mathbf{R})$, $H, P_1, P_2 \in C(\mathbf{I} \times$

$\mathbf{R^2},\mathbf{R}),\beta(t),r(t),\alpha(t),\gamma(t),p(t)\in C(\mathbf{I},\mathbf{R}_+)$, $\mathbf{I}=(0,\infty)$, $\mathbf{R}_+=[0,\infty)$, $\mathbf{R}=(-\infty,\infty)$.

2. Preliminaries

In this section we present some assumptions, definitions, lemmas and theorems to make this paper self-dependent. For convenience, we list the following general assumptions:
let

- i $H(t,u(t),u'(t)) = \phi(t)\varpi(u(t))(u'(t))^n$, where n a positive integer, $\varpi \in C(\mathbf{R}_+,\mathbf{R}_+)$ $\phi(t)\in C(\mathbf{R}_+)$,
- ii $P_1(t,u(t),u'(t)) = h(t)\kappa(u(t))b(u'(t))^4$, where $\kappa(u(t))$, $b(u'(t))\in C(\mathbf{R}_+,\mathbf{R}_+)$, $h(t)\in C(\mathbf{I},\mathbf{R}_+)$,
- iii $P_2(t,u(t),u'(t)) = y(t)v(t)\omega(u(t))(u'(t))^4$ where $\omega(u(t))\in C(\mathbf{R}_+,\mathbf{R}_+)$, $y(t)$, $v(t)\in C(\mathbf{I},\mathbf{R}_+)$, $u'(t)\in C^1(\mathbf{I},\mathbf{R}_+)$
- iv $Q(t,u(t)) = \psi(t)\sigma(u(t))$ where $\psi(t)\in C(\mathbf{R}_+)$, $\sigma\in C(\mathbf{R}_+,\mathbf{R}_+)$

Definition 1. We say that equation (1) has the Hywrs-Ulam stability, if there exists a constant $K_1\geq 0$ with the following property: for every $\varepsilon>0$, $u(t)\in C^3(\mathbf{R}_+)$, if

$$\begin{aligned}|(\alpha(t)p(u(t))u''(t))' + (\gamma(t)f(u(t))u'(t))' + \beta(t)g(u(t))u'(t)\\ +r(t)\rho(u(t)) - H(t,u(t),u'(t))| \leq \varepsilon,\end{aligned} \tag{4}$$

then, there exists some $u_0(t)\in C^3(\mathbf{R}_+)$ such that

$$|u(t)-u_0(t)|\leq K_1\varepsilon,$$

we call such K_1 the Hyers-Ulam constant.

Definition 2. The differential equation (2) has the Hyers-Ulam stability, if there exists a positive constant $K_2\geq 0$ with the following property: for every $\varepsilon>0$, $u(t)\in C^3(\mathbf{R}_+)$, which satisfies

$$\begin{aligned}|(P_1(t,u(t),u'(t))u''(t))' + np(t)P_2(t,u(t),u'(t))u'(t)\\ +Q(t,u(t)) - H(t,u(t),u'(t))| \leq \varepsilon,\end{aligned} \tag{5}$$

then there exists a function $u_0(t)\in C^3(\mathbf{R}_+)$ satisfies (2) with initial condition (3) such that

$$|u(t)-u_0(t)| \le K_2\varepsilon,$$

we call such K_2 a Hyers-Ulam stability for the differential equation (2).

Lemma 1. [7] Let $u(t)$, $f(t)$ be positive continuous functions defined on $t_0 \le t \le b, (\le \infty)$ and $K > 0$, $M \ge 0$, further let $\omega(u)$ be a nonnegative nondecreasing continuous function for $u \ge 0$, then the inequality

$$u(t) \le N + M\int_{t_0}^{t} f(s)\omega(u(s))ds,\;\; t_0 \le t < b, \tag{6}$$

implies the inequality

$$u(t) \le \Omega^{-1}\left(\Omega(N) + M\int_{t_0}^{t} f(s)ds\right),\;\; t_0 \le t \le b' \le b, \tag{7}$$

where

$$\Omega(u) = \int_{u_0}^{u} \frac{dt}{\omega(t)}, \quad 0 < u_0 < u. \tag{8}$$

In the case $\omega(0) > 0$ or $\Omega(0+)$ is finite, one may take $u_0 = 0$ and Ω^{-1} is the inverse function of Ω and t must be in the subinterval $[t_0, b']$ of $[t_0, b]$ such that

$$\Omega(N) + M\int_{t_0}^{t} f(s)ds \in Dom(\Omega^{-1}).$$

Lemma 2. [24] Let $r(t)$ be an integrable function then the n successive integration of r over the interval $[t_0, t]$ is given by

$$\int_{t_0}^{t} \cdots \int_{t_0}^{t} r(s)ds^n = \frac{1}{(n-1)!}\int_{t_0}^{t} (t-s)^{n-1} r(s)ds \tag{9}$$

Theorem 1. [39] If $f(t)$ and $g(t)$ are continuous in $[t_0, t] \subseteq \mathbf{I}$ and $f(t)$ does not change sign in the interval, then there is a point $\xi \in [t_0, t]$ such that $\int_{t_0}^{t} g(s)f(s)ds = g(\xi)\int_{t_0}^{t} f(s)ds$

Theorem 2. [14, 15] Suppose $u(t), r(t), h(t) \in C(\mathbf{I}, \mathbf{R}_+)$ are nonnegative, monotonic, nondecreasing, continuous and $\omega(u)$ be submultiplicative for $u > 0$. Let

$$u(t) \le N + T\int_{t_0}^{t} r(s)\beta(u(s))ds + L\int_{t_0}^{t} h(s)\varpi(u(s))ds \tag{10}$$

for N, T and L positive constants, then

$$u(t) \leq \Omega^{-1}\left(\Omega(K)+L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1)+T\int_{t_0}^{s} r(\alpha)d\alpha\right)\right)ds\right) \quad (11)$$
$$F^{-1}\left(F(1)+T\int_{t_0}^{t} r(s)ds\right)$$

where $\beta(u) \neq \varpi(u)$, Ω is defined in equation (7) and $F(u)$ is defined as

$$F(u)=\int_{u_0}^{u}\frac{ds}{\beta(s)},\ \ 0<u_0\leq u, \quad (12)$$

F^{-1}, Ω^{-1} are the inverses of F, Ω respectively and t is in the subinterval $(0,b) \in \mathbf{I}$ so that

$$F(1)+T\int_{t_0}^{t} r(s)ds \in Dom(F^{-1})$$

and

$$\Omega(N)+L\int_{t_0}^{t} h(s)\varpi\left(F^{-1}\left(F(1)+T\int_{t_0}^{t} r(\alpha)d\alpha\right)\right)ds \in Dom(\Omega^{-1})$$

Theorem 3. [14, 15] If $u(t),r(t),h(t),\rho(t),g(t) \in C(\mathbf{R}_+)$ be nonnegative, monotonic, nondecreasing continuous functions. Let γ be submultiplicative. If

$$u(t) \leq \rho(t)+A\int_{t_0}^{t} r(s)\beta(u(s))ds+B\int_{t_0}^{t} h(s)\varpi(u(s))ds+ \quad (13)$$
$$L\int_{t_0}^{t} g(s)\gamma(u(s))ds$$

for $A,B,L>0$, then

$$u(t) \leq \rho(t)\Upsilon^{-1}$$
$$\left[\Upsilon(1)+L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1)+B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds\right] \quad (14)$$
$$\Omega^{-1}\left(\Omega(1)+B\int_{t_0}^{t} h(s)\varpi\left(T(s)\right)ds\right)T(t)$$

where $T(t)$ is given as

$$T(t)=F^{-1}\left(F(1)+A\int_{t_0}^{t} r(s)ds\right) \quad (15)$$

and

$$\Upsilon(r)=\int_{r_0}^{r}\frac{ds}{\gamma(s)},\quad 0<r_0\leq r, \quad (16)$$

and F^{-1}, Ω^{-1} and Υ^{-1} are the inverses of F, Ω, Υ respectively $t \in (0,b) \subset (I)$. So that

$$\Upsilon(1) + L\int_{t_0}^{t} g(s)\gamma\left[\Omega^{-1}\left(\Omega(1) + B\int_{t_0}^{s} h(\alpha)\varpi\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds \in Dom(\Upsilon^{-1}$$

3. Main Results

In this section, we will state our main results and give their proofs.

Theorem 4. Suppose that

i setting $|u'(t)| \leq \lambda$ where $\lambda > 0$,
ii $\lim_{t_0 \to \infty} \int_{t_0}^{t} |u'(s)|ds = L$, where $L > 0$,
iii $\lim_{t_0 \to \infty} \int_{t_0}^{t} \phi(s)ds \leq n_1 < \infty$, where $n_1 > 0$,
iv $\lim_{t_0 \to \infty} \int_{t_0}^{t} \beta(s)ds \leq n_2 < \infty$, where $n_2 > 0$,
v $\lim_{t_0 \to \infty} \int_{t_0}^{t} r(s)ds \leq n_3 < \infty$, where $n_3 > 0$,
vi $|F(u(t))| \geq |u(t)|$,
vii setting $F(u(t)) = \int_{u(t_0)}^{u(t)} f(u(s)ds < \infty$,

are satisfied. In addition, let $\varpi(u(t))$ be continuous, nondecreasing and monotonic, then equation (1) has the Hyer-Ulam stability with initial conditions (3), if for Hyers-Ulam constant $K_1 \geq 0$ and for each approximate solution $u(t) \in C^3(\mathbf{R}_+)$ of (1) satisfying (4), there exists any solution $u_0(t) \in C^3(\mathbf{R}_+)$ of (1) such that

$$|u(t) - u_0(t)| \leq K_1\varepsilon,$$

thus, the Hyers-Ulam constant is given as

$$K_1 = \frac{L}{\sigma\lambda}\Upsilon^{-1}\left[\Upsilon(1) + \frac{\lambda^n}{\sigma}n_1\varpi\left[\Omega^{-1}\left(\Omega(1) + \frac{1}{\sigma\lambda}n_2\rho\left(T^*\right)\right)T^*\right]\right]$$
$$\Omega^{-1}\left(\Omega(1) + \frac{1}{\sigma\lambda}n_2\rho\left(T^*\right)\right)T^*.$$

Proof. Multiplying inequality(4) by $u'(t)$ to obtain

$$\begin{aligned} -\varepsilon u'(t) &\leq (\alpha(t)p(u(t))u''(t))'u'(t) + (\gamma(t)f(u(t))u'(t))'u'(t) \\ &+\beta(t)g(u(t))(u'(t))^2 + r(t)\rho(u(t))u'(t) - H(t,u(t),u'(t))u'(t) \leq \varepsilon u'(t). \end{aligned} \tag{17}$$

Integrating twice and applying Lemma 2, since $u'(t)$ is differentiable and nondecreasing on $\mathbf{R}_+$, then $u''(t) \geq 0\ \forall t \in \mathbf{I}$, we use this condition on the second term of (17) to have

$$\begin{aligned} -\varepsilon \int_{t_0}^t u'(s)ds \leq \int_{t_0}^t (\gamma(s)f(u(s))u'(s))'u'(s)ds + \int_{t_0}^t \beta(s)g(u(s))(u'(s))^2 ds \\ + \int_{t_0}^t r(s)\rho(u(s))u'(s)ds - \int_{t_0}^t H(s,u(s),u'(s))u'(s)ds \leq \varepsilon \int_{t_0}^t u'(s)ds, \quad t \geq 0. \end{aligned} \tag{18}$$

We apply the assumption (i) to fifth term of (18) and use the mean value Theorem 1 for integrals, that is, there exist $\xi, \nu, \eta, \delta \in [t_0, t]$ such that

$$\begin{aligned} u'(\xi)\int_{t_0}^t (\gamma(s)f(u(s))u'(s))'ds + u'(\nu)^2 \int_{t_0}^t \beta(s)g(u(s))ds \\ +u'(\eta)\int_{t_0}^t r(s)\rho(u(s))ds - u'(\delta)^{n+1}\int_{t_0}^t \phi(t)\varpi(u(t))ds \leq \varepsilon \int_{t_0}^t u'(s)ds. \end{aligned} \tag{19}$$

Employing the condition (vii) of Theorem 4 to the first term of (19), integrating by parts, let $\gamma(t)$ be differentiable on $\mathbf{R}_+$, if $\gamma'(t) \geq 0$ for all $t \in \mathbf{I}$, then $\gamma(t)$ is nondecreasing on $\mathbf{R}_+$, in addition $\gamma'(t) \geq 0$, since $\gamma(t) > 0$ there exists constant $\sigma > 0$ such that $\gamma(t) \geq \sigma$, thus, we have

$$\begin{aligned} u'(\xi)\sigma F(u(s)) \leq \varepsilon \int_{t_0}^t u'(s)ds - u'(\nu)^2 \int_{t_0}^t \beta(s)g(u(s))ds \\ -u'(\eta)\int_{t_0}^t r(s)\rho(u(s))ds + u'(\delta)^{n+1}\int_{t_0}^t \phi(t)\varpi(u(t))ds, \quad \forall t \geq 0. \end{aligned}$$

We apply the conditions (i), (vi), (vii) of Theorem 4 to obtain

$$\begin{aligned} |(u(s))| \leq \frac{L\varepsilon}{\sigma\lambda}L + \frac{\lambda}{\sigma}\int_{t_0}^t \beta(s)g(|u(s)|)ds \\ + \frac{1}{\sigma\lambda}\int_{t_0}^t r(s)\rho(|u(s)|)ds + \frac{\lambda^n}{\sigma}\int_{t_0}^t \phi(t)\varpi(|u(t)|)ds, \forall t \geq 0. \end{aligned}$$

Furthermore, we apply the Theorem 3 to arrive at

$$\begin{aligned} |u(t)| \leq \frac{L\varepsilon}{\sigma\lambda}\Upsilon^{-1}\Bigg[\Upsilon(1) + \frac{\lambda^n}{\sigma} \\ \int_{t_0}^t \phi(s)\varpi\left[\Omega^{-1}\left(\Omega(1) + \frac{1}{\sigma\lambda}\int_{t_0}^s r(\alpha)\rho\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds\Bigg] \\ \Omega^{-1}\left(\Omega(1) + \frac{1}{\sigma\lambda}\int_{t_0}^t r(s)\rho\left(T(s)\right)ds\right)T(t), \end{aligned} \tag{20}$$

for

$$T(t) = F^{-1}\left(F(1) + \frac{\lambda}{\sigma}\int_{t_0}^t \beta(s)ds\right).$$

Applying the conditions (iii)- (v) of Theorem 4, we get

$$|u(t)| \leq \frac{L\varepsilon}{\sigma\lambda}\Upsilon^{-1}\left[\Upsilon(1)+\frac{\lambda^n}{\sigma}n_1\varpi\left[\Omega^{-1}\left(\Omega(1)+\frac{1}{\sigma\lambda}n_2\rho\left(T^*\right)\right)T^*\right]\right] \\ \Omega^{-1}\left(\Omega(1)+\frac{1}{\sigma\lambda}n_2\rho\left(T^*\right)\right)T^*$$

and

$$T^* = F^{-1}\left(F(1)+\frac{\lambda}{\sigma}n_3\right).$$

Hence,

$$|u(t)-u(t_0)| \leq |u(t)| \leq K_1\varepsilon,$$

where

$$K_1 = \frac{L}{\sigma\lambda}\Upsilon^{-1}\left[\Upsilon(1)+\frac{\lambda^n}{\sigma}n_1\varpi\left[\Omega^{-1}\left(\Omega(1)+\frac{1}{\sigma\lambda}n_2\rho\left(T^*\right)\right)T^*\right]\right] \\ \Omega^{-1}\left(\Omega(1)+\frac{1}{\sigma\lambda}n_2\rho\left(T^*\right)\right)T^*.$$

□

Theorem 5. Suppose the assumptions (i)-(iv) and conditions (i), (vii) of Theorem 4 remained satisfied. In addition, let

- i' $\Phi(u(t)) = \int_{u(t_0)}^{u(t)} \sigma(u(s))ds$,
- ii' setting $|\Phi(u(t))| \geq |u(t)|$,
- iii' let $|u''(t)| \leq \rho$ where $\rho > 0$,
- iv' $\lim_{t_0\to\infty}\int_{t_0}^{t} p(s)y(s)v(s)ds \leq n_4 < \infty$, where $n_4 > 0$,
- v' $\lim_{t_0\to\infty}\int_{t_0}^{t} h(s)ds \leq n_5$ where $n_5 > 0$,

hold, then equation (2) is stable in the sense of Hyers-Ulam, if $K_2 \geq 0$ and for each approximate solution $u(t) \in C^3(\mathbf{R}_+)$ satisfying (5), there exists any solution $u_0(t) \in C^3(\mathbf{R}_+)$ of (2) such that

$$|u(t)-u_0(t)| \leq K_2\varepsilon,$$

for Hyers-Ulam constant is given as

$$K_2 = \frac{L}{v}\Upsilon^{-1}\left[\Upsilon(1)+\frac{\lambda^{n+1}}{v}k_1\varpi\left[\Omega^{-1}\left(\Omega(1)+\frac{n\lambda^6}{v}k_4\omega\left(T_2^*\right)\right)T_2^*\right]\right] \\ \Omega^{-1}\left(\Omega(1)+\frac{n\lambda^6}{v}k_4\omega\left(T_2^*\right)\right)T_2^*.$$

Proof. In this case we simplify (5), consider the right hand side of first inequality and multiplying by $u'(t)$ where we apply on the first term the Theorem 1, there exists $\xi \in [t_0, t]$ such that

$$u'(\xi)\int_{t_0}^{t}(P_1(s,u(s),u'(s)u''(s))'ds + n\int_{t_0}^{t}p(s)P_2(s,u(s),u'(s))(u'(s))^2 ds$$
$$+\int_{t_0}^{t}Q(s,u(s))u'(s)ds - \int_{t_0}^{t}H(s,u(s),u'(s))u'(s)ds \le \varepsilon\int_{t_0}^{t}u'(s)ds.$$

Using the assumptions (i)-(iv) to get

$$\begin{aligned}\int_{t_0}^{t}\psi(s)\sigma(u(s))u'(s)ds &\le \varepsilon\int_{t_0}^{t}u'(s)ds - u'(\xi)\int_{t_0}^{t}h(s)\kappa(u(s))b(u'(s))^4u''(s)ds\\ &-n\int_{t_0}^{t}p(s)y(s)v(s)\omega(u(s))(u'(s))^6ds + \int_{t_0}^{t}\phi(s)\varpi(u(s))(u'(s))^{n+1}ds,\ \forall t>0.\end{aligned}\tag{21}$$

Applying the condition (i') of Theorem 5 to the first term of (21) and Theorem 1 there exists $\imath,\ \varepsilon,\ \delta \in [t_0, t]$ such that

$$\begin{aligned}\int_{t_0}^{t}\psi(s)\frac{d}{ds}\Phi(u(s))ds &\le \varepsilon\int_{t_0}^{t}u'(s)ds - u'(\xi)b(u'(\imath)^4)u''(\imath)\int_{t_0}^{t}h(s)\kappa(u(s))ds\\ &-nu'(\varepsilon)^6\int_{t_0}^{t}p(s)y(s)v(s)\omega(u(s))ds + u'(\delta)^{n+1}\int_{t_0}^{t}\phi(s)\varpi(u(s))ds,\ \forall t>0.\end{aligned}\tag{22}$$

Integrate by parts the first term of (22) by letting $\psi(t)$ be nondecreasing function on $\mathbf{R}_+$ implies $\psi'(t) \ge 0$ and using the fact that since $\psi(t) > 0$ there exists a positive constant ν such that $\psi(t) \ge \nu$ with application of the conditions (i), (vii) of Theorem 5 together with the conditions (ii'), (iii') of Theorem 5 to obtain

$$|u(t)| \le \frac{\varepsilon}{\nu}L + \frac{\lambda b(\lambda^4)\rho}{\nu}\int_{t_0}^{t}h(s)\kappa(|u(s)|)ds$$
$$+\frac{n\lambda^6}{\nu}\int_{t_0}^{t}p(s)y(s)v(s)\omega(|u(s)|)ds + \frac{\lambda^{n+1}}{\nu}\int_{t_0}^{t}\phi(s)\varpi(|u(s)|)ds,\ \forall t>0.$$

As before, we apply the Theorem 3 to arrive at

$$\begin{aligned}|u(t)| \le \frac{L\varepsilon}{\nu}\Upsilon^{-1}\Bigg[\Upsilon(1) &+ \frac{\lambda^{n+1}}{\nu}\int_{t_0}^{t}\phi(s)\varpi\left[\Omega^{-1}\left(\Omega(1)\right.\right.\\ &\left.\left.+\frac{n\lambda^6}{\nu}\int_{t_0}^{s}p(\alpha)y(\alpha)v(\alpha)\omega\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds\Bigg]\\ \Omega^{-1}\left(\Omega(1) + \frac{n\lambda^6}{\nu}\int_{t_0}^{t}p(s)y(s)v(s)\omega\left(T(s)\right)ds\right)T(t),&\end{aligned}\tag{23}$$

where

$$T(t) = F^{-1}\left(F(1) + \frac{\lambda b(\lambda^4)\rho}{\nu}\int_{t_0}^{t} h(s)ds\right).$$

Simplifying (23) using conditions (i) of Theorem 4 and (iv'),(v') of Theorem 5 to obtain

$$|u(t)| \leq \frac{L\varepsilon}{\nu}\Upsilon^{-1}\left[\Upsilon(1) + \frac{\lambda^{n+1}}{\nu}n_1\varpi\left[\Omega^{-1}\left(\Omega(1) + \frac{n\lambda^6}{\nu}n_4\omega\left(T_2^*\right)\right)T_2^*\right]\right]$$
$$\Omega^{-1}\left(\Omega(1) + \frac{n\lambda^6}{\nu}n_4\omega\left(T_2^*\right)\right)T_2^*,$$

where

$$T_2^* = F^{-1}\left(F(1) + \frac{\lambda b(\lambda^4)\rho}{\nu}n_5\right).$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq K_2\varepsilon$$

and

$$K_2 = \frac{L}{\nu}\Upsilon^{-1}\left[\Upsilon(1) + \frac{\lambda^{n+1}}{\nu}k_1\varpi\left[\Omega^{-1}\left(\Omega(1) + \frac{n\lambda^6}{\nu}k_4\omega\left(T_2^*\right)\right)T_2^*\right]\right]$$
$$\Omega^{-1}\left(\Omega(1) + \frac{n\lambda^6}{\nu}k_4\omega\left(T_2^*\right)\right)T_2^*.$$

□

Next, we consider the equations (1) and (2) in the following forms:

$$(\alpha(t)p(u(t))u''(t))' + (\gamma(t)f(u(t))u'(t))' + \beta(t)g(u(t))u'(t) + r(t)\rho(u(t)) = 0 \tag{24}$$

and

$$(P_1(t,u(t),u'(t)u''(t))' + np(t)P_2(t,u(t),u'(t))u'(t) + Q(t,u(t)) = 0. \tag{25}$$

where the term $H(t,u(t,)u'(t))$ is replaced by 0

Theorem 6. Supposed the conditions (ii)-(iv) of Theorem 4 and conditions (i'), (iii') of Theorem 5 remain valid. In addition to the above conditions, let the following conditions:

i" $\Lambda(u(t)) = \int_{u(t_0)}^{u(t)} \rho(u(s))ds,$

ii" $\lim_{t_0\to\infty}\int_{t_0}^{t} \gamma(s)ds \leq n_6 < \infty,$ where $n_6 > 0,$

iii" $\lim_{t_0\to\infty}\int_{t_0}^{t} \alpha(s)ds \leq n_7 < \infty,$ where $n_7 > 0.$

hold. Then, equation (24) is Hyers-Ulam stable with Hyera-Ulam constant given as

$$K_3 = \frac{L}{\eta}\Upsilon^{-1}\left[\Upsilon(1) + \frac{\lambda^2}{\eta}n_2 g\left[\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\eta}n_6 f\left(T_3^*\right)\right)\right.\right.$$
$$\left.\left. T_3^*\right]\right]\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\eta}n_6 f\left(T_3^*\right)\right)T_3^*,$$

Proof. We proceed as in Theorem 4 by allowing $H(t,u(t,)u'(t) = 0$ in inequality (4) and using the Theorem 1 there exists $\xi, \nu, \delta \in [t_0,t]$ such that

$$\begin{aligned} &u'(\delta)\int_{t_0}^{t}(\alpha(s)p(u(s))u''(s))'ds + u'(\xi)\int_{t_0}^{t}(\gamma(s)f(u(s))u'(s))'ds \\ &+u'(\nu)^2\int_{t_0}^{t}\beta(s)g(u(s))ds + \int_{t_0}^{t}r(s)\rho(u(s))u'(s)ds \le \varepsilon\int_{t_0}^{t}u'(s)ds. \end{aligned} \tag{26}$$

Integrating again (26) from t_0 to t, using Lemma 2 and condition (i”) we obtain

$$\begin{aligned} &u'(\delta)\int_{t_0}^{t}\alpha(s)p(u(s))u''(s)ds + u'(\xi)\int_{t_0}^{t}\gamma(s)f(u(s))u'(s)ds \\ &+u'(\nu)^2\int_{t_0}^{t}\beta(s)g(u(s))ds + \int_{t_0}^{t}r(s)\frac{d}{ds}\Lambda(u(s)))ds \le \varepsilon\int_{t_0}^{t}u'(s)ds, \quad t \ge 0. \end{aligned} \tag{27}$$

Integrating by parts of the fourth term of (27), since $r(t)$ a nondecreasing on $\mathbf{R}_+$, then $r'(t) \ge 0$ and using the fact that $r(t) > 0$ so $r(t) \ge \eta$ where constant $\eta > 0$, we obtain

$$\begin{aligned} \eta\Phi(u(t)) \le \varepsilon\int_{t_0}^{t}u'(s)ds - u'(\delta)\int_{t_0}^{t}\alpha(s)p(u(s))u''(s)ds \\ -u'(\xi)\int_{t_0}^{t}(\gamma(s)f(u(s))u'(s)ds - u'(\nu)^2\int_{t_0}^{t}\beta(s)g(u(s))ds, \quad \forall t > 0. \end{aligned}$$

Using conditions (ii), (iii) of Theorem 4, (iii’) of Theorem 5 and Theorem 1, there exists $\chi, \varphi \in [t_0,t]$ such that

$$\begin{aligned} |u(t)| \le \frac{\varepsilon L}{\eta} + \frac{\lambda\rho}{\eta}\int_{t_0}^{t}\alpha(s)p(|u(s)|)ds \\ +\frac{\lambda^2}{\eta}\int_{t_0}^{t}(\gamma(s)f(|u(s)|)ds + \frac{\lambda^2}{\eta}\int_{t_0}^{t}\beta(s)g(|u(s)|)ds, \quad \forall t > 0. \end{aligned} \tag{28}$$

As before, we apply the Theorem3 to (28) leads to

$$|u(t)| \le \frac{\varepsilon L}{\eta}\Upsilon^{-1}\left[\Upsilon(1) + \frac{\lambda^2}{\eta}\int_{t_0}^{t}\beta(s)g\left[\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\eta}\int_{t_0}^{s}\gamma(\alpha)f\left(T_3(\alpha)\right)d\alpha\right)\right.\right.$$
$$\left.\left. T_3(s)\right]ds\right]\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\eta}\int_{t_0}^{t}\gamma(s)f\left(T_3(s)\right)ds\right)T_3(t),$$

for

$$T_3(t) = F^{-1}\left(F(1) + \frac{\lambda^2\rho}{\eta}\int_{t_0}^{t}\alpha(s)ds\right).$$

Using conditions (iv) of Theorem 4 and (ii”), (iii”) of Theorem 6 , we arrive at

$$|u(t)| \leq \frac{\varepsilon L}{\eta}\Upsilon^{-1}\left[\Upsilon(1) + \frac{\lambda^2}{\eta}n_2 g\left[\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\eta}n_6 f\left(T_3^*\right)\right)\right.\right.$$
$$\left.\left. T_3^*\right]\right]\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\eta}n_6 f\left(T_3^*\right)\right)T_3^*,$$

and

$$T_3^*(t) = F^{-1}\left(F(1) + \frac{\lambda^2\rho}{\eta}n_7\right).$$

Hence,

$$|u(t) - u(t_0)| \leq |u(t)| \leq K_3\varepsilon,$$

where

$$K_3 = \frac{L}{\eta}\Upsilon^{-1}\left[\Upsilon(1) + \frac{\lambda^2}{\eta}n_2 g\left[\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\eta}n_6 f\left(T_3^*\right)\right)\right.\right.$$
$$\left.\left. T_3^*\right]\right]\Omega^{-1}\left(\Omega(1) + \frac{\lambda^2}{\eta}n_6 f\left(T_3^*\right)\right)T_3^*,$$

□

Theorem 7. Let the assumptions (ii)-(iv), conditions (i) of Theorem 4 and (ii’), (v’) of Theorem 5 hold. Then, the equation (25) has the Hyers-Ulam stability, if $u(t) \in C^3(\mathbf{R}_+)$ is any solution satisfying

$$|(P_1(t,u(t),u'(t)u''(t))' + np(t)P_2(t,u(t),u'(t))u'(t)$$
$$+Q(t,u(t))| \leq \varepsilon, \tag{29}$$

there exists a solution $u_0(t) \in C^3(\mathbf{R}_+)$ of equation (25) such that

$$|u(t) - u(t_0)| \leq |u(t)| \leq K_4\varepsilon,$$

where Hyers-Ulam constant of (25) is given as

$$K_4 = \frac{L}{\delta}\Omega^{-1}\left(\Omega(1) + \frac{n\lambda^6}{\delta}n_5\omega\left(F^{-1}\left(F(1) + \frac{\lambda b(\lambda)^4\rho}{\delta}n_4\right)\right)\right)$$
$$F^{-1}\left(F(1) + \frac{\lambda b(\lambda)^4\rho}{\delta}n_4\right).$$

Proof. We evaluate and simplify the inequality (29) to get

$$\begin{aligned} -\varepsilon\int_{t_0}^{t} u'(s)ds \leq \int_{t_0}^{t}(P_1(s,u(s),u'(s))u'')'u'(s)ds+ \\ n\int_{t_0}^{t} p(s)P_2(s,u(s),u'(s))(u'(s))^2ds+\int_{t_0}^{t} Q(s,u(s))u'(s)ds \leq \varepsilon\int_{t_0}^{t} u'(s)ds. \end{aligned} \tag{30}$$

Using assumptions (i), (ii), (iii) and considering the right hand side of first inequality of (30) in the form

$$\begin{aligned} \int_{t_0}^{t}(h(s)\kappa(u(s))b(u'(s))^4u''(s))'u'(s)ds+n\int_{t_0}^{t} p(s)y(s)v(s)\omega(u(s))(u'(s))^6ds \\ +\int_{t_0}^{t}\psi(s)\sigma(u(s))ds \leq \varepsilon\int_{t_0}^{t} u'(s)ds. \end{aligned} \tag{31}$$

Integrating inequality (31), as before, applying Lemma 2 and Theorem 1 there exists $\xi \in [t_0,t]$ such that

$$\begin{aligned} u'(\xi)\int_{t_0}^{t} h(s)\kappa(u(s))b(u'(s))^4u''(s)ds+n\int_{t_0}^{t} p(s)y(s)v(s)\omega(u(s))(u'(s))^6ds \\ +\int_{t_0}^{t}\psi(s)\sigma(u(s))ds \leq \varepsilon\int_{t_0}^{t} u'(s)ds, \quad t\geq 0. \end{aligned} \tag{32}$$

Now, we use condition (1') of Theorem 6 on the third term of (32) and reapplying mean value Theorem 1 for the integrals to the first and second terms to obtain

$$\begin{aligned} \int_{t_0}^{t}\psi(s)\frac{d}{ds}\Phi(u(s))ds \leq \varepsilon\int_{t_0}^{t} u'(s)ds - u'(\xi)b(u'(\tau))^4u''(\tau)\int_{t_0}^{t} h(s)\kappa(u(s))ds \\ -nu(\mu)^6\int_{t_0}^{t} p(s)y(s)v(s)\omega(u(s))ds, \quad t>0. \end{aligned} \tag{33}$$

where $t_0 \leq \tau \leq t$ and $t_0 \leq \mu \leq t$.

Integrating by parts the first term of (33), simplify further, since $\psi(t)$ a nondecreasing, then $\psi'(t) \geq 0$ and $\psi(t) > 0$ there exists constant $\delta > 0$ such that $\psi \geq \delta$ and using conditions (i) of Theorem 4, (ii') of Theorem 6, we have

$$|u(t)| \leq \frac{L\varepsilon}{\delta} + \frac{\lambda b(\lambda)^4\rho}{\delta}\int_{t_0}^{t} h(s)\kappa(|u(s)|)ds + \frac{n\lambda^6}{\delta}\int_{t_0}^{t} p(s)y(s)v(s)\omega(|u(s|))ds, \quad t>0$$

As before, applying the Theorem 2 we obtain

$$\begin{aligned} |u(t)| \leq \frac{L\varepsilon}{\delta}\Omega^{-1}\left(\Omega(1)+\frac{n\lambda^6}{\delta}\int_{t_0}^{t} p(s)y(s)v(s)\omega\left(F^{-1}\left(F(1)\right.\right.\right. \\ \left.\left.\left.+\frac{\lambda b(\lambda)^4\rho}{\delta}\int_{t_0}^{s} h(\alpha)d\alpha\right)\right)ds\right)F^{-1}\left(F(1)+\frac{\lambda b(\lambda)^4\rho}{\delta}\int_{t_0}^{t} h(s)ds\right) \end{aligned}$$

and using the conditions (iv'), (v') of Theorem 5 to have

$$|u(t)| \leq \frac{L\varepsilon}{\delta}\Omega^{-1}\left(\Omega(1)+\frac{n\lambda^6}{\delta}n_5\omega\left(F^{-1}\left(F(1)+\frac{\lambda b(\lambda)^4\rho}{\delta}n_4\right)\right)\right)$$
$$F^{-1}\left(F(1)+\frac{\lambda b(\lambda)^4\rho}{\delta}k_4\right).$$

Hence,

$$|u(t)-u(t_0)| \leq |u(t)| \leq K_4\varepsilon,$$

for

$$K_4 = \frac{L}{\delta}\Omega^{-1}\left(\Omega(1)+\frac{n\lambda^6}{\delta}n_5\omega\left(F^{-1}\left(F(1)+\frac{\lambda b(\lambda)^4\rho}{\delta}n_4\right)\right)\right)$$
$$F^{-1}\left(F(1)+\frac{\lambda b(\lambda)^4\rho}{\delta}n_4\right).$$

□

Example 1. Consider the following equation

$$\left(\frac{1}{t^4}u^4(t)(u''(t)\right)' + \left(\frac{1}{t^4}u^6(t)u'(t)\right)' + \frac{1}{t^2}u^2(t)(u'(t)) + \frac{1}{t^4}u^2(t) = \frac{1}{t^6}u^2(t)(u'(t))^8, \ t>0,$$

where $H(t,u(t),u'(t)) = \frac{1}{t^6}u^2(t)(u'(t))^8$, $\alpha(t)=\frac{1}{t^4}$, $\gamma(t)=\frac{1}{t^4}$, $\beta(t)=\frac{1}{t^2}$, $r(t)=\frac{1}{t^4}$, $\phi(t)=\frac{1}{t^6}$. By criteria of Theorem 4 and inequality (20) we have

$$|u(t)| \leq \frac{L\varepsilon}{\sigma\lambda}\Upsilon^{-1}\left[\Upsilon(1)+\frac{\lambda^n}{\sigma}\right.$$
$$\left.\int_{t_0}^{t}\frac{1}{s^6}\varpi\left[\Omega^{-1}\left(\Omega(1)+\frac{1}{\sigma\lambda}\int_{t_0}^{s}\frac{1}{\alpha^4}\rho\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds\right]$$
$$\Omega^{-1}\left(\Omega(1)+\frac{1}{\sigma\lambda}\int_{t_0}^{t}\frac{1}{s^4}\rho\left(T(s)\right)ds\right)T(t)$$

where

$$T(t) = F^{-1}\left(F(1)+\frac{\lambda}{\sigma}\int_{t_0}^{t}\frac{1}{s^2}ds\right).$$

Further simplification by using the conditions (iii)- (v) of Theorem 4, we arrive at

$$|u(t)| \leq \frac{L\varepsilon}{\sigma\lambda}\Upsilon^{-1}\left[\Upsilon(1)+\frac{\lambda^n}{\sigma}n_1\varpi\left[\Omega^{-1}\left(\Omega(1)+\frac{1}{\sigma\lambda}n_2\rho\left(T^*\right)\right)T^*\right]\right]$$
$$\Omega^{-1}\left(\Omega(1)+\frac{1}{\sigma\lambda}n_2\rho\left(T^*\right)\right)T^*,$$

where

$$T^{*}=F^{-1}\left(F(1)+\frac{\lambda}{\sigma}n_{3}\right).$$

where

i $\int_{t_0}^{t}\frac{1}{s^6}ds\leq n_1$
ii $\int_{t_0}^{t}\frac{1}{s^6}ds\leq n_2$
iii $\int_{t_0}^{t}\frac{1}{s^2}ds\leq n_3$

Therefore, Hyers-Ulam constant is given as

$$K=\frac{L}{\sigma\lambda}\Upsilon^{-1}\left[\Upsilon(1)+\frac{\lambda^{n}}{\sigma}n_{1}\varpi\left[\Omega^{-1}\left(\Omega(1)+\frac{1}{\sigma\lambda}n_{2}\rho\left(T^{*}\right)\right)T^{*}\right]\right]$$
$$\Omega^{-1}\left(\Omega(1)+\frac{1}{\sigma\lambda}n_{2}\rho\left(T^{*}\right)\right)T^{*},$$

Example 2. Consider the following equation

$$\left(\frac{1}{t^4}u^4(t)(u''(t)\right)'+n\frac{1}{t^6}u^6(t)u'(t)^4+t^4u^2(t)=\frac{1}{t^6}u^2(t)(u'(t))^8,\ t>0,$$

where $P_1(t,u(t),u'(t))=\frac{1}{t^4}u^4(t)(u'(t))^2$, $P_2(t,u(t),u'(t))=\frac{1}{t^6}u^2(t)(u'(t))^3$, $Q(t,u(t))=\frac{1}{t^4u^2(t)}$, $H(t,u(t),u'(t))=\frac{1}{t^6}u^2(t)(u'(t))^8$, $h(t)=\frac{1}{t^4}$, $y(t)p(t)v(t)=\frac{1}{t^6}$, $\psi(t)=\frac{1}{t^4}$, $\phi(t)=\frac{1}{t^6}$. By criteria of Theorem 5 and using (23) to obtain

$$|u(t)|\leq\frac{L\varepsilon}{\nu}\Upsilon^{-1}\left[\Upsilon(1)+\frac{\lambda^{n+1}}{\nu}\int_{t_0}^{t}\frac{1}{s^6}\varpi\left[\Omega^{-1}\left(\Omega(1)\right.\right.\right.$$
$$\left.\left.\left.+\frac{n\lambda^6}{\nu}\int_{t_0}^{s}\frac{1}{\alpha^6}\omega\left(T(\alpha)\right)d\alpha\right)T(s)\right]ds\right]$$
$$\Omega^{-1}\left(\Omega(1)+\frac{n\lambda^6}{\nu}\int_{t_0}^{t}\frac{1}{s^6}\omega\left(T(s)\right)ds\right)T(t),$$

where

$$T(t)=F^{-1}\left(F(1)+\frac{\lambda b(\lambda^4)\rho}{\nu}\int_{t_0}^{t}\frac{1}{s^4}ds\right).$$

Then, we obtain

$$|u(t)|\leq\frac{L\varepsilon}{\nu}\Upsilon^{-1}\left[\Upsilon(1)+\frac{\lambda^{n+1}}{\nu}n_1\varpi\left[\Omega^{-1}\left(\Omega(1)+\frac{n\lambda^6}{\nu}n_4\omega\left(T_2^{*}\right)\right)T_2^{*}\right]\right]$$
$$\Omega^{-1}\left(\Omega(1)+\frac{n\lambda^6}{\nu}n_4\omega\left(T_2^{*}\right)\right)T_2^{*},$$

where

$$T_2^* = F^{-1}\left(F(1) + \frac{\lambda b(\lambda^4)\rho}{\nu} n_5\right).$$

The limits are taking as:

i $\int_{t_0}^{t} \frac{1}{s^6} ds \leq n_1$

ii $\int_{t_0}^{t} \frac{1}{s^6} ds \leq n_4$

iii $\int_{t_0}^{t} \frac{1}{s^4} ds \leq n_5$

The Hyers-Ulam constant is given as

$$K = \frac{L}{\nu}\Upsilon^{-1}\left[\Upsilon(1) + \frac{\lambda^{n+1}}{\nu} n_1 \varpi \left[\Omega^{-1}\left(\Omega(1) + \frac{n\lambda^6}{\nu} n_4 \omega\left(T_2^*\right)\right) T_2^*\right]\right]$$
$$\Omega^{-1}\left(\Omega(1) + \frac{n\lambda^6}{\nu} n_4 \omega\left(T_2^*\right)\right) T_2^*,$$

4. CONCLUSION

In this work, Hyers-Ulam stability criteria of third order nonlinear differential equations with nonlinear damping which is very prominent in finding the stability of some problems such as hereditary, the surge in birth-rates, spreading of certain contagious diseases and so on. These problems appear directly in terms of integral equations and in terms of differential equations with certain criteria which can be reduced to integral equations whereby Gronwall-Bellman-Bihari type inequality is useful to determine the stability.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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