Available online at http://scik.org
Adv. Inequal. Appl. 2025, 2025:6
https://doi.org/10.28919/aia/9545
ISSN: 2050-7461

MAJORIZED PROOF FOR GENERALIZED LUO-GENG HUA INEQUALITY

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Abstract. In this paper, we use the Schur convexity of function and majorization inequality give the majorized proof for generalized Luo-geng Hua inequality.

Keywords: Luo-geng Hua inequality; Schur convexity; majorization.

2020 AMS Subject Classification: 26E60, 26D15, 26A51, 34K38.

1. INTRODUCTION

Let δ and α be normal constant numbers, and

$$C = \{x = (x_1, \ldots, x_n) | x_i \geq 0, x_1 + \cdots + x_n \leq \delta\}.$$

Then for any $x \in C$, we have

$$(\delta - x_1 + \cdots + x_n)^2 + \alpha(x_1^2 + \cdots + x_n^2) \geq \alpha(n+\alpha)^{-1} \tag{1}$$

holds.

Inequality(1) is Luo-geng Hua inequality (see[1]).

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Received August 08, 2025

Z. L. Wang use dynamic programming method extended Luo-geng Hua inequality to the following form (see[2]):

Let δ and α be normal constant numbers, and

$$C = \{x = (x_1, \ldots, x_n) \mid x_i \geq 0, x_1 + \cdots + x_n \leq \delta\}.$$

If $p > 1$ or $p < 0$, for any $x \in C$, then

$$F(x) := (\delta - x_1 + \cdots + x_n)^p + \alpha^{p-1}(x_1^2 + \cdots + x_n^2) \geq k_n^{p-1}\delta^p \tag{2}$$

holds. Where $k_n = \alpha(n+\alpha)^{-1}$, when $0 < p < 1$ inequality (2) reverse.

Pearće and Pečarić use weighted Jensen inequality to generalize Luo-geng Hua inequality into function from(see[3]).

Theorem 1. *Let f be a real convex function on the $I \subseteq \mathbb{R}$, $\alpha; x_1, \ldots, x_n$ be real numbers. If $\alpha > 0$ and $\delta - x_1 - \cdots - x_n, \alpha x_1, \ldots, \alpha x_n \in I$, then*

$$f\left(\delta - \sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} \alpha^{-1} f(\alpha x_i) \geq \frac{\alpha + n}{\alpha} f\left(\frac{\alpha\delta}{\alpha + n}\right) \tag{3}$$

if $f(x)$ is concave function inequality (3) reverse.

Schur convexity was introduced by Schur in 1923, and it has many important applications in analytic inequalities, linear regression, graphs and matrices, combinatorial optimization, information-theoretic topics, Gamma functions, stochastic orderings, reliability, and other related fields.

By studying the Schur convexity of corresponding functions and combining the majorization inequalities, to discover and prove various kinds of analytic inequalities, it is a hot topic of the inequalities research in recent years.

In this paper, a new proof of Theorem 1 is given by using the majorization theory, to prove Theorem 1, we need the following definitions and lemmas.

2. Preliminaries

We introduce some definitions and lemmas, which will be used in the proofs of the main results in subsequent sections.

Definition 1 ([4, 6, 7])**.** Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$.

(i) x is said to be majorized by y (in symbols $x \prec y$) if $\sum_{i=1}^{k} x_{[i]} \le \sum_{i=1}^{k} y_{[i]}$ for $k = 1, 2, \ldots, n-1$ and $\sum_{i=1}^{n} x_i = \sum_{i=1}^{n} y_i$, where $x_{[1]} \ge \cdots \ge x_{[n]}$ and $y_{[1]} \ge \cdots \ge y_{[n]}$ are rearrangements of x and y in a descending order.

(ii) $\Omega \subset \mathbb{R}^n$ is called a convex set if $(\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n) \in \Omega$ for any x and $y \in \Omega$, where α and $\beta \in [0, 1]$ with $\alpha + \beta = 1$.

(iii) let $\Omega \subset \mathbb{R}^n$, φ: $\Omega \to \mathbb{R}$ is said to be a Schur convex function on Ω if $x \prec y$ on Ω implies $\varphi(x) \le \varphi(y)$. φ is said to be a Schur concave function on Ω if and only if $-\varphi$ is Schur convex function.

Lemma 1 ([5, 7])**.** *Let $\Omega \subset \mathbb{R}^n$ is convex set, and has a nonempty interior set Ω°. Let $\varphi : \Omega \to \mathbb{R}$ is continuous on Ω and differentiable in Ω°. Then φ is the Schur convex (Schur concave) function, if and only if it is symmetric on Ω and if*

$$(x_1 - x_2)\left(\frac{\partial \varphi}{\partial x_1} - \frac{\partial \varphi}{\partial x_2}\right) \ge 0 (or \le 0; respectively)$$

holds for any $x = (x_1, \cdots, x_n) \in \Omega^\circ$.

Lemma 2 ([5, 7])**.** *Let $x = (x_1, \ldots, x_n) \in \mathbb{R}^n_{++}$. Then*

$$\left(\underbrace{A_n(x), \cdots, A_n(x)}_{n}\right) \prec (x_1, \cdots, x_n). \tag{4}$$

3. Proof of Main Results

Proof of Theorem 1.

Proof. Let

$$L(x) = f\left(\delta - \sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} \alpha^{-1} f(\alpha x_i).$$

Obviously $L(x)$ is symmetry with $x_1, \ldots, x_n$, it may be assumed that $x_1 \ge x_2$. Write $\theta = \delta - \sum_{i=1}^{n} x_i, \omega_1 = \alpha x_1, \omega_2 = \alpha x_2$. Because $\alpha > 0$, so $\omega_1 \le \omega_2$. We have

$$\frac{\partial L}{\partial x_1} = \frac{df(\theta)}{d\theta}\frac{d\theta}{dx_1} + \alpha^{-1}\frac{df(\omega_1)}{d\omega_1}\frac{d\omega_1}{dx_1} = -\frac{df(\theta)}{d\theta} + \frac{df(\omega_1)}{d\omega_1}$$

$$\frac{\partial L}{\partial x_2} = \frac{df(\theta)}{d\theta}\frac{d\theta}{dx_2} + \alpha^{-1}\frac{df(\omega_2)}{d\omega_2}\frac{d\omega_2}{dx_2} = -\frac{df(\theta)}{d\theta} + \frac{df(\omega_2)}{d\omega_2}$$

Due to $f(x)$ is convex function, hence

$$\Delta := (x_1 - x_2)\left(\frac{\partial L}{\partial x_1} - \frac{\partial L}{\partial x_2}\right) = (x_1 - x_2)\left(\frac{df(\omega_1)}{d\omega_1} - \frac{df(\omega_2)}{d\omega_2}\right) \geq 0,$$

by Lemma 1 $L(x)$ is Schur convex with $(x_1, \ldots, x_n)$ on $I^n \subseteq \mathbb{R}^n$. By the majorization inequality in Lemma 2

$$\left(\underbrace{A_n(x), \cdots, A_n(x)}_{n}\right) \prec (x_1, \cdots, x_n)$$

and Definition 1, we have

$$f\left(\delta - \sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} \alpha^{-1} f(\alpha x_i) \geq f(\delta - nA_n(x)) + n\alpha^{-1} f(\alpha A_n(x)).$$

Let $t = A_n(x), z(t) = f(\delta - nt) + n\beta^{-1} f(\alpha t)$, write $\theta_1 = \delta - nt, \theta_2 = \alpha t$, then

$$z^{'}(t) = -n\frac{df(\theta_1)}{d\theta_1} + n\frac{df(\theta_2)}{d\theta_2}.$$

Let $z^{'}(t) = 0$, we get

$$\frac{df(\theta_1)}{d\theta_1} = \frac{df(\theta_2)}{d\theta_2}. \tag{5}$$

Obviously, when $\theta_1 = \theta_2$ formula (5) holds, therefore, $t_0 = \frac{\delta}{\alpha+n}$ is a solution to $z^{'}(t) = 0$. Because $z^{''}(t) = n^2 f^{''}(\theta_1) + \alpha n f^{''}(\theta_2) \geq 0$, so $z(t_0) = z\left(\frac{\delta}{\alpha+n}\right)$ is minimum of $z(t)$, hence

$$\begin{aligned} f(\delta - n) + \sum_{i=1}^{n} \alpha^{-1} f(\alpha x_i) &\geq f(\delta - nA_n(x)) + n\alpha^{-1} f(\alpha A_n(x)) \\ &\geq \frac{\delta}{\alpha + n} f\left(\frac{\alpha\delta}{\alpha + n}\right). \end{aligned}$$

The proof of Theorem 1 is complete. □

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