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SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH WRIGHT FUNCTION

V.S. SHINDE[1,*], S.B. CHAVHAN[2]

[1]Department of Mathematics, K. N. B. A. C. and V. P. S. College, Bhosare (Kurduwadi) Dist. Solapur- 413208, Maharastra, India

[2]Department of Mathematics, D.B.A.C. and S. College, Bhokar, Dist.-Nanded Maharashtra, India

Abstract. The Wright function is a special function with notable applications in several branches of mathematics, including geometric function theory. It helps in constructing and studying classes of analytic and univalent functions, particularly due to its connection with fractional calculus and differential subordinations. The target of this paper is to discuss a new subclass $TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$ of univalent functions with negative coefficients related to Wright distributation in the unit disk $U=\{z:|z|<1\}$. We obtain basic properties like coefficient inequality, distortion and covering theorem, radii of starlikeness, convexity and close-to-convexity, extreme points, Hadamard product, and closure theorems for functions belonging to our class.

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1. Introduction

In 1933, Wright [15] introduced a special function which is named as Wright function and defined in the following way

*Corresponding author
E-mail address: viharss@gmail.com
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(1) $$W_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{z^n}{n!\Gamma(\lambda n+\mu)},$$

where $\lambda>-1$, $\mu\in\mathbb{C}$ and $\Gamma(\cdot)$ stands for the usual Gamma function. The series given by (1) is absolutely convergent for all $z\in\mathbb{C}$ while for $\lambda=-1$ this is absolutely convergent in U. He also proved that it is an entire function for $\lambda>-1$. For more basic properties on Wright functions one may refer to Gorenflo et al.[5] and Mustafa[6]. It is easy to see that the series (1) is not in normalized form so we norimalized it as

$$\mathbb{W}_{\lambda,\mu}(z)=\Gamma(\mu)zW_{\lambda,\mu}(z)$$

(2) $$\mathbb{W}_{\lambda,\mu}(z)=\sum_{n=0}^{\infty}\frac{\Gamma(\mu)z^{n+1}}{n!\Gamma(\lambda n+\mu)}$$

where $\lambda>-1$, $\mu>0$, $z\in\mathbb{U}$. Now, we introduce Wright distribution in the following way, first we define the series

(3) $$\mathbb{W}_{\lambda,\mu}(m)=\sum_{n=0}^{\infty}\frac{\Gamma(\mu)m^{n+1}}{n!\Gamma(\lambda n+\mu)}$$

which is convergent for all $\lambda,\mu,m>0$.

The probability mass function of Wright distribution is given by

(4) $$p(n)=\frac{\Gamma(\mu)m^{n+1}}{n!\Gamma(\lambda n+\mu)\,\mathbb{W}_{\lambda,\mu}(m)},\ m,\mu,\lambda>0,\ n=0,1,2,3,\ldots$$

It is worthy to note that for $\lambda=0$ it reduces to the Poisson distribution.

Let $\mathscr{A}$ signify the class of all functions $u(z)$ of the type

(5) $$u(z)=z+\sum_{n=2}^{\infty}a_nz^n$$

in the open unit disc $U=\{z\in\mathbb{C}:|z|<1\}$. Let S be the subclass of $\mathscr{A}$ consisting of univalent functions and satisfy the following usual normalization condition $u(0)=u'(0)-1=0$. We denote by S the subclass of $\mathscr{A}$ consisting of functions $u(z)$ which are all univalent in U. A function $u\in\mathscr{A}$ is a starlike function of the order ς, $0\le\varsigma<1$, if it satisfy

(6) $$\Re\left\{\frac{zu'(z)}{u(z)}\right\}>\varsigma,z\in U.$$

We denote this class with $S^*(\varsigma)$. A function $u \in \mathscr{A}$ is a convex function of the order $\varsigma,\ 0 \leq \varsigma < 1$, if it fulfil

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \varsigma, z \in U. \tag{7}$$

We denote this class with $K(\varsigma)$. Note that $S^*(0) = S^*$ and $K(0) = K$ are the usual classes of starlike and convex functions in U respectively. Let T denote the class of functions analytic in U that are of the form

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n,\ a_n \geq 0\ z \in U \tag{8}$$

and let $T^*(\varsigma) = T \cap S^*(\varsigma)$, $C(\varsigma) = T \cap K(\varsigma)$. The class $T^*(\varsigma)$ and allied classes possess some interesting properties and have been extensively studied by Silverman [12].

In 2014, by using the definition of Poisson distribution, Porwal [8] introduced Poisson distribution series and gave a nice application of it on certain classes of univalent functions and opened up a new direction of research in the geometric function theory. After the investigation of this series several researchers investigated various distribution series like Hypergeometric distribution series [1], Pascal distribution series [3], Mittag-Leffler type Poisson distribution series [4], Binomial distribution series [7], generalized distribution series [9] Hypergeometric type distribution series [10], confluent hypergeometric distribution series [11], generalized hypergeometric distribution series [13], Borel distribution series [14] (see also [2]) and obtained various interesting results on certain classes of univalent functions for these series. Now, using the definition of Wright distribution, we introduce the Wright distribution series as follows:

$$K(\lambda,\mu,m,z) = z + \sum_{n=2}^{\infty} \frac{\Gamma(\mu)m^n}{(n-1)!\Gamma(\lambda(n-1)+\mu)\mathbb{W}_{\lambda,\mu}(m)} z^n.$$

The convolution of two power series $u(z)$ of the form (5) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$ is defined as the power series

$$(u * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n.$$

Now, we introduce the linear operator $I^{\lambda}_{\mu,m} : \mathscr{A} \to \mathscr{A}$ defined as

$$\begin{aligned} I^{\lambda}_{\mu,m} u(z) &= u(z) * K(\lambda,\mu,m,z) \\ &= z + \sum_{n=2}^{\infty} \Phi_n a_n z^n \end{aligned} \tag{9}$$

where

$$\Phi_n = \frac{\Gamma(\mu)m^n}{\Gamma(\lambda(n-1)+\mu)(n-1)!\mathbb{W}_{\lambda,\mu}(m))} a_n z^n. \tag{10}$$

Now, by making use of the Wright distributation, we define a new subclass of functions motivated by the recent work of [5, 6, 8, 9].

Definition 1.1 For $0 \leq \hbar < 1, 0 \leq \sigma < 1,$ and $0 < \varsigma < 1$, we let $TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$ be the subclass of u consisting of functions of the form (8) and its geometrical condition satisfy

$$\left| \frac{\hbar\left((I^{\lambda}_{\mu,m}u(z))' - \frac{I^{\lambda}_{\mu,m}u(z)}{z}\right)}{\sigma(I^{\lambda}_{\mu,m}u(z))' + (1-\hbar)\frac{I^{\lambda}_{\mu,m}u(z)}{z}} \right| < \varsigma, \ (z \in \mathbb{U}),$$

where $I^{\lambda}_{\mu,m}u$ is given by (9).

2. Coefficient Inequality

In this section, we obtain a necessary and sufficient condition for function to be in the class $TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$.

Theorem 2.1 Let the function u be defined by (8). Then $u \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$ if and only if

$$\sum_{n=2}^{\infty}[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Phi_n a_n \leq \varsigma(\sigma+(1-\hbar)) \tag{11}$$

where $0 < \varsigma < 1, 0 \leq \hbar < 1,$ and $0 \leq \sigma < 1$. The result (11) is sharp for the function

$$u(z) = z - \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Phi_n} z^n, \ n \geq 2.$$

Proof. Suppose that the inequality (11) holds true and $|z| = 1$. Then we obtain

$$\begin{aligned}
&\left|\hbar\left((I^{\lambda}_{\mu,m}u(z))' - \frac{I^{\lambda}_{\mu,m}u(z)}{z}\right)\right| - \varsigma\left|\sigma\left(I^{\lambda}_{\mu,m}u(z))' + (1-\hbar)\frac{I^{\lambda}_{\mu,m}u(z)}{z}\right)\right| \\
&= \left|-\hbar\sum_{n=2}^{\infty}(n-1)\Phi_n a_n z^{n-1}\right| \\
&\quad - \varsigma\left|\sigma + (1-\hbar) - \sum_{n=2}^{\infty}(n\sigma+1-\hbar)\Phi_n a_n z^{n-1}\right| \\
&\leq \sum_{n=2}^{\infty}[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Phi_n a_n - \varsigma(\sigma+(1-\hbar)) \\
&\leq 0.
\end{aligned}$$

Hence, by maximum modulus principle, $u \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$. Now assume that $u \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$ so that

$$\left| \frac{\hbar\left((I^{\lambda}_{\mu,m}u(z))' - \frac{I^{\lambda}_{\mu,m}u(z)}{z} \right)}{\sigma(I^{\lambda}_{\mu,m}u(z))' + (1-\hbar)\frac{I^{\lambda}_{\mu,m}u(z)}{z}} \right| < \varsigma,\ z \in \mathbb{U}.$$

Hence

$$\left| \hbar\left((I^{\lambda}_{\mu,m}u(z))' - \frac{I^{\lambda}_{\mu,m}u(z)}{z} \right) \right| < \varsigma \left| \sigma\left(I^{\lambda}_{\mu,m}u(z))' + (1-\hbar)\frac{I^{\lambda}_{\mu,m}u(z)}{z} \right) \right|.$$

Therefore, we get

$$\left| -\sum_{n=2}^{\infty} \hbar(n-1)\Phi_n a_n z^{n-1} \right|$$
$$< \ \varsigma \left| \sigma + (1-\hbar) - \sum_{n=2}^{\infty} (n\sigma + 1 - \hbar)\Phi_n a_n z^{n-1} \right|.$$

Thus

$$\sum_{n=2}^{\infty} [\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Phi_n a_n \leq \varsigma(\sigma + (1-\hbar))$$

and this completes the proof.

Corollary 2.1 Let the function $u \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$. Then

$$a_n \leq \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Phi_n} z^n,\ n \geq 2.$$

3. Distortion and Covering Theorem

We introduce the growth and distortion theorems for the functions in the class $TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$.

Theorem 3.1 Let the function $u \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$. Then

$$|z| - \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar + \varsigma(2\sigma + 1 - \hbar)]\Phi_2}|z|^2 \leq |u(z)| \leq |z| + \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar + \varsigma(2\sigma + 1 - \hbar)]\Phi_2}|z|^2.$$

The result is sharp and attained

$$u(z) = z - \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar + \varsigma(2\sigma + 1 - \hbar)\Phi_2]} z^2.$$

Proof.

$$|u(z)| = \left| z - \sum_{n=2}^{\infty} a_n z^n \right| \leq |z| + \sum_{n=2}^{\infty} a_n |z|^n$$
$$\leq |z| + |z|^2 \sum_{n=2}^{\infty} a_n.$$

By Theorem 2.1, we get

$$\sum_{n=2}^{\infty} a_n \leq \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar + \varsigma(2\sigma + 1 - \hbar)]\Phi_n}. \tag{12}$$

Thus

$$|u(z)| \leq |z| + \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar + \varsigma(2\sigma + 1 - \hbar)]\Phi_2}|z|^2.$$

Also

$$|u(z)| \geq |z| - \sum_{n=2}^{\infty} a_n |z|^n$$
$$\geq |z| - |z|^2 \sum_{n=2}^{\infty} a_n$$
$$\geq |z| - \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar + \varsigma(2\sigma + 1 - \hbar)]\Phi_2}|z|^2.$$

Theorem 3.2 Let $u \in TS^{\lambda}_{\mu,m}(\hbar, \sigma, \varsigma)$. Then

$$1 - \frac{2\varsigma(\sigma + (1-\hbar))}{[\hbar + \varsigma(2\sigma + 1 - \hbar)\Phi_2]}|z| \leq |u'(z)| \leq 1 + \frac{2\varsigma(\sigma + (1-\hbar))}{[\hbar + \varsigma(2\sigma + 1 - \hbar)]\Phi_2}|z|$$

with equality for

$$u(z) = z - \frac{2\varsigma(\sigma + (1-\hbar))}{[\hbar + \varsigma(2\sigma + 1 - \hbar)]\Phi_2}z^2.$$

Proof. Notice that

$$[\hbar + \varsigma(2\sigma + 1 - \hbar)]\Phi_2 \sum_{n=2}^{\infty} n a_n$$
$$\leq \sum_{n=2}^{\infty} n[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Phi_n a_n$$
$$\leq \varsigma(\sigma + (1-\hbar)), \tag{13}$$

from Theorem 2.1. Thus

$$|u'(z)| = \left| 1 - \sum_{n=2}^{\infty} n a_n z^{n-1} \right|$$

$$\leq 1+\sum_{n=2}^{\infty} na_n|z|^{n-1}$$

$$\leq 1+|z|\sum_{n=2}^{\infty} na_n$$

$$\leq 1+|z|\frac{2\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\Phi_2}. \tag{14}$$

On the other hand

$$|u'(z)| = \left|1-\sum_{n=2}^{\infty} na_n z^{n-1}\right|$$

$$\geq 1-\sum_{n=2}^{\infty} na_n|z|^{n-1}$$

$$\geq 1-|z|\sum_{n=2}^{\infty} na_n$$

$$\geq 1-|z|\frac{2\varsigma(\sigma+(1-\hbar))}{[\hbar+\varsigma(2\sigma+1-\hbar)]\Phi_2}. \tag{15}$$

Combining (14) and (15), we get the result.

4. Radii of Starlikeness, Convexity and Close-to-Convexity

In the following theorems, we obtain the radii of starlikeness, convexity and close-to-convexity for the class $TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$.

Theorem 4.1 Let $u \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$. Then u is starlike in $|z| < R_1$ of order δ, $0 \leq \delta < 1$, where

$$R_1 = \inf_n \left\{ \frac{(1-\delta)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Phi_n}{(n-\delta)\varsigma(\sigma+(1-\hbar))} \right\}^{\frac{1}{n-1}}, \; n \geq 2. \tag{16}$$

Proof. u is starlike of order $\delta, 0 \leq \delta < 1$ if

$$\Re\left\{\frac{zu'(z)}{u(z)}\right\} > \delta.$$

Thus it is enough to show that

$$\left|\frac{zu'(z)}{u(z)}-1\right| = \left|\frac{-\sum_{n=2}^{\infty}(n-1)a_n z^{n-1}}{1-\sum_{n=2}^{\infty}a_n z^{n-1}}\right| \leq \frac{\sum_{n=2}^{\infty}(n-1)a_n|z|^{n-1}}{1-\sum_{n=2}^{\infty}a_n|z|^{n-1}}.$$

Thus

$$\left|\frac{zu'(z)}{u(z)}-1\right| \le 1-\delta \text{ if } \sum_{n=2}^{\infty}\frac{(n-\delta)}{(1-\delta)}a_n|z|^{n-1} \le 1. \tag{17}$$

Hence by Theorem 2.1, (17) will be true if

$$\frac{n-\delta}{1-\delta}|z|^{n-1} \le \frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Phi_n}{\varsigma(\sigma+(1-\hbar)}$$

or if

$$|z| \le \left[\frac{(1-\delta)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Phi_n}{(n-\delta)\varsigma(\sigma+(1-\hbar))}\right]^{\frac{1}{n-1}}, n \ge 2. \tag{18}$$

The Theorem 4.1 follows easily from (18).

Theorem 4.2 Let $u \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$. Then u is convex in $|z| < R_2$ of order $\delta, 0 \le \delta < 1$, where

$$R_2 = \inf_n \left\{\frac{(1-\delta)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Phi_n}{n(n-\delta)\varsigma(\sigma+(1-\hbar))}\right\}^{\frac{1}{n-1}}, \ n \ge 2. \tag{19}$$

Proof. u is convex of order $\delta, 0 \le \delta < 1$ if

$$\Re\left\{1+\frac{zu''(z)}{u'(z)}\right\} > \delta.$$

Thus it is enough to show that

$$\left|\frac{zu''(z)}{u'(z)}\right| = \left|\frac{-\sum\limits_{n=2}^{\infty} n(n-1)a_n z^{n-1}}{1-\sum\limits_{n=2}^{\infty} na_n z^{n-1}}\right| \le \frac{\sum\limits_{n=2}^{\infty} n(n-1)a_n|z|^{n-1}}{1-\sum\limits_{n=2}^{\infty} na_n|z|^{n-1}}.$$

Thus

$$\left|\frac{zu''(z)}{u'(z)}\right| \le 1-\delta \text{ if } \sum_{n=2}^{\infty}\frac{n(n-\delta)}{(1-\delta)}a_n|z|^{n-1} \le 1. \tag{20}$$

Hence by Theorem 2.1, (20) will be true if

$$\frac{n(n-\delta)}{1-\delta}|z|^{n-1} \le \frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Phi_n}{\varsigma(\sigma+(1-\hbar)}$$

or if

$$|z| \le \left[\frac{(1-\delta)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Phi_n}{n(n-\delta)\varsigma(\sigma+(1-\hbar))}\right]^{\frac{1}{n-1}}, n \ge 2. \tag{21}$$

The theorem follows easily from (21).

Theorem 4.3 Let $u \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$. Then u is close-to-convex in $|z| < R_3$ of order δ, $0 \leq \delta < 1$, where

$$R_3 = \inf_n \left\{ \frac{(1-\delta)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Phi_n}{n\varsigma(\sigma+(1-\hbar))} \right\}^{\frac{1}{n-1}}, \ n \geq 2. \tag{22}$$

Proof. u is close-to-convex of order $\delta, 0 \leq \delta < 1$ if

$$\Re\left\{u'(z)\right\} > \delta.$$

Thus it is enough to show that

$$|u'(z)-1| = \left| -\sum_{n=2}^{\infty} n a_n z^{n-1} \right| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

Thus

$$|u'(z)-1| \leq 1-\delta \text{ if } \sum_{n=2}^{\infty} \frac{n}{(1-\delta)} a_n |z|^{n-1} \leq 1. \tag{23}$$

Hence by Theorem 2.1, (23) will be true if

$$\frac{n}{1-\delta}|z|^{n-1} \leq \frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Phi_n}{\varsigma(\sigma+(1-\hbar)}$$

or if

$$|z| \leq \left[\frac{(1-\delta)(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Phi_n}{n\varsigma(\sigma+(1-\hbar))} \right]^{\frac{1}{n-1}}, n \geq 2. \tag{24}$$

The theorem follows easily from (24).

5. Extreme Points

In the following theorem, we obtain extreme points for the class $TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$.

Theorem 5.1 Let $u_1(z) = z$ and

$$u_n(z) = z - \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Phi_n} z^n, \text{ for } n = 2,3,\cdots.$$

Then $u \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$ if and only if it can be expressed in the form

$$u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z), \ \textit{where } \theta_n \geq 0 \textit{ and } \sum_{n=1}^{\infty} \theta_n = 1.$$

Proof. Assume that $u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z)$, hence we get

$$u(z) = z - \sum_{n=2}^{\infty} \frac{\varsigma(\sigma + (1-\hbar))\theta_n}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Phi_n} z^n.$$

Now, $u \in TS^{\lambda}_{\mu,m}(\hbar, \sigma, \varsigma)$, since

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Phi_n}{\varsigma(\sigma + (1-\hbar))} \\ &\quad \times \frac{\varsigma(\sigma + (1-\hbar))\theta_n}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Phi_n} \\ &= \sum_{n=2}^{\infty} \theta_n = 1 - \theta_1 \leq 1. \end{aligned}$$

Conversely, suppose $u \in TS^{\lambda}_{\mu,m}(\hbar, \sigma, \varsigma)$. Then we show that u can be written in the form $\sum_{n=1}^{\infty} \theta_n u_n(z)$.
Now $u \in TS^{\lambda}_{\mu,m}(\hbar, \sigma, \varsigma)$ implies from Theorem 2.1

$$a_n \leq \frac{\varsigma(\sigma + (1-\hbar))}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Phi_n}.$$

Setting $\theta_n = \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Phi_n}{\varsigma(\sigma + (1-\hbar))} a_n, n = 2, 3, \cdots$
and $\theta_1 = 1 - \sum_{n=2}^{\infty} \theta_n$, we obtain $u(z) = \sum_{n=1}^{\infty} \theta_n u_n(z)$.

6. HADAMARD PRODUCT

In the following theorem, we obtain the convolution result for functions belongs to the class $TS^{\lambda}_{\mu,m}(\hbar, \sigma, \varsigma)$.

Theorem 6.1 Let $u, g \in TS(\hbar, \sigma, \varsigma, \vartheta)$. Then $u * g \in TS(\hbar, \sigma, \zeta, \vartheta)$ for

$$u(z) = z - \sum_{n=2}^{\infty} a_n z^n, g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } (u * g)(z) = z - \sum_{n=2}^{\infty} a_n b_n z^n,$$

where

$$\zeta \geq \frac{\varsigma^2(\sigma + (1-\hbar))\hbar(n-1)}{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]^2\Phi_n - \varsigma^2(\sigma + (1-\hbar))(n\sigma + 1 - \hbar)}.$$

Proof. $u \in TS^{\lambda}_{\mu,m}(\hbar, \sigma, \varsigma)$ and so

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1) + \varsigma(n\sigma + 1 - \hbar)]\Phi_n}{\varsigma(\sigma + (1-\hbar))} a_n \leq 1 \tag{25}$$

and

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Phi_n}{\varsigma(\sigma+(1-\hbar))} b_n \leq 1. \tag{26}$$

We have to find the smallest number ζ such that

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\Phi_n}{\zeta(\sigma+(1-\hbar))} a_n b_n \leq 1. \tag{27}$$

By Cauchy-Schwarz inequality

$$\sum_{n=2}^{\infty} \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Phi_n}{\varsigma(\sigma+(1-\hbar))} \sqrt{a_n b_n} \leq 1. \tag{28}$$

Therefore it is enough to show that

$$\begin{aligned} &\frac{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\Phi_n}{\zeta(\sigma+(1-\hbar))} a_n b_n \\ \leq &\frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Phi_n}{\varsigma(\sigma+(1-\hbar))} \sqrt{a_n b_n}. \end{aligned}$$

That is

$$\sqrt{a_n b_n} \leq \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\zeta}{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\varsigma}. \tag{29}$$

From (28),

$$\sqrt{a_n b_n} \leq \frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Phi_n}.$$

Thus it is enough to show that

$$\frac{\varsigma(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Phi_n} \leq \frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\zeta}{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\varsigma},$$

which simplifies to

$$\zeta \geq \frac{\varsigma^2(\sigma+(1-\hbar))\hbar(n-1)}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]^2\Phi_n - \varsigma^2(\sigma+(1-\hbar))(n\sigma+1-\hbar)}.$$

7. Closure Theorems

We shall prove the following closure theorems for the class $TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$.

Theorem 7.1 Let $u_j \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$, $j=1,2,\cdots,s$. Then

$$g(z)=\sum_{j=1}^{s} c_j u_j(z) \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma).$$

For $u_j(z)=z-\sum\limits_{n=2}^{\infty} a_{n,j}z^n$, where $\sum\limits_{j=1}^{s} c_j=1$.

Proof.

$$\begin{aligned} g(z) &= \sum_{j=1}^{s} c_j u_j(z) \\ &= z-\sum_{n=2}^{\infty}\sum_{j=1}^{s} c_j a_{n,j} z^n \\ &= z-\sum_{n=2}^{\infty} e_n z^n, \end{aligned}$$

where $e_n=\sum\limits_{j=1}^{s} c_j a_{n,j}$. Thus $g(z)\in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$ if

$$\sum_{n=2}^{\infty}\frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Phi_n}{\varsigma(\sigma+(1-\hbar))}e_n \le 1,$$

that is, if

$$\begin{aligned} &\sum_{n=2}^{\infty}\sum_{j=1}^{s}\frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Phi_n}{\varsigma(\sigma+(1-\hbar))}c_j a_{n,j} \\ &= \sum_{j=1}^{s} c_j \sum_{n=2}^{\infty}\frac{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]\Phi_n}{\varsigma(\sigma+(1-\hbar))}a_{n,j} \\ &\le \sum_{j=1}^{s} c_j = 1. \end{aligned}$$

Theorem 7.2 Let $u,g \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$. Then

$$h(z)=z-\sum_{n=2}^{\infty}(a_n^2+b_n^2)z^n \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma),$$

where $\zeta \ge \dfrac{2\hbar(n-1)\varsigma^2(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]^2\Phi_n-2\varsigma^2(\sigma+(1-\hbar))(n\sigma+1-\hbar)}$.

Proof. Since $u, g \in TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$, so Theorem 2.1 yields

$$\sum_{n=2}^{\infty}\left[\frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Phi_n}{\varsigma(\sigma+(1-\hbar))}a_n\right]^2 \leq 1$$

and

$$\sum_{n=2}^{\infty}\left[\frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Phi_n}{\varsigma(\sigma+(1-\hbar))}b_n\right]^2 \leq 1.$$

We obtain from the last two inequalities

$$\sum_{n=2}^{\infty}\frac{1}{2}\left[\frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Phi_n}{\varsigma(\sigma+(1-\hbar))}\right]^2(a_n^2+b_n^2) \leq 1. \tag{30}$$

But $h(z) \in TS(\hbar,\sigma,\zeta,q,m)$, if and only if

$$\sum_{n=2}^{\infty}\frac{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\Phi_n}{\zeta(\sigma+(1-\hbar))}(a_n^2+b_n^2) \leq 1, \tag{31}$$

where $0<\zeta<1$, however (30) implies (31) if

$$\frac{[\hbar(n-1)+\zeta(n\sigma+1-\hbar)]\Phi_n}{\zeta(\sigma+(1-\hbar))} \leq \frac{1}{2}\left[\frac{(\hbar(n-1)+\varsigma(n\sigma+1-\hbar))\Phi_n}{\varsigma(\sigma+(1-\hbar))}\right]^2.$$

Simplifying, we get

$$\zeta \geq \frac{2\hbar(n-1)\varsigma^2(\sigma+(1-\hbar))}{[\hbar(n-1)+\varsigma(n\sigma+1-\hbar)]^2\Phi_n-2\varsigma^2(\sigma+(1-\hbar))(n\sigma+1-\hbar)}.$$

CONCLUSION

The Wright function is an important special function with significant applications in various areas of mathematics, particularly in geometric function theory. Its close relationship with fractional calculus and differential subordinations makes it a useful tool for constructing and analyzing new classes of analytic and univalent functions.

In this paper, we introduced and studied a new subclass

$$TS^{\lambda}_{\mu,m}(\hbar,\sigma,\varsigma)$$

of univalent functions with negative coefficients associated with the Wright distribution in the open unit disk

$$U=\{z:|z|<1\}.$$

A comprehensive investigation of this class was carried out by establishing several fundamental geometric properties. In particular, we derived coefficient inequalities, distortion and covering theorems, and determined the radii of starlikeness, convexity, and close-to-convexity. Furthermore, we examined the extreme points, closure properties, and behavior under the Hadamard product for functions belonging to this subclass.

The results obtained in this work not only extend several known outcomes in the theory of univalent functions but also highlight the effectiveness of the Wright function in generating and studying new analytic structures. These findings may serve as a foundation for further investigations involving fractional operators, related special functions, and their applications in geometric function theory.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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