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EXPLORING FIXED POINT THEOREMS IN C^*-ALGEBRA VALUED FUZZY SOFT METRIC SPACES USING CYCLIC (α,β)-$\mathscr{F}_G(\varphi,\wp,\varpi)$-RATIONAL CONTRACTIONS VIA $\mathscr{C}_{G^\star}$-CLASS FUNCTIONS

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Abstract. This study primarily aims to develop generalized $\mathscr{C}_{G^\star}$-class functions and establish fixed point results for cyclic (α,β)-$\mathscr{F}_G(\varphi,\wp,\varpi)$-rational contraction mappings within the framework of C^*-algebra-valued fuzzy soft metric spaces ($\mathscr{C}^*$-$\mathscr{A}\mathbb{VFSMS}$). The analysis incorporates key properties of various control functions to support the theoretical development. The findings not only extend previous work but also align with and enhance existing results in the literature. To illustrate the validity of our theorems, we present concrete examples. Additionally, we explore applications to homotopy theory and integral equations, highlighting the broader significance of the results.

Keywords: $\mathscr{C}_{G^\star}$-class functions; cyclic (α,β)-$\mathscr{F}_G(\varphi,\wp,\varpi)$-rational contraction maps; $\mathscr{C}^*$-$\mathscr{A}\mathbb{VFSMS}$; fixed points.

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1. Introduction

Uncertainty and imprecision are inherent in many real-world problems,often making classi-

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cal mathematical approaches inadequate for effective modeling. To address these challenges, Zadeh introduced fuzzy set theory [1], and Molodtsov later developed soft set theory [2], both offering flexible tools for handling ambiguity in various applications. Building on these foundations, Thangaraj Beaula et al. [3] proposed fuzzy soft metric spaces based on fuzzy soft points, establishing key theoretical results that have since led to a growing body of research (see [4]-[6]). In a related direction, Ma et al. [7] introduced the concept of $C^{\star}$-algebra-valued metric spaces in 2006 and explored fixed and coupled fixed point theorems under contraction conditions, prompting further developments in this area (see [8]-[12]). Combining these frameworks, Agarwal et al. [13] in 2018 introduced $C^{\star}$-algebra-valued fuzzy soft metric spaces and investigated fixed point results within this enriched structure, which has since attracted considerable interest and ongoing research efforts (see [14]-[17]).

Recent developments in fixed point theory have focused on generalizing contractive conditions. Khan et al. [18] and Ansari et al. [19, 20] introduced altering and ultra altering distance functions to relax classical contraction requirements. Ansari later proposed C-class functions [19], leading to unique fixed point results and further research. Kumssa [21] extended this work by studying Suzuki-type rational contractions in $b_v(s)$-metric spaces and introducing generalized C_G-class functions. Additionally, Samet et al. [22] developed α-admissible and α-ψ-contractive mappings, which inspired further generalizations such as those by Isik et al. [23] and Yamaod and Sintunavarat [24].

The objective of this paper is to establish unique fixed point (UFP) theorems in the context of $\mathscr{C}^*$-$\mathscr{A}\mathbb{VFSMS}$, specifically for cyclic (α,β)-$\mathscr{F}_G(\varphi,\wp,\varpi)$-rational contraction mappings using generalized $\mathscr{C}_{G^\star}$-class functions. Furthermore, we present applications to integral equations and homotopy theory, along with a discussion on the relevance and impact of the results obtained.

2. Preliminaries

Definition 2.1:([13]) Let $\mathbb{E} \subseteq \mathbb{V}$, and consider $\tilde{\mathbb{V}}$ as the absolute fuzzy soft set, where $\mathfrak{s}_{\mathbb{V}}(\mathfrak{v}) = \tilde{1}$ for every $\mathfrak{v} \in \mathbb{V}$. Denote the associated C^*-algebra by $\tilde{\mathscr{A}}$. A mapping $\tilde{\rho_{c^*}} : \tilde{\mathbb{V}} \times \tilde{\mathbb{V}} \to \tilde{\mathscr{A}}$ that satisfies the required conditions is referred to as a C^*-algebra-valued fuzzy soft metric defined via fuzzy soft points.

(i) $\tilde{0}_{\tilde{\mathscr{A}}} \preceq \tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})$ for all $\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2} \in \tilde{\mathbb{V}}$.

(ii) $\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2}) = \tilde{0}_{\tilde{\mathscr{A}}} \Leftrightarrow \mathfrak{s}_{\mathfrak{v}_1} = \mathfrak{s}_{\mathfrak{v}_2}$

(iii) $\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2}) = \tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_2},\mathfrak{s}_{\mathfrak{v}_1})$

(iv) $\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_3}) \preceq \tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2}) + \tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_2},\mathfrak{s}_{\mathfrak{v}_3})\ \forall\ \mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2},\mathfrak{s}_{\mathfrak{v}_3} \in \tilde{\mathbb{V}}$.

A $\mathscr{C}^*$-$\mathscr{A}\mathbb{VFSMS}$ consists of a fuzzy soft set $\tilde{\mathbb{V}}$ along with a corresponding fuzzy soft metric $\tilde{\rho_{c^*}}$. This structure is denoted by $(\tilde{\mathbb{V}},\tilde{\mathscr{A}},\tilde{\rho_{c^*}})$, where $\tilde{\mathscr{A}}$ represents the underlying C^*-algebra.

Example 2.2: ([13]) If $\mathbb{E}$ and $\mathbb{V}$ are subsets of $\mathscr{R}$, then $\tilde{\mathbb{V}}$ is an absolute fuzzy soft set, where $\tilde{\mathbb{V}}(\mathfrak{v}) = \tilde{1}$ for every $\mathfrak{v}$ in $\mathbb{V}$, and $\tilde{\mathscr{A}}$ is defined as $M_2(\mathscr{R}(C)^*)$. Define $\tilde{\rho_{c^*}}: \tilde{\mathbb{V}} \times \tilde{\mathbb{V}} \to \tilde{\mathscr{A}}$ by $\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2}) = \begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix}$, where $\kappa = \inf\{|\mu^a_{\mathfrak{s}_{\mathfrak{v}_1}}(t) - \mu^a_{\mathfrak{s}_{\mathfrak{v}_2}}(t)|/t \in C\}$ and $\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2} \in \tilde{\mathbb{V}}$.Then, by the completeness of $\mathscr{R}(C)^*$, $(\tilde{\mathbb{V}},\tilde{\mathscr{A}},\tilde{\rho_{c^*}})$ is a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{VFSMS}$. $\tilde{\rho_{c^*}}$ is a C^* - algebra valued fuzzy soft metric.

Definition 2.3:([13]) Assume that $(\tilde{\mathbb{V}},\tilde{\mathscr{A}},\tilde{\rho_{c^*}})$ is a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VFSMS}$. According to $\tilde{\mathscr{A}}$ a sequence $\{\mathfrak{s}_{\mathfrak{v}_k}\}$ in $\tilde{\mathbb{V}}$ is defined as:

(1) C^*-algebra valued fuzzy soft Cauchy sequence if, for each $\tilde{0}_{\tilde{\mathscr{A}}} \prec \tilde{\varepsilon}$, there exist $\tilde{0}_{\tilde{\mathscr{A}}} \prec \tilde{\delta}$ and a positive integer $N = N(\tilde{\varepsilon})$ such that $||\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_k},\mathfrak{s}_{\mathfrak{v}_l})|| < \tilde{\delta}$ implies that $||\mu^a_{\mathfrak{s}_{\mathfrak{v}_k}}(t) - \mu^a_{\mathfrak{s}_{\mathfrak{v}_l}}(s)|| < \tilde{\varepsilon}$ whenever $k,l \geq N$. That is $||\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_k},\mathfrak{s}_{\mathfrak{v}_l})||_{\tilde{\mathscr{A}}} \to \tilde{0}_{\tilde{\mathscr{A}}}$as $k,l \to \infty$.

(2) C^*-algebra valued fuzzy soft convergent to a point $\mathfrak{s}_{\mathfrak{v}^{\shortmid}} \in \tilde{\mathbb{V}}$ if, for each $\tilde{0}_{\tilde{\mathscr{A}}} \prec \tilde{\varepsilon}$, there exist $\tilde{0}_{\tilde{\mathscr{A}}} \prec \tilde{\delta}$ and a positive integer $N = N(\tilde{\varepsilon})$ such that $||\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_k},\mathfrak{s}_{\mathfrak{v}^{\shortmid}})|| < \tilde{\delta}$ implies $||\mu^a_{\mathfrak{s}_{\mathfrak{v}_k}}(t) - \mu^a_{\mathfrak{s}_{\mathfrak{v}^{\shortmid}}}(t)|| < \tilde{\varepsilon}$ whenever $k \geq N$. It is usually denoted as $\lim_{k\to\infty}\mathfrak{s}_{\mathfrak{v}_k} = \mathfrak{s}_{\mathfrak{v}^{\shortmid}}$.

(3) It is referred to as being complete when a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VFSMS}$ $(\tilde{\mathbb{V}},\tilde{\mathscr{A}},\tilde{\rho_{c^*}})$ is present. If each Cauchy sequence in $\tilde{\mathbb{V}}$ converges to a fuzzy soft point in $\tilde{\mathbb{V}}$.

Lemma 2.4: ([13]) Let $\tilde{\mathscr{A}}$ be a C^*-algebra with the identity element $\tilde{I}_{\tilde{\mathscr{A}}}$ and $\tilde{\mathfrak{v}}$ be a positive element of $\tilde{\mathscr{A}}$. If $\tilde{\lambda} \in \tilde{\mathscr{A}}$ is such that $||\tilde{\lambda}|| < 1$ then for $p < q$, we have

(a) $\lim_{q\to\infty}\sum_{k=p}^{q}(\tilde{\lambda}^*)^k\tilde{\mathfrak{b}}(\tilde{\lambda})^k = \tilde{I}_{\tilde{\mathscr{A}}}||\widetilde{(\mathfrak{v})^{\frac{1}{2}}}||^2\left(\frac{||\tilde{\lambda}||^p}{1-||\tilde{\lambda}||}\right)$.

(b) $\sum_{k=p}^{q}(\tilde{\lambda}^*)^k\tilde{\mathfrak{v}}(\tilde{\lambda})^k \to \tilde{0}_{\tilde{\mathscr{A}}}$ as $q \to \infty$.

For more properties of a C^*-algebra valued fuzzy soft metric and C^* -algebra we refer the reader to [13, 25].

3. Main Results

For $\mathscr{C}_{G^\star}$-class functions in C^*-algebra valued fuzzy soft metric spaces, we will demonstrate various fixed point theorems in this section.

Definition 3.1: Let $\tilde{\mathscr{A}}$ is a unital C^*-algebra. Then a continuous function $\mathscr{F}:\tilde{\mathscr{A}}_+\times\tilde{\mathscr{A}}_+\to\tilde{\mathscr{A}}_+$ is called a C_*-class function if for all $\tilde{a},\tilde{b}\in\tilde{\mathscr{A}}_+$,

- (a) $\mathscr{F}(\tilde{a},\tilde{b})\preceq\tilde{a}$;
- (b) $\mathscr{F}(\tilde{a},\tilde{b})=\tilde{a}\Rightarrow\tilde{a}=\tilde{0}_{\tilde{\mathscr{A}}}$ or $\tilde{b}=\tilde{0}_{\tilde{\mathscr{A}}}$.

We denote Υ as the family of all C_*-class functions.

Definition 3.2: A mapping $\mathscr{F}_G:\tilde{\mathscr{A}}_+\times\tilde{\mathscr{A}}_+\times\tilde{\mathscr{A}}_+\to\tilde{\mathscr{A}}_+$ is called a generalized $\mathscr{C}_{G^\star}$-class function if for all $\tilde{\mathfrak{s}},\tilde{\mathfrak{v}},\tilde{\mathfrak{e}}\in\tilde{\mathscr{A}}_+$

1. $\mathscr{F}_G$ is continuous;
2. $\mathscr{F}_G(\tilde{\mathfrak{s}},\tilde{\mathfrak{v}},\tilde{\mathfrak{e}})\leq\max\{\tilde{\mathfrak{s}},\tilde{\mathfrak{v}}\}$;
3. $\mathscr{F}_G(\tilde{\mathfrak{s}},\tilde{\mathfrak{v}},\tilde{\mathfrak{e}})=\tilde{\mathfrak{s}}$ or $\tilde{\mathfrak{v}}\Rightarrow$ either of $\tilde{\mathfrak{s}},\tilde{\mathfrak{v}}$ or $\tilde{\mathfrak{e}}$ is $\tilde{0}_{\tilde{\mathscr{A}}}$.

Γ represents the family of all $\mathscr{C}_{G^\star}$-class functions.

Example 3.3: In the following, we give some members of $\mathscr{C}_{G^\star}$ where $\mathscr{F}_G:\tilde{\mathscr{A}}_+{}^3\to\tilde{\mathscr{A}}_+$ is a mapping:

1. $\mathscr{F}_G(\tilde{\mathfrak{e}},\tilde{\varsigma},\tilde{\rho})=\tilde{\varsigma}-\frac{\tilde{\mathfrak{e}}}{1+||\tilde{\rho}||_{\tilde{\mathscr{A}}}}$ where $||.||_{\tilde{\mathscr{A}}}$ is the fuzzy soft norm induced by the C^*-algebra and $\mathscr{F}_G(\tilde{\mathfrak{e}},\tilde{\varsigma},\tilde{\rho})=\tilde{\varsigma}$ implies $\tilde{\mathfrak{e}}=\tilde{0}_{\tilde{\mathscr{A}}}$.
2. $\mathscr{F}_G(\tilde{\mathfrak{e}},\tilde{\varsigma},\tilde{\rho})=\tilde{\varsigma}-\sigma(\tilde{\mathfrak{e}})\tilde{\rho}$ where $\sigma(.)$ denote the fuzzy soft spectral radius of an element and $\mathscr{F}_G(\tilde{\mathfrak{e}},\tilde{\varsigma},\tilde{\rho})=\tilde{\varsigma}$ implies $\sigma(\tilde{\mathfrak{e}})=\tilde{0}_{\tilde{\mathscr{A}}}$ or $\tilde{\rho}=\tilde{0}_{\tilde{\mathscr{A}}}$.
3. $\mathscr{F}_G(\tilde{\mathfrak{e}},\tilde{\varsigma},\tilde{\rho})=\tilde{\varsigma}-\phi(\tilde{\mathfrak{e}}+\tilde{\rho})$ where $\phi:\tilde{\mathscr{A}}_+\to\tilde{\mathscr{A}}_+$ is a fuzzy soft linear operator such that $\phi(\tilde{0}_{\tilde{\mathscr{A}}})=\tilde{0}_{\tilde{\mathscr{A}}}$ and $\mathscr{F}_G(\tilde{\mathfrak{e}},\tilde{\varsigma},\tilde{\rho})=\tilde{\varsigma}$ then $\phi(\tilde{\mathfrak{e}}+\tilde{\rho})=\tilde{0}_{\tilde{\mathscr{A}}}$ implying $\tilde{\mathfrak{e}}=\tilde{\rho}=\tilde{0}_{\tilde{\mathscr{A}}}$.

Definition 3.4: A function $\varphi:\tilde{\mathscr{A}}_+\to\tilde{\mathscr{A}}_+$ is called an altering distance function if the following properties are satisfied:

- (a) φ is nondecreasing and continuous;
- (b) $\varphi(\tilde{\mathfrak{s}})=\tilde{0}_{\tilde{\mathscr{A}}}$ if and only if $\tilde{\mathfrak{s}}=\tilde{0}_{\tilde{\mathscr{A}}}$.

The family of all altering distance functions is denoted by Ω

Definition 3.5: The mapping $\varpi : \tilde{\mathscr{A}_+} \to \tilde{\mathscr{A}_+}$ is called the ultra altering distance function if the following properties are met:

(1) ϖ is continuous;

(2) $\varpi(\tilde{\mathfrak{s}}) \succ \tilde{0}_{\tilde{\mathscr{A}}}\ \forall\ \tilde{\mathfrak{s}} \succ \tilde{0}_{\tilde{\mathscr{A}}}$;

Θ represents the class of all ultra-altering distance functions.

For convenience, we set: $\Delta = \{\wp / \wp : \tilde{\mathscr{A}_+} \to \tilde{\mathscr{A}_+}\}$ be a family of functions that satisfy the following properties;

(i) $\wp$ is a non-decreasing, upper semi-continuous from the right;

(ii) $\wp(\tilde{\mathfrak{e}}) = \tilde{0}_{\tilde{\mathscr{A}}}$ iff $\tilde{\mathfrak{e}} = \tilde{0}_{\tilde{\mathscr{A}}}$;

Definition 3.6: Let $\tilde{\mathbb{V}}$ be a absolute fuzzy soft set, and $\alpha,\beta : \tilde{\mathbb{V}} \to \tilde{\mathscr{A}_+}$. If $\mathfrak{f},\mathfrak{g} \colon \tilde{\mathbb{V}} \to \tilde{\mathbb{V}}$ then the mapping $\mathfrak{f}$ is $\mathfrak{g}$-cyclic-(α,β)-admissible if:

(i) $\alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ implies $\beta(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ for some $\mathfrak{s}_{\mathfrak{v}} \in \tilde{\mathbb{V}}$;

(ii) $\beta(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ implies $\alpha(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ for some $\mathfrak{s}_{\mathfrak{v}} \in \tilde{\mathbb{V}}$.

Example 3.7: Let $\mathbb{V} = \{\mathfrak{v}_1,\mathfrak{v}_2,\mathfrak{v}_3\}, U = \{x,y,z,w\}$ and $\mathbb{C}$ be a subset of $\mathbb{V}$ where $\mathbb{C} = \{\mathfrak{v}_1,\mathfrak{v}_2\}$. Define fuzzy soft set as,

$$(\mathfrak{s}_{\mathbb{V}},\mathbb{C}) = \left\{ \ \mathfrak{v}_1 = \{x_{0.7},y_{0.6},z_{0.6},w_{0.5}\},\mathfrak{v}_2 = \{x_{0.3},y_{0.7},z_{0.4},w_{0.6}\}, \ \right\}$$

$$\text{then,}\mathfrak{s}_{\mathfrak{v}_1} = \mu_{\mathfrak{s}_{\mathfrak{v}_1}} = \{x_{0.7},y_{0.6},z_{0.6},w_{0.5}\},\mathfrak{s}_{\mathfrak{v}_2} = \mu_{\mathfrak{s}_{\mathfrak{v}_2}} = \{x_{0.3},y_{0.7},z_{0.4},w_{0.6}\}$$

and $FSC(F_{\mathbb{V}}) = \{\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2}\}$, let $\tilde{\mathbb{V}}$ be absolute fuzzy soft set, that is $\tilde{\mathbb{V}}(\mathfrak{v}) = \tilde{1}$ for all $\mathfrak{v} \in \mathbb{V}$ and $\tilde{\mathscr{A}} = M_2(C)$, the algebra of 2×2 complex matrices. Define fuzzy soft mappings $\alpha(\mathfrak{s}_{\mathfrak{v}_1}) = 0.6.I$, $\alpha(\mathfrak{s}_{\mathfrak{v}_2}) = 0.3.I$ and $\beta(\mathfrak{s}_{\mathfrak{v}_1}) = 0.7.I$, $\beta(\mathfrak{s}_{\mathfrak{v}_2}) = 0.4.I$ and let us define two mappings $\mathfrak{f}$ and $\mathfrak{g}$ as $\mathfrak{f}(\mathfrak{s}_{\mathfrak{v}_1}) = \mathfrak{s}_{\mathfrak{v}_2}$ and $\mathfrak{g}(\mathfrak{s}_{\mathfrak{v}_1}) = \mathfrak{s}_{\mathfrak{v}_1}$ then for some $\mathfrak{s}_{\mathfrak{v}_1} \in \tilde{\mathbb{V}}$;

(i) $\alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1}) = \alpha(\mathfrak{s}_{\mathfrak{v}_1}) = 0.6.I \succ \tilde{0}_{\tilde{\mathscr{A}}}$ implies $\beta(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1}) = \beta(\mathfrak{s}_{\mathfrak{v}_2}) = 0.4.I \succ \tilde{0}_{\tilde{\mathscr{A}}}$

(ii) $\beta(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1}) = \beta(\mathfrak{s}_{\mathfrak{v}_1}) = 0.7.I \succ \tilde{0}_{\tilde{\mathscr{A}}}$ implies $\alpha(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1}) = \alpha(\mathfrak{s}_{\mathfrak{v}_2}) = 0.3.I \succ \tilde{0}_{\tilde{\mathscr{A}}}$.

Therefore, the mapping $\mathfrak{f}$ is $\mathfrak{g}$-cyclic-(α,β)-admissible.

Definition 3.8: Let $(\tilde{\mathbb{V}},\tilde{\mathscr{A}},\tilde{\rho_{c^*}})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VFSMS}$ and $\alpha,\beta : \tilde{\mathbb{V}} \to \tilde{\mathscr{A}_+}$ be two mappings. Let $\mathfrak{f},\mathfrak{g} \colon \tilde{\mathbb{V}} \to \tilde{\mathbb{V}}$ be two self mappings satisfying $\mathfrak{f}$ is a $\mathfrak{g}$ cyclic (α,β)-admissible mapping. Then $\mathfrak{f}$ is a $\mathfrak{g}$-cyclic-(α,β)-$\mathscr{F}_G(\varphi,\wp,\varpi)$-rational contraction type-I and type-II if for all $\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2} \in \tilde{\mathbb{V}}$ and $\tilde{a} \in \tilde{\mathscr{A}_+}$ with $||\tilde{a}|| < 1$

type-I:

$$\alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1})\beta(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2}) \succ \tilde{0}_{\tilde{\mathscr{A}}} \Rightarrow \varphi\left(\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2})\right)$$

$$\preceq \mathscr{F}_G\Big(\ \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star}), \wp(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star}), \varpi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star})\ \Big) \tag{1}$$

type-II:

(2)

$$\alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1})\beta(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2})\varphi\left(\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2})\right) \preceq \mathscr{F}_G\Big(\ \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star}), \wp(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star}), \varpi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star})\ \Big)$$

$$\text{where } \mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2}) = \max\left\{\begin{array}{c} \tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2}), \frac{1}{2}\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2}, \mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1}), \\ \frac{\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2}).\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2})}{2[1+\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2})]}, \frac{\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1}).\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1})}{2[1+\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2})]} \\ \frac{\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2}).\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2})\right]} \end{array}\right\}$$

and $\varphi \in \Omega$ $\wp \in \Delta$, $\varpi \in \Theta$, $\mathscr{F}_G \in \Gamma$.

Theorem 3.9: Let $(\tilde{\mathbb{V}}, \tilde{\mathscr{A}}, \tilde{\rho_{c^*}})$be a C^*-algebra valued fuzzy soft metric space and $\alpha, \beta : \tilde{\mathbb{V}} \to \tilde{\mathscr{A}}_+$ be two mappings. Let $\mathfrak{f}$ and $\mathfrak{g}$ be two self mappings on $\tilde{\mathbb{V}}$ and $\mathfrak{f}$ is a $\mathfrak{g}$-cyclic-(α,β)-admissible mapping such that $\mathfrak{f}$ is a $\mathfrak{g}$-cyclic-(α,β)-$\mathscr{F}_G(\varphi,\wp,\varpi)$-rational type-I and type-II contraction and with regard to a $\mathscr{C}_{G^\star}$-class functions $\mathscr{F}_G$ with $\varphi(\tilde{\mathfrak{e}}) > \wp(\tilde{\mathfrak{e}})$ for all $\tilde{\mathfrak{e}} \succ \tilde{0}_{\tilde{\mathscr{A}}}$ satisfying the following conditions:

(3.9.1) $\mathfrak{f}(\tilde{\mathbb{V}}) \subseteq \mathfrak{g}(\tilde{\mathbb{V}})$ with $\mathfrak{g}(\tilde{\mathbb{V}})$ is closed subspace of $\tilde{\mathbb{V}}$;

(3.9.2) there exists $\mathfrak{s}_{\mathfrak{v}_0} \in \tilde{\mathbb{V}}$ with $\alpha\left(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_0}\right) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ and $\beta\left(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_0}\right) \succ \tilde{0}_{\tilde{\mathscr{A}}}$;

(3.9.3) if $\{\mathfrak{s}_{\mathfrak{v}_n}\}_{n=1}^{\infty}$ is a sequence in $\tilde{\mathbb{V}}$ with $\beta(\mathfrak{s}_{\mathfrak{v}_n}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ for all n and $\lim_{n\to\infty} \mathfrak{s}_{\mathfrak{v}_n} = \mathfrak{s}_{\mathfrak{v}'}$ then $\beta(\mathfrak{s}_{\mathfrak{v}'}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$;

(3.9.4) $\alpha\left(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1}\right) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ and $\beta\left(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2}\right) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ whenever, $\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1} = \mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1}$ and $\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2} = \mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2}$.

Then, $\mathfrak{f}$ and $\mathfrak{g}$ have a unique point of coincidence in $\tilde{\mathbb{V}}$. Furthermore, if $\mathfrak{f}$ and $\mathfrak{g}$ are weakly compatible, then $\mathfrak{f}$ and $\mathfrak{g}$ have a unique common fixed point in $\tilde{\mathbb{V}}$.

Proof Let $\mathfrak{s}_{\mathfrak{v}_0} \in \tilde{\mathbb{V}}$. From condition (3.9.1) and (3.9.2) we can construct the sequences $\{\mathfrak{s}_{\mathfrak{v}_n}\}_{n=1}^{\infty}$, $\{\xi_{\mathfrak{v}_n}\}_{n=1}^{\infty}$ such that

$$\mathfrak{f}(\mathfrak{s}_{\mathfrak{v}_n}) = \mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n+1}} = \xi_{\mathfrak{v}_n}\ \forall\, n \in \mathbb{N} \cup \{0\}. \tag{3}$$

Observes that in C^*-algebra, if $\tilde{\kappa}, \tilde{b} \in \tilde{\mathscr{A}}_+$ and $\tilde{\kappa} \preceq \tilde{b}$, then for any $\tilde{x} \in \tilde{\mathscr{A}}_+$ both $\tilde{x}^*\tilde{\kappa}\tilde{x}$ and $\tilde{x}^*\tilde{b}\tilde{x}$ are positive. If $\xi_{\mathfrak{v}_n} = \xi_{\mathfrak{v}_{n+1}}$, then $\xi_{\mathfrak{v}_n}$ is a point of coincidence of $\mathfrak{f}$ and $\mathfrak{g}$. Therefore, we assume that

$\xi_{\mathfrak{v}_n} \neq \xi_{\mathfrak{v}_{n+1}}$ for all $n = 0,1,2,\ldots.$ Since, $\alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_0}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ and $\mathfrak{f}$ is a $\mathfrak{g}$-cyclic-(α,β)-admissible mapping, we have

$$\beta(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1}) = \beta(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_0}) \succ \tilde{0}_{\tilde{\mathscr{A}}} \implies \alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2}) = \alpha(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$$

$$\text{and } \beta(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_3}) = \beta(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2}) \succ \tilde{0}_{\tilde{\mathscr{A}}} \implies \alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_4}) = \alpha(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_3}) \succ \tilde{0}_{\tilde{\mathscr{A}}}.$$

By continuing this procedure, we obtain that:

$$\alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{2i}}) \succ \tilde{0}_{\tilde{\mathscr{A}}} \text{ and } \beta\left(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{2i+1}}\right) \succ \tilde{0}_{\tilde{\mathscr{A}}} \ \forall\, i \in \mathbb{N}\cup\{0\}. \tag{4}$$

Similarly, since, $\beta(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_0}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ and $\mathfrak{f}$ is a $\mathfrak{g}$-cyclic-(α,β)-admissible mapping, we have

$$\alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1}) = \alpha(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_0}) \succ \tilde{0}_{\tilde{\mathscr{A}}} \implies \beta(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2}) = \beta(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$$

$$\text{and } \alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_3}) = \alpha(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2}) \succ \tilde{0}_{\tilde{\mathscr{A}}} \implies \beta(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_4}) = \beta(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_3}) \succ \tilde{0}_{\tilde{\mathscr{A}}}.$$

By continuing this procedure, we obtain that:

$$\beta(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{2i}}) \succ \tilde{0}_{\tilde{\mathscr{A}}} \text{ and } \alpha\left(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{2i+1}}\right) \succ \tilde{0}_{\tilde{\mathscr{A}}} \ \forall\, i \in \mathbb{N}\cup\{0\}. \tag{5}$$

From Eq.(4) and Eq.(5), it follows that:

$$\alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_n}) \succ \tilde{0}_{\tilde{\mathscr{A}}} \text{ and } \beta(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_n}) \succ \tilde{0}_{\tilde{\mathscr{A}}} \ \forall\, n \in \mathbb{N}\cup\{0\}.$$

Consequently,

$$\alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_n}) \succ \tilde{0}_{\tilde{\mathscr{A}}} \text{ and } \beta\left(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n+1}}\right) \succ \tilde{0}_{\tilde{\mathscr{A}}} \implies \alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_n})\,\beta\left(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n+1}}\right) \succ \tilde{0}_{\tilde{\mathscr{A}}} \forall\, n \in \mathbb{N}\cup\{0\}. \tag{6}$$

Then from Eq.(1), we get

$$\begin{aligned} \varphi\left(\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{n+1}})\right) &\preceq \mathscr{F}_G\Big(\ \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{s}_{\mathfrak{v}_{n+1}})\tilde{a^\star}), \wp(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{s}_{\mathfrak{v}_{n+1}})\tilde{a^\star}), \varpi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{s}_{\mathfrak{v}_{n+1}})\tilde{a^\star})\ \Big) \\ &\preceq \max\Big\{\ \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{s}_{\mathfrak{v}_{n+1}})\tilde{a^\star}), \wp(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{s}_{\mathfrak{v}_{n+1}})\tilde{a^\star})\ \Big\} \end{aligned} \tag{7}$$

where

$$\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{s}_{\mathfrak{v}_{n+1}}) = \max\left\{\begin{array}{c} \tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n+1}}), \frac{1}{2}\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n+1}},\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_n}), \\ \frac{\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n+1}}).\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{n+1}})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n+1}})\right]}, \frac{\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_n}).\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{n+1}},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_n})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n+1}})\right]} \\ \frac{\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{n+1}},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n+1}}).\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n+1}})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n+1}})\right]} \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} \tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n-1}},\xi_{\mathfrak{v}_n}), \frac{1}{2}\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_n},\xi_{\mathfrak{v}_n}), \\ \frac{\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_n},\xi_{\mathfrak{v}_n})\cdot\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n-1}},\xi_{\mathfrak{v}_{n+1}})}{2\left[1+\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n-1}},\xi_{\mathfrak{v}_n})\right]}, \frac{\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_n},\xi_{\mathfrak{v}_{n-1}})\cdot\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n+1}},\xi_{\mathfrak{v}_{n-1}})}{2\left[1+\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n-1}},\xi_{\mathfrak{v}_n})\right]} \\ \frac{\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n+1}},\xi_{\mathfrak{v}_n})\cdot\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_n},\xi_{\mathfrak{v}_n})}{2\left[1+\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n-1}},\xi_{\mathfrak{v}_n})\right]} \end{array}\right\}$$

$$= \max\left\{ \tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n-1}},\xi_{\mathfrak{v}_n}), \tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_n},\xi_{\mathfrak{v}_{n+1}}) \right\}.$$

If $\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_i},\mathfrak{s}_{\mathfrak{v}_{i+1}}) = \tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_i},\xi_{\mathfrak{v}_{i+1}})$ for some $i \in \mathbb{N}\cup\{0\}$. Accordingly, we conclude that

$$\begin{aligned} \varphi\left(\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_i},\xi_{\mathfrak{v}_{i+1}})\right) &= \varphi\left(\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_i},\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{i+1}})\right) \\ &\preceq \max\left\{ \varphi(\tilde{a}\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_i},\xi_{\mathfrak{v}_{i+1}})\tilde{a^\star}), \wp(\tilde{a}\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_i},\xi_{\mathfrak{v}_{i+1}})\tilde{a^\star}) \right\} \\ &\prec \varphi(\tilde{a}\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_i},\xi_{\mathfrak{v}_{i+1}})\tilde{a^\star}). \end{aligned}$$

From non-decreasing property of φ, we have $\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_i},\xi_{\mathfrak{v}_{i+1}}) \prec \tilde{a}\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_i},\xi_{\mathfrak{v}_{i+1}})\tilde{a^\star}$ this implies that $||\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_i},\xi_{\mathfrak{v}_{i+1}})|| < ||\tilde{a}||^2||\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_i},\xi_{\mathfrak{v}_{i+1}})||$ which is a contradiction.
Accordingly, we conclude that $\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{s}_{\mathfrak{v}_{n+1}}) = \tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n-1}},\xi_{\mathfrak{v}_n})$ for all $n \in \mathbb{N}\cup\{0\}$.
Hence, from Eq.(7),we can write

$$\begin{aligned} \varphi\left(\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_n},\xi_{\mathfrak{v}_{n+1}})\right) &= \varphi\left(\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{n+1}})\right) \\ &\preceq \max\left\{ \varphi(\tilde{a}\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n-1}},\xi_{\mathfrak{v}_n})\tilde{a^\star}), \wp(\tilde{a}\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n-1}},\xi_{\mathfrak{v}_n})\tilde{a^\star}) \right\} \\ &\prec \varphi(\tilde{a}\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n-1}},\xi_{\mathfrak{v}_n})\tilde{a^\star}). \end{aligned}$$

By the definition of φ, we have

$$\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_n},\xi_{\mathfrak{v}_{n+1}}\right) \prec \tilde{a}^*\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n-1}},\xi_{\mathfrak{v}_n})\tilde{a}$$

which implies that

$$||\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_n},\xi_{\mathfrak{v}_{n+1}}\right)|| < ||\tilde{a}||^2||\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n-1}},\xi_{\mathfrak{v}_n})|| < ||\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n-1}},\xi_{\mathfrak{v}_n})||\forall n \in \mathbb{N}\cup\{0\}. \tag{8}$$

Thus,the sequence $\{\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_n},\xi_{\mathfrak{v}_{n+1}}\right)\}$ is decreasing bounded below in $\tilde{\mathbb{V}}$ with respect $\tilde{\mathscr{A}}$,and hence, there exists some $\tilde{\delta} \succeq \tilde{0}_{\tilde{\mathscr{A}}}$ such that $\lim_{n\to\infty}||\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_n},\xi_{\mathfrak{v}_{n+1}}\right)|| = \tilde{\delta}$. Now we claim that

$\tilde{\delta}=\tilde{0}$ and letting $n\to\infty$ in Eq.(7), we have that

$$\begin{aligned}\varphi\left(\tilde{\delta}\right) &\leq \mathscr{F}_G\left(\ \varphi(\tilde{\delta}),\wp(\tilde{\delta}),\varpi(\tilde{\delta})\ \right)\\ &\leq \max\left\{\ \varphi(\tilde{\delta}),\wp(\tilde{\delta})\ \right\}=\varphi(\tilde{\delta}).\end{aligned}$$

Therefore, $\mathscr{F}_G\left(\ \varphi(\tilde{\delta}),\wp(\tilde{\delta}),\varpi(\tilde{\delta})\ \right)=\varphi(\tilde{\delta})$. By the condition (3) of Definition (3.2), it can be deduced that either $\varphi(\tilde{\delta})=\tilde{0}$ or $\wp(\tilde{\delta})=\tilde{0}$. Hence, $\tilde{\delta}=\tilde{0}$. Suppose $\tilde{\delta}\succ\tilde{0}_{\tilde{\mathscr{A}}}$ and letting $n\to\infty$ in Eq. (8), we have that $\tilde{\delta}<\tilde{\delta}$, is a contradiction . Hence $\tilde{\delta}=\tilde{0}$.

Thus $\lim_{n\to\infty}||\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_n},\xi_{\mathfrak{v}_{n+1}}\right)||=\tilde{0}$. Next, we will show that $\{\xi_{\mathfrak{v}_n}\}$ is a Cauchy sequence in $\tilde{\mathbb{V}}$ with regard to $\tilde{\mathscr{A}}$. Let on contrary $\{\xi_{\mathfrak{v}_n}\}$ not be a Cauchy sequence, then for some $\tilde{\varepsilon}\succ\tilde{0}_{\tilde{\mathscr{A}}}$, there exists two subsequences $\{\xi_{\mathfrak{v}_{m(j)}}\}$ and $\{\xi_{\mathfrak{v}_{n(j)}}\}$ of $\{\xi_{\mathfrak{v}_n}\}$ such that

$$\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)}}\right)\succeq\tilde{\varepsilon} \tag{9}$$

where $n(j)>m(j)\geq j$ with $n(j)$ is odd and $m(j)$ is even. Corresponding to $m(j)$, one can choose the smallest number $n(j)$ with $n(j)>m(j)$ such that

$$\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)-1}}\right)\prec\tilde{\varepsilon}. \tag{10}$$

Using Inequalities Eq.(9) and Eq.(10) and the triangle inequity, we have

$$\begin{aligned}\tilde{\varepsilon} &\preceq \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)}}\right)\\ &\preceq \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)-1}}\right)+\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{n(j)-1}},\xi_{\mathfrak{v}_{n(j)}}\right)\\ &\preceq \tilde{\varepsilon}+\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{n(j)-1}},\xi_{\mathfrak{v}_{n(j)}}\right).\end{aligned}$$

Letting $n\to\infty$,we get

$$\lim_{n\to\infty}\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)}}\right)=\tilde{\varepsilon}. \tag{11}$$

It follows from the triangle inequity that

$$\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)}}\right) \preceq \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)+1}}\right)+\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{n(j)+1}},\xi_{\mathfrak{v}_{n(j)}}\right).$$

Letting $n\to\infty$, we get $\tilde{\varepsilon}\preceq\lim_{n\to\infty}\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)+1}}\right)$ and

$$\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)+1}}\right) \preceq \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)}}\right)+\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{n(j)+1}},\xi_{\mathfrak{v}_{n(j)}}\right).$$

Letting $n \to \infty$, we get $\lim\limits_{n\to\infty} \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}}, \xi_{\mathfrak{v}_{n(j)+1}}\right) \preceq \tilde{\varepsilon}$. Therefore,

$$\lim_{n\to\infty} \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}}, \xi_{\mathfrak{v}_{n(j)+1}}\right) = \tilde{\varepsilon}. \tag{12}$$

Similarly,we can show that

$$\lim_{n\to\infty} \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{n(j)}}, \xi_{\mathfrak{v}_{m(j)+1}}\right) = \tilde{\varepsilon}. \tag{13}$$

Again using the triangular inequality,we get

$$\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}}, \xi_{\mathfrak{v}_{n(j)+1}}\right) \preceq \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}}, \xi_{\mathfrak{v}_{m(j)+1}}\right) + \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)+1}}, \xi_{\mathfrak{v}_{n(j)+1}}\right).$$

Letting $n \to \infty$, we get $\tilde{\varepsilon} \preceq \lim\limits_{n\to\infty} \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)+1}}, \xi_{\mathfrak{v}_{n(j)+1}}\right)$ and

$$\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)+1}}, \xi_{\mathfrak{v}_{n(j)+1}}\right) \preceq \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)+1}}, \xi_{\mathfrak{v}_{m(j)}}\right) + \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)}}, \xi_{\mathfrak{v}_{n(j)+1}}\right).$$

Letting $n \to \infty$, we get $\lim\limits_{n\to\infty} \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)+1}}, \xi_{\mathfrak{v}_{n(j)+1}}\right) \preceq \tilde{\varepsilon}$.
Therefore,

$$\lim_{n\to\infty} \tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)+1}}, \xi_{\mathfrak{v}_{n(j)+1}}\right) = \tilde{\varepsilon}. \tag{14}$$

From Eq. (6), we obtain $\alpha\left(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{m(j)+1}}\right)\beta\left(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n(j)+1}}\right) \succ \tilde{0}_{\mathscr{A}}$, and from Eq. (1), we can write

$$\begin{aligned}
\varphi(\tilde{\rho_{c^*}}\left(\xi_{\mathfrak{v}_{m(j)+1}}, \xi_{\mathfrak{v}_{n(j)+1}}\right)) &= \varphi\left(\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{m(j)+1}}, \mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{n(j)+1}})\right) \\
&\preceq \mathscr{F}_G\left(\begin{array}{c} \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_{m(j)+1}}, \mathfrak{s}_{\mathfrak{v}_{n(j)+1}})\tilde{a^\star}), \wp(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_{m(j)+1}}, \mathfrak{s}_{\mathfrak{v}_{n(j)+1}})\tilde{a^\star}), \\ \varpi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_{m(j)+1}}, \mathfrak{s}_{\mathfrak{v}_{n(j)+1}})\tilde{a^\star}) \end{array}\right) \\
&\preceq \max\left\{ \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_{m(j)+1}}, \mathfrak{s}_{\mathfrak{v}_{n(j)+1}})\tilde{a^\star}), \wp(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_{m(j)+1}}, \mathfrak{s}_{\mathfrak{v}_{n(j)+1}})\tilde{a^\star}) \right\} \\
&\prec \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_{m(j)+1}}, \mathfrak{s}_{\mathfrak{v}_{n(j)+1}})\tilde{a^\star}) \\
&\prec \varphi\left(\tilde{a}\max\left\{\begin{array}{c} \tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)}}, \xi_{\mathfrak{v}_{n(j)}}), \frac{1}{2}\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n(j)}}, \xi_{\mathfrak{v}_{m(j)+1}}), \\ \frac{\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)+1}}, \xi_{\mathfrak{v}_{n(j)}}).\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)}}, \xi_{\mathfrak{v}_{n(j)+1}})}{2\left[1+\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)}}, \xi_{\mathfrak{v}_{n(j)}})\right]}, \\ \frac{\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)+1}}, \xi_{\mathfrak{v}_{m(j)}}).\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n(j)+1}}, \xi_{\mathfrak{v}_{m(j)}})}{2\left[1+\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)}}, \xi_{\mathfrak{v}_{n(j)}})\right]} \\ \frac{\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n(j)+1}}, \xi_{\mathfrak{v}_{n(j)}}).\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)+1}}, \xi_{\mathfrak{v}_{n(j)}})}{2\left[1+\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)}}, \xi_{\mathfrak{v}_{n(j)}})\right]} \end{array}\right\}\tilde{a^\star}\right).
\end{aligned}$$

Because of

$$\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_{m(j)+1}},\mathfrak{s}_{\mathfrak{v}_{n(j)+1}}) = \max\left\{\begin{array}{c} \tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{m(j)+1}},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n(j)+1}}),\frac{1}{2}\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n(j)+1}},\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{m(j)+1}}),\\ \frac{\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{m(j)+1}},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n(j)+1}}).\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{m(j)+1}},\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{n(j)+1}})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{m(j)+1}},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n(j)+1}})\right]},\\ \frac{\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{m(j)+1}},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{m(j)+1}}).\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{n(j)+1}},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{m(j)+1}})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{m(j)+1}},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n(j)+1}})\right]}\\ \frac{\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{n(j)+1}},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n(j)+1}}).\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_{m(j)+1}},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n(j)+1}})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{m(j)+1}},\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n(j)+1}})\right]} \end{array}\right\}$$

$$= \max\left\{\begin{array}{c} \tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)}}),\frac{1}{2}\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n(j)}},\xi_{\mathfrak{v}_{m(j)+1}}),\\ \frac{\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)+1}},\xi_{\mathfrak{v}_{n(j)}}).\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)+1}})}{2\left[1+\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)}})\right]},\\ \frac{\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)+1}},\xi_{\mathfrak{v}_{m(j)}}).\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n(j)+1}},\xi_{\mathfrak{v}_{m(j)}})}{2\left[1+\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)}})\right]}\\ \frac{\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{n(j)+1}},\xi_{\mathfrak{v}_{n(j)}}).\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)+1}},\xi_{\mathfrak{v}_{n(j)}})}{2\left[1+\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}_{m(j)}},\xi_{\mathfrak{v}_{n(j)}})\right]} \end{array}\right\}.$$

Using $n\to\infty$ with respect $\mathscr{A}$ and $||a||<1$, Eq.(11), Eq.(12), Eq.(13), Eq.(14) we obtain that

$$\varphi(||\tilde{\varepsilon}||) \quad \leq \mathscr{F}_G\Big(\ \varphi(||\tilde{\varepsilon}||),\wp(||\tilde{\varepsilon}||),\varpi(||\tilde{\varepsilon}||)\ \Big) < \varphi(||\tilde{\varepsilon}||).$$

Therefore, $\mathscr{F}_G\Big(\ \varphi(||\tilde{\varepsilon}||),\wp(||\tilde{\varepsilon}||),\varpi(||\tilde{\varepsilon}||)\ \Big) = \varphi(||\tilde{\varepsilon}||)$. By the condition (3) of Definition (3.2), it can be deduced that either $\varphi(||\tilde{\varepsilon}||)=0$ or $\wp(||\tilde{\varepsilon}||)=0$ implies $\tilde{\varepsilon}=0_{\tilde{\mathscr{A}}}$ is contradiction. As a result, $\{\xi_{\mathfrak{v}_n}\}$ is a Cauchy sequence in $\tilde{\mathbb{V}}$ with regard to $\tilde{\mathscr{A}}$. However, $(\tilde{\mathbb{V}},\tilde{\mathscr{A}},\tilde{\rho_{c^*}})$ is complete, so there exists $\xi_{\mathfrak{v}'}\in\tilde{\mathbb{V}}$ such that $\lim_{n\to\infty}\xi_{\mathfrak{v}_n}=\xi_{\mathfrak{v}'}$ and hence, from Eq.(3)

$$\lim_{n\to\infty}\mathfrak{f}(\mathfrak{s}_{\mathfrak{v}_n}) = \lim_{n\to\infty}\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n+1}} = \lim_{n\to\infty}\xi_{\mathfrak{v}_n} = \xi_{\mathfrak{v}'}. \tag{15}$$

Since $\mathfrak{g}(\tilde{\mathbb{V}})$ is closed subspace of $\tilde{\mathbb{V}}$ so in view of Eq.(15), $\xi_{\mathfrak{v}'}\in\mathfrak{g}(\tilde{\mathbb{V}})$ and therefore, one can find $\mathfrak{s}_{\mathfrak{v}'}\in\tilde{\mathbb{V}}$ such that $\mathfrak{g}(\mathfrak{s}_{\mathfrak{v}'})=\xi_{\mathfrak{v}'}$. Now, we will show that $\mathfrak{f}(\mathfrak{s}_{\mathfrak{v}'})=\xi_{\mathfrak{v}'}$. For this, since $\xi_{\mathfrak{v}_n}\to\xi_{\mathfrak{v}'}$, so from Eq.(3), it follows that $\beta(\xi_{\mathfrak{v}_n})=\beta(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_{n+1}})\succ 0_{\tilde{\mathscr{A}}}$ for all $n\in\mathbb{N}$. From Condition (3.9.2), we have $\beta(\xi_{\mathfrak{v}'})=\beta(\mathfrak{g}(\mathfrak{s}_{\mathfrak{v}'}))\succ 0_{\tilde{\mathscr{A}}}$ and thus, $\alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_n})\beta(\mathfrak{g}(\mathfrak{s}_{\mathfrak{v}'}))\succ 0_{\tilde{\mathscr{A}}}$ then, from Eq.(1), we have

$$\begin{aligned}\varphi\left(\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{f}\mathfrak{s}_{\mathfrak{v}'})\right) &\preceq \mathscr{F}_G\Big(\ \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{s}_{\mathfrak{v}'})\tilde{a^\star}),\wp(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{s}_{\mathfrak{v}'})\tilde{a^\star}),\varpi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{s}_{\mathfrak{v}'})\tilde{a^\star})\ \Big)\\ &\preceq \max\Big\{\ \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{s}_{\mathfrak{v}'})\tilde{a^\star}),\wp(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{s}_{\mathfrak{v}'})\tilde{a^\star})\ \Big\}\\ &\prec \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_n},\mathfrak{s}_{\mathfrak{v}'})\tilde{a^\star})\end{aligned}$$

$$\prec \varphi\left(\tilde{a}\max\left\{\begin{array}{c}\tilde{\rho_{c^*}}(\mathfrak{gs}_{\mathfrak{v}_n},\mathfrak{gs}_{\mathfrak{v}'}),\frac{1}{2}\tilde{\rho_{c^*}}(\mathfrak{gs}_{\mathfrak{v}'},\mathfrak{fs}_{\mathfrak{v}_n}),\\ \frac{\tilde{\rho_{c^*}}(\mathfrak{fs}_{\mathfrak{v}_n},\mathfrak{gs}_{\mathfrak{v}'}).\tilde{\rho_{c^*}}(\mathfrak{gs}_{\mathfrak{v}_n},\mathfrak{fs}_{\mathfrak{v}'})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{gs}_{\mathfrak{v}_n},\mathfrak{gs}_{\mathfrak{v}'})\right]},\frac{\tilde{\rho_{c^*}}(\mathfrak{fs}_{\mathfrak{v}_n},\mathfrak{gs}_{\mathfrak{v}_n}).\tilde{\rho_{c^*}}(\mathfrak{fs}_{\mathfrak{v}'},\mathfrak{gs}_{\mathfrak{v}_n})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{gs}_{\mathfrak{v}_n},\mathfrak{gs}_{\mathfrak{v}'})\right]}\\ \frac{\tilde{\rho_{c^*}}(\mathfrak{fs}_{\mathfrak{v}'},\mathfrak{gs}_{\mathfrak{v}'}).\tilde{\rho_{c^*}}(\mathfrak{fs}_{\mathfrak{v}_n},\mathfrak{gs}_{\mathfrak{v}'})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{gs}_{\mathfrak{v}_n},\mathfrak{gs}_{\mathfrak{v}'})\right]}\end{array}\right\}\tilde{a^\star}\right).$$

Now, taking the limit as $n\to\infty$ with $||\tilde{a}||<1$ and using the above Eq. (15), we get

$$\varphi\left(||\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}'},\mathfrak{fs}_{\mathfrak{v}'})||\right)<\varphi(0)=0$$

which is possible only if $\varphi\left(||\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}'},\mathfrak{fs}_{\mathfrak{v}'})||\right)=0$. Thus, $||\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}'},\mathfrak{fs}_{\mathfrak{v}'})||=0\Rightarrow\mathfrak{fs}_{\mathfrak{v}'}=\xi_{\mathfrak{v}'}$ and hence, $\mathfrak{fs}_{\mathfrak{v}'}=\mathfrak{gs}_{\mathfrak{v}'}=\xi_{\mathfrak{v}'}$. Next, to show that $\mathfrak{f}$ and $\mathfrak{g}$ have a unique point of coincidence $\xi_{\mathfrak{v}'}$, let $\mathfrak{f}$ and $\mathfrak{g}$ have another point of coincidence $\xi_{\mathfrak{v}'}\neq\xi_{\mathfrak{v}''}$. Then, there exists $\mathfrak{e}_{\mathfrak{v}'}\in\tilde{\mathbb{V}}$ so that $\mathfrak{fe}_{\mathfrak{v}'}=\mathfrak{ge}_{\mathfrak{v}'}=\xi_{\mathfrak{v}''}$. Using Condition (3.9.4), we get $\alpha(\mathfrak{gs}_{\mathfrak{v}'})\,\beta(\mathfrak{ge}_{\mathfrak{v}'})\succ\tilde{0}_{\tilde{\mathscr{A}}}$, then from Eq.(1), we get

$$\begin{aligned}\varphi\left(\tilde{\rho_{c^*}}(\mathfrak{fs}_{\mathfrak{v}'},\mathfrak{fe}_{\mathfrak{v}'})\right)&\preceq\mathscr{F}_G\left(\ \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{e}_{\mathfrak{v}'})\tilde{a^\star}),\wp(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{e}_{\mathfrak{v}'})\tilde{a^\star}),\varpi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{e}_{\mathfrak{v}'})\tilde{a^\star})\ \right)\\&\preceq\max\left\{\ \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{e}_{\mathfrak{v}'})\tilde{a^\star}),\wp(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{e}_{\mathfrak{v}'})\tilde{a^\star})\ \right\}\\&\prec\varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{e}_{\mathfrak{v}'})\tilde{a^\star})\\&\prec\varphi\left(\tilde{a}\max\left\{\begin{array}{c}\tilde{\rho_{c^*}}(\mathfrak{gs}_{\mathfrak{v}'},\mathfrak{ge}_{\mathfrak{v}'}),\frac{1}{2}\tilde{\rho_{c^*}}(\mathfrak{ge}_{\mathfrak{v}'},\mathfrak{fs}_{\mathfrak{v}'}),\\ \frac{\tilde{\rho_{c^*}}(\mathfrak{fs}_{\mathfrak{v}'},\mathfrak{ge}_{\mathfrak{v}'}).\tilde{\rho_{c^*}}(\mathfrak{gs}_{\mathfrak{v}'},\mathfrak{fe}_{\mathfrak{v}'})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{gs}_{\mathfrak{v}'},\mathfrak{ge}_{\mathfrak{v}'})\right]},\frac{\tilde{\rho_{c^*}}(\mathfrak{fs}_{\mathfrak{v}'},\mathfrak{gs}_{\mathfrak{v}'}).\tilde{\rho_{c^*}}(\mathfrak{fe}_{\mathfrak{v}'},\mathfrak{gs}_{\mathfrak{v}'})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{gs}_{\mathfrak{v}'},\mathfrak{ge}_{\mathfrak{v}'})\right]}\\ \frac{\tilde{\rho_{c^*}}(\mathfrak{fe}_{\mathfrak{v}'},\mathfrak{ge}_{\mathfrak{v}'}).\tilde{\rho_{c^*}}(\mathfrak{fs}_{\mathfrak{v}'},\mathfrak{ge}_{\mathfrak{v}'})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{gs}_{\mathfrak{v}'},\mathfrak{ge}_{\mathfrak{v}'})\right]}\end{array}\right\}\tilde{a^\star}\right)\\&\prec\varphi(\tilde{a}\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}'},\xi_{\mathfrak{v}''})\tilde{a^\star}).\end{aligned}$$

By the definition of φ, we have

$$\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}'},\xi_{\mathfrak{v}''})\prec\tilde{a}\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}'},\xi_{\mathfrak{v}''})\tilde{a^\star}$$

which implies that

$$||\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}'},\xi_{\mathfrak{v}''})||<||\tilde{a}||^2||\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}'},\xi_{\mathfrak{v}''})||<||\tilde{\rho_{c^*}}(\xi_{\mathfrak{v}'},\xi_{\mathfrak{v}''})||$$

which is a contradiction, unless $\xi_{\mathfrak{v}'}=\xi_{\mathfrak{v}''}$. Finally, since the pair $(\mathfrak{f},\mathfrak{g})$ is weakly compatible, since $\mathfrak{fs}_{\mathfrak{v}'}=\mathfrak{gs}_{\mathfrak{v}'}=\xi_{\mathfrak{v}'}$ then $\mathfrak{f}\xi_{\mathfrak{v}'}=\mathfrak{fgs}_{\mathfrak{v}'}=\mathfrak{gfs}_{\mathfrak{v}'}=\mathfrak{g}\xi_{\mathfrak{v}'}$ i.e., $\mathfrak{f}\xi_{\mathfrak{v}'}=\mathfrak{g}\xi_{\mathfrak{v}'}$ is a point of coincidence of $\mathfrak{f}$ and $\mathfrak{g}$. But $\xi_{\mathfrak{v}'}$ is the only point of coincidence of $\mathfrak{f}$ and $\mathfrak{g}$, so $\mathfrak{f}\xi_{\mathfrak{v}'}=\mathfrak{g}\xi_{\mathfrak{v}'}=\xi_{\mathfrak{v}'}$. Moreover if $\mathfrak{f}\vartheta_{\mathfrak{v}'} = \mathfrak{g}\vartheta_{\mathfrak{v}'} = \vartheta_{\mathfrak{v}'}$, then $\vartheta_{\mathfrak{v}'}$ is a point of coincidence of $\mathfrak{f}$ and $\mathfrak{g}$, and therefore $\xi_{\mathfrak{v}'} = \vartheta_{\mathfrak{v}'}$ by uniqueness. Thus $\xi_{\mathfrak{v}'}$ is a unique common fixed point of $\mathfrak{f}$ and $\mathfrak{g}$ in $\tilde{\mathbb{V}}$.

Let $\alpha(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_1})\beta(\mathfrak{g}\mathfrak{s}_{\mathfrak{v}_2}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$, then from Eq.(2), we get

$$\varphi\left(\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2})\right) \preceq \mathscr{F}_G\Big(\ \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star}), \wp(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star}), \varpi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star})\ \Big).$$

Thus, the Eq.(1) is satisfied, hence, the proof easily follows similar lines of above and $\mathfrak{f}$ and $\mathfrak{g}$ have a unique common fixed point in $\tilde{\mathbb{V}}$.

Corollary 3.10: Let $(\tilde{\mathbb{V}},\tilde{\mathscr{A}},\tilde{\rho_{c^*}})$ be a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VFSMS}$ and $\alpha,\beta : \tilde{\mathbb{V}} \to \tilde{\mathscr{A}}_+$ be two mappings. Let $\mathfrak{f} : \tilde{\mathbb{V}} \to \tilde{\mathbb{V}}$ be a cyclic (α,β)-admissible mapping with regard to a $\mathscr{C}_{G^\star}$-class functions $\mathscr{F}_G$ with $\varphi(\tilde{\mathfrak{e}}) > \wp(\tilde{\mathfrak{e}})$ for all $\tilde{\mathfrak{e}} \succ \tilde{0}_{\tilde{\mathscr{A}}}$ such that

$$\begin{aligned}\alpha(\mathfrak{s}_{\mathfrak{v}_1})\beta(\mathfrak{s}_{\mathfrak{v}_2}) \succ \tilde{0}_{\tilde{\mathscr{A}}} \Rightarrow \varphi\left(\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2})\right)&\\ \preceq \mathscr{F}_G\Big(\ \varphi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star}), \wp(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star}), \varpi(\tilde{a}\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star})\ \Big)&\end{aligned}$$

where $\mathbb{M}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2}) = \max\left\{\begin{array}{c}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2}), \frac{1}{2}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_2},\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1}),\\ \frac{\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2}).\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2})}{2[1+\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})]}, \frac{\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_1}).\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2},\mathfrak{s}_{\mathfrak{v}_1})}{2[1+\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})]}\\ \frac{\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2},\mathfrak{s}_{\mathfrak{v}_2}).\tilde{\rho_{c^*}}(\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})}{2\left[1+\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})\right]}\end{array}\right\}$,

$\varphi \in \Omega$ $\wp \in \Delta$, $\varpi \in \Theta$, $\mathscr{F}_G \in \Gamma$. If the following assumptions hold:

(3.10.1) there exists $\mathfrak{s}_{\mathfrak{v}_0} \in \tilde{\mathbb{V}}$ with $\alpha(\mathfrak{s}_{\mathfrak{v}_0}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ and $\beta(\mathfrak{s}_{\mathfrak{v}_0}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$;

(3.10.2) if $\{\mathfrak{s}_{\mathfrak{v}_n}\}_{n=1}^{\infty}$ is a sequence in $\tilde{\mathbb{V}}$ with $\beta(\mathfrak{s}_{\mathfrak{v}_n}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ for all n and $\lim_{n\to\infty}\mathfrak{s}_{\mathfrak{v}_n} = \mathfrak{s}_{\mathfrak{v}'}$ then $\beta(\mathfrak{s}_{\mathfrak{v}'}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$;

(3.10.3) $\alpha(\mathfrak{s}_{\mathfrak{v}_1}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ and $\beta(\mathfrak{s}_{\mathfrak{v}_2}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ whenever, $\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_1} = \mathfrak{s}_{\mathfrak{v}_1}$ and $\mathfrak{f}\mathfrak{s}_{\mathfrak{v}_2} = \mathfrak{s}_{\mathfrak{v}_2}$.

Then, $\mathfrak{f}$ has a unique fixed point in $\tilde{\mathbb{V}}$.

proof Using the identity map on $\tilde{\mathbb{V}}$ and $\mathfrak{g} = \tilde{I}_{\tilde{\mathscr{A}}}$, we can determine from Theorem (3.9) that $\mathfrak{f}$ has a UFP.

Example 3.11: Let $\mathbb{U} = \{\mathfrak{x}_1,\mathfrak{x}_2\}$ be a universe and $\mathbb{E} = \{\mathfrak{v}_1,\mathfrak{v}_2\}$ be a parameter set and $\tilde{\mathbb{V}} = \{\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2}\}$ be fuzzy soft set, for each parameter $\mathfrak{v}_i$, define fuzzy soft elements $\mathfrak{s}_{\mathfrak{v}_1} = \{(\mathfrak{x}_1,0.6),(\mathfrak{x}_2,0.8)\}$, $\mathfrak{s}_{\mathfrak{v}_2} = \{(\mathfrak{x}_1,0.4),(\mathfrak{x}_2,0.9)\}$. Let $A = \begin{bmatrix}\mathfrak{s}_{\mathfrak{v}_1} & \mathfrak{s}_{\mathfrak{v}_2}\\ \mathfrak{s}_{\mathfrak{v}_2} & \mathfrak{s}_{\mathfrak{v}_1}\end{bmatrix} \in \tilde{\mathscr{A}}$ be the set of 2×2 matrices whose entries are fuzzy soft elements with $||A|| = \max_{i,j}\left(\max_{\mathfrak{x}\in\mathbb{U}}\mu_{\mathfrak{v}_k}(\mathfrak{x})\right)$.

Define $\tilde{\rho_{c^*}} : \tilde{\mathbb{V}} \times \tilde{\mathbb{V}} \to \tilde{\mathscr{A}}$ by $\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_i}, \mathfrak{s}_{\mathfrak{v}_j}) = \begin{cases} \tilde{0}_{\tilde{\mathscr{A}}} \text{ if } i = j \\ \begin{bmatrix} \mathfrak{s}_{\mathfrak{v}_1} & 0 \\ 0 & \mathfrak{s}_{\mathfrak{v}_2} \end{bmatrix} \text{ if } i \neq j. \end{cases}$, then obviously $(\tilde{\mathbb{V}}, \tilde{\mathscr{A}}, \tilde{d}_{c^*})$ is a complete C^*-algebra valued fuzzy soft metric space.

We define $\mathfrak{f}, \mathfrak{g} \colon \tilde{\mathbb{V}} \to \tilde{\mathbb{V}}$ by $\mathfrak{f}(\mathfrak{s}_{\mathfrak{v}_1}) = \mathfrak{s}_{\mathfrak{v}_2}$, $\mathfrak{f}(\mathfrak{s}_{\mathfrak{v}_2}) = \mathfrak{s}_{\mathfrak{v}_1}$ and $\mathfrak{g}(\mathfrak{s}_{\mathfrak{v}_1}) = \mathfrak{s}_{\mathfrak{v}_2}$, $\mathfrak{g}(\mathfrak{s}_{\mathfrak{v}_2}) = \mathfrak{s}_{\mathfrak{v}_1}$. So $\mathfrak{f} = \mathfrak{g}$, and they are cyclic and weakly compatible.

Let $\alpha, \beta : \tilde{\mathbb{V}} \to \tilde{\mathscr{A}}_+$ be as $\alpha(\mathfrak{s}_{\mathfrak{v}_i}) = \begin{bmatrix} \mathfrak{s}_{\mathfrak{v}_1} & 0 \\ 0 & \mathfrak{s}_{\mathfrak{v}_1} \end{bmatrix}$ and $\beta(\mathfrak{s}_{\mathfrak{v}_i}) = \begin{bmatrix} \mathfrak{s}_{\mathfrak{v}_2} & 0 \\ 0 & \mathfrak{s}_{\mathfrak{v}_2} \end{bmatrix}$, then clearly, $\alpha(\mathfrak{s}_{\mathfrak{v}_i}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ and $\beta(\mathfrak{s}_{\mathfrak{v}_i}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$. Let $\varphi(\tilde{b}) = \tilde{b}$, $\wp(\tilde{b}) = \frac{1}{2}\tilde{b}$, $\varpi(\tilde{b}) = \frac{1}{3}\tilde{b}$ for all $\tilde{b} \in \tilde{\mathscr{A}}$ and $\mathscr{F}_G(\mathfrak{s}, \mathfrak{e}, \mathfrak{z}) = \max\{\mathfrak{s}, \mathfrak{e}, \mathfrak{z}\}$ then clearly, $\varphi(\tilde{b}) > \wp(\tilde{b})$ for all $\tilde{b} \succ \tilde{0}_{\tilde{\mathscr{A}}}$. Then, all the conditions of Theorem 3.9 satisfied with $\tilde{a} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$ with $||\tilde{a}|| < 1$ and $\mathfrak{s}_{\mathfrak{v}^\star} = \{(\mathfrak{x}_1, \frac{0.4+0.6}{2} = 0.5), (\mathfrak{x}_2, \frac{0.9+0.8}{2} = 0.85)\}$ is unique common fixed point of $\mathfrak{f}$ and $\mathfrak{g}$.

4. Applications

4.1. Application to Integral Equations.

In this section, we study the existence of an unique solution to an initial value problem, as an application to Corollary 3.10.

Let us consider the boundary value problem of ODE:

$$\mathfrak{s}'_{\mathfrak{v}_1}(\mathfrak{e}) = \mathscr{G}(\mathfrak{e}, \mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e})),\ \mathfrak{e} \in \mathbb{E},\ \mathfrak{s}_{\mathfrak{v}_1}(0) = \mathfrak{s}_{\mathfrak{v}_1}(1) = 0 \tag{16}$$

where $\mathscr{G} : \mathbb{E} \times \mathbb{R}(\mathbb{E})_+ \to \mathbb{R}(\mathbb{E})_+$ is a continuous function where $\mathbb{R}(\mathbb{E})_+$ is a set of non-negative fuzzy soft real numbers. The Green's function to Eq.(16) is given by

$$\mathbb{I}(\mathfrak{e}, \mathfrak{z}) = \begin{cases} \mathfrak{e}(1-\mathfrak{z}) \text{ if } 0 \leq \mathfrak{e} \leq \mathfrak{z} \\ \mathfrak{z}(1-\mathfrak{e}) \text{ if } \mathfrak{z} \leq \mathfrak{e} \leq 1. \end{cases}$$

Let $\mathbb{V} = \mathbb{E} = [0,1]$, $\mathscr{G} = L^2(\mathbb{E})$, and the absolute fuzzy soft set $\tilde{\mathbb{V}} = L^\infty(\mathbb{E})$ where $\mathbb{E}$ is a Lebesgue measurable set. The set of bounded linear operators on Hilbert space $\mathscr{G}$ is denoted by $L(\mathscr{G})$. $L(\mathscr{G})$ is undoubtedly a C^*-algebra with the standard operator norm.

Define $\tilde{\rho_{c^*}} : \tilde{\mathbb{V}} \times \tilde{\mathbb{V}} \to L(\mathscr{G})$ by $\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2}) = \mathbb{M}_{\inf\{|\mu^a_{\mathfrak{s}_{\mathfrak{v}_1}}(s) - \mu^a_{\mathfrak{s}_{\mathfrak{v}_2}}(s)|/s \in \mathbb{E}\}}$ for all $\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2} \in \tilde{\mathbb{V}}$, where $\mathbb{M}_u : \mathscr{G} \to \mathscr{G}$ is the multiplication operator defined by $\mathbb{M}_u(\eta) = u.\eta$ for $\eta \in \mathscr{G}$. Then $\tilde{\rho_{c^*}}$

is a $\mathscr{C}^*$-$\mathscr{A}\mathbb{VFSM}$ and $(\tilde{\mathbb{V}}, L(\mathscr{G}), \tilde{\rho_{c^*}})$ is a complete $\mathscr{C}^*$-$\mathscr{A}\mathbb{VFSMS}$.
Let $\varphi, \wp, \varpi : \tilde{\mathscr{A}}_+ \to \tilde{\mathscr{A}}_+$ as $\varphi(\tilde{b}) = \tilde{b}$, $\wp(\tilde{b}) = \frac{3\tilde{b}}{2}$ and $\varpi(\tilde{b}) = \frac{\tilde{b}}{6}$ and $\mathscr{F}_G : \tilde{\mathscr{A}}_+ \times \tilde{\mathscr{A}}_+ \times \tilde{\mathscr{A}}_+ \to \tilde{\mathscr{A}}_+$ as $\mathscr{F}_G(\tilde{\mathfrak{e}}, \tilde{\mathfrak{z}}, \tilde{\mathfrak{x}}) = \tilde{\mathfrak{z}} - \mathfrak{e} - \tilde{\mathfrak{x}}$ where $\frac{\tilde{\mathfrak{z}}-\mathfrak{e}}{2} > \tilde{\mathfrak{x}}$.

Consider the following conditions,

(i) if there exist a functions $\psi, \phi : \mathbb{R}(\mathbb{E})_+ \to \mathbb{R}(\mathbb{E})^\star$ such that there is an $\mathfrak{e} \in \mathbb{V}$, for all $\mathfrak{s}_{\mathfrak{v}_1} \in \mathbb{R}(\mathbb{E})_+$, with $\psi(\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e})) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ and $\phi(\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e})) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ and $\theta \in (0,1)$, we have

$$\inf\{|\mathscr{G}(\mathfrak{e}, \mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e})) - \mathscr{G}(\mathfrak{e}, \mathfrak{s}_{\mathfrak{v}_2}(\mathfrak{e}))|/\mathfrak{e} \in \mathbb{E}\} \leq \frac{\theta}{3} \inf\{|\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e}) - \mathfrak{s}_{\mathfrak{v}_2}(\mathfrak{e})|/\mathfrak{e} \in \mathbb{E}\}$$

(ii) there is $\mathfrak{s}_{\mathfrak{v}_1} \in L^\infty(\mathbb{E})$ and $\mathfrak{z} \in \mathbb{V}$ such that

$$\psi(\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{z})) \succ \tilde{0}_{\tilde{\mathscr{A}}} \implies \psi\left(\int_{\mathbb{E}} \mathbb{I}(\mathfrak{e}, \mathfrak{z}) \mathscr{G}(\mathfrak{z}, \mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{z})) d\mathfrak{z}\right) \succ \tilde{0}_{\tilde{\mathscr{A}}}$$

and

$$\phi(\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{z})) \succ \tilde{0}_{\tilde{\mathscr{A}}} \implies \phi\left(\int_{\mathbb{E}} \mathbb{I}(\mathfrak{e}, \mathfrak{z}) \mathscr{G}(\mathfrak{z}, \mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{z})) d\mathfrak{z}\right) \succ \tilde{0}_{\tilde{\mathscr{A}}}$$

(iii) for any point $\mathfrak{s}_{\mathfrak{v}'}$ of a sequence $\{\mathfrak{s}_{\mathfrak{v}_n}\}$ of points in $L^\infty(\mathbb{E})$ with $\psi(\mathfrak{s}_{\mathfrak{v}_n}(\mathfrak{z})) \succ \tilde{0}_{\tilde{\mathscr{A}}}$,
$\lim_{n\to\infty} \inf \psi(\mathfrak{s}_{\mathfrak{v}_n}(\mathfrak{z})) = \inf \psi(\mathfrak{s}_{\mathfrak{v}'}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ and $\phi(\mathfrak{s}_{\mathfrak{v}_n}(\mathfrak{z})) \succ \tilde{0}_{\tilde{\mathscr{A}}}$,
$\lim_{n\to\infty} \inf \phi(\mathfrak{s}_{\mathfrak{v}_n}(\mathfrak{z})) = \inf \phi(\mathfrak{s}_{\mathfrak{v}'}) \succ \tilde{0}_{\tilde{\mathscr{A}}}$.

Theorem 4.1: Suppose that conditions (i) - (iii) are satisfied. Then BVP Eq.(16) has minimum one solution $\mathfrak{s}_{\mathfrak{v}'} \in \tilde{\mathbb{V}}$.

Proof Since we know that the solution of (16) is exists iff the solution of the integral equation

$$\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e}) = \int_{\mathbb{E}} \mathbb{I}(\mathfrak{e}, \mathfrak{z}) \mathscr{G}(\mathfrak{e}, \mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e})) d\mathfrak{e}, \quad \mathfrak{e} \in \mathbb{E}.$$

is exist and the same. Define $\mathscr{T} : \tilde{\mathbb{V}} \to \tilde{\mathbb{V}}$ by $\mathscr{T}(\mathfrak{s}_{\mathfrak{v}_1})(\mathfrak{e}) = \int_{\mathbb{E}} \mathbb{I}(\mathfrak{e}, \mathfrak{z}) \mathscr{G}(\mathfrak{e}, \mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e})) d\mathfrak{e}$, $\forall\ \mathfrak{e} \in \mathbb{E}$. Clearly $\mathfrak{s}_{\mathfrak{v}'} \in \tilde{\mathbb{V}}$ that is a fixed point of $\mathscr{T}$. Let $\tilde{a} = \theta \tilde{I}_{\mathscr{A}}$ then $\tilde{a} \in L(\mathscr{G})$ with $||\tilde{a}|| < 1$. For any $u \in \mathscr{G}$ and let $\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2} \in \tilde{\mathbb{V}}$ such that $\psi(\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e})) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ and $\phi(\mathfrak{s}_{\mathfrak{v}_2}(\mathfrak{e})) \succ \tilde{0}_{\tilde{\mathscr{A}}}$ for all $\mathfrak{z} \in \mathbb{E}$. Then from (i), we have

$$\tilde{\rho_{c^*}}(\mathscr{T}\mathfrak{s}_{\mathfrak{v}_1}, \mathscr{T}\mathfrak{s}_{\mathfrak{v}_2}) = \mathbb{M}_{\inf\{|\mu^a_{\mathscr{T}\mathfrak{s}_{\mathfrak{v}_1}}(s) - \mu^a_{\mathscr{T}\mathfrak{s}_{\mathfrak{v}_2}}(s)|/s \in \mathbb{E}\}}$$

then

$$||\tilde{\rho_{c^*}}(\mathscr{T}\mathfrak{s}_{\mathfrak{v}_1},\mathscr{T}\mathfrak{s}_{\mathfrak{v}_2})|| = \sup_{||h||=1}(\mathbb{M}_{\inf\{|\mu^a_{\mathscr{T}\mathfrak{s}_{\mathfrak{v}_1}}(s)-\mu^a_{\mathscr{T}\mathfrak{s}_{\mathfrak{v}_2}}(s)|/s\in\mathbb{E}\}}u,u)$$

$$\begin{aligned}
&= \sup_{||u||=1}\int_{\mathbb{E}}\left[\inf\{|\int_{\mathbb{E}}\mathbb{I}(\mathfrak{e},\mathfrak{z})(\mathscr{G}(\mathfrak{e},\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e}))-\mathscr{G}(\mathfrak{e},\mathfrak{s}_{\mathfrak{v}_2}(\mathfrak{e})))|\}d\mathfrak{e}\right]u(t)\overline{u(t)}dt\\
&\leq \sup_{||u||=1}\int_{\mathbb{E}}\left[\inf\{\int_{\mathbb{E}}|\mathbb{I}(\mathfrak{e},\mathfrak{z})|\}\frac{\theta}{3}\inf\{|\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e})-\mathfrak{s}_{\mathfrak{v}_2}(\mathfrak{e})|/\mathfrak{e}\in\mathbb{E}\}d\mathfrak{e}\right]|u(t)|^2dt\\
&\leq \sup_{||u||=1}\int_{\mathbb{E}}\left[\inf\{\int_{\mathbb{E}}|\mathbb{I}(\mathfrak{e},\mathfrak{z})|d\mathfrak{e}\}\right]||u(t)||^2dt||\frac{\theta}{3}\inf\{|\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e})-\mathfrak{s}_{\mathfrak{v}_2}(\mathfrak{e})|/\mathfrak{e}\in\mathbb{E}\}||\\
&\leq \frac{\theta}{3}\sup_{||u||=1}\left[\inf\{\int_{\mathbb{E}}|\mathbb{I}(\mathfrak{e},\mathfrak{z})|d\mathfrak{e}\}\right].\sup_{||u||=1}\int_{\mathbb{E}}|u(t)|^2dt||\inf\{|\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e})-\mathfrak{s}_{\mathfrak{v}_2}(\mathfrak{e})|/\mathfrak{e}\in\mathbb{E}\}||_\infty\\
&\leq \frac{\theta}{3}||\inf\{|\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e})-\mathfrak{s}_{\mathfrak{v}_2}(\mathfrak{e})|/\mathfrak{e}\in\mathbb{E}\}||_\infty\\
&\leq ||\tilde{a}||^2||\frac{\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})}{3}||.
\end{aligned}$$

Thus,

$$\varphi(\tilde{\rho_{c^*}}(\mathscr{T}\mathfrak{s}_{\mathfrak{v}_1},\mathscr{T}\mathfrak{s}_{\mathfrak{v}_2})) \preceq \mathscr{F}_G\Big(\ \varphi(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star}),\wp(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star}),\varpi(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star})\ \Big).$$

For all $\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2}\in\tilde{\mathbb{V}}$, with $\psi(\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e}))\succ\tilde{0}_{\mathscr{A}}$ and $\phi(\mathfrak{s}_{\mathfrak{v}_2}(\mathfrak{e}))\succ\tilde{0}_{\mathscr{A}}$ $\forall$ $\mathfrak{e}\in\mathbb{E}$, define $\alpha,\beta:\tilde{\mathbb{V}}\to\tilde{\mathscr{A}}_+$ as $\alpha(\mathfrak{s}_{\mathfrak{v}_1})=\begin{cases}\tilde{I}_{\tilde{\mathscr{A}}} \text{ if } \mathfrak{e}\in\mathbb{E},\ \psi(\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{e}))\succ\tilde{0}_{\tilde{\mathscr{A}}}\\ \tilde{0}_{\tilde{\mathscr{A}}} \text{ otherwise}\end{cases}$ and $\beta(\mathfrak{s}_{\mathfrak{v}_2})=\begin{cases}\tilde{I}_{\tilde{\mathscr{A}}} \text{ if } \mathfrak{e}\in\mathbb{E},\ \phi(\mathfrak{s}_{\mathfrak{v}_2}(\mathfrak{e}))\succ\tilde{0}_{\tilde{\mathscr{A}}}\\ \tilde{0}_{\tilde{\mathscr{A}}} \text{ otherwise.}\end{cases}$

Then for all $\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2}\in\tilde{\mathbb{V}}$, we have

$$\begin{aligned}
\alpha(\mathfrak{s}_{\mathfrak{v}_1})\beta(\mathfrak{s}_{\mathfrak{v}_2})\succ\tilde{0}_{\tilde{\mathscr{A}}}\Rightarrow\varphi(\tilde{\rho_{c^*}}(\mathscr{T}\mathfrak{s}_{\mathfrak{v}_1},\mathscr{T}\mathfrak{s}_{\mathfrak{v}_2}))&\\
\preceq\mathscr{F}_G\Big(\ \varphi(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star}),\wp(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star}),\varpi(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1},\mathfrak{s}_{\mathfrak{v}_2})\tilde{a^\star})\ \Big)&.
\end{aligned}$$

Obviously, $\mathscr{T}$ is cyclic (α,β)-admissible mapping . From (ii) there is there is $\mathfrak{s}_{\mathfrak{v}_1}\in L^\infty(\mathbb{E})$ and $\mathfrak{z}\in\mathbb{V}$ such that $\alpha(\mathfrak{s}_{\mathfrak{v}_1}(\mathfrak{z}))=\tilde{I}_{\tilde{\mathscr{A}}}$ and $\mathfrak{s}_{\mathfrak{v}_2}\in L^\infty(\mathbb{E})$ and $\mathfrak{z}\in\mathbb{V}$ such that $\beta(\mathfrak{s}_{\mathfrak{v}_2}(\mathfrak{z}))=\tilde{I}_{\tilde{\mathscr{A}}}$. By (iii), we have that for any cluster point $\mathfrak{s}_{\mathfrak{v}'}$ of a sequence $\{\mathfrak{s}_{\mathfrak{v}_n}\}$ of points in $L^\infty(\mathbb{E})$ with $\alpha(\mathfrak{s}_{\mathfrak{v}_n}(\mathfrak{z}))=\tilde{I}_{\tilde{\mathscr{A}}}$, $\lim_{n\to\infty}\inf\alpha(\mathfrak{s}_{\mathfrak{v}_n}(\mathfrak{z}))=\inf\alpha(\mathfrak{s}_{\mathfrak{v}'})=\tilde{I}_{\tilde{\mathscr{A}}}$ and $\beta(\mathfrak{s}_{\mathfrak{v}_n}(\mathfrak{z}))=\tilde{I}_{\tilde{\mathscr{A}}}$, $\lim_{n\to\infty}\inf\beta(\mathfrak{s}_{\mathfrak{v}_n}(\mathfrak{z}))=\inf\beta(\mathfrak{s}_{\mathfrak{v}'})=\tilde{I}_{\tilde{\mathscr{A}}}$.

By applying Corollary 3.10, $\mathscr{T}$ has fixed point in $L^\infty(\mathbb{E})$. Also $\mathfrak{s}_{\mathfrak{v}'} \in L^\infty(\mathbb{E})$ is a solution of BVP Eq.(16).

4.2. Application to Homotopy.

In this part, we examine the possibility that homotopy theory has a unique solution.

Theorem 4.2: Let $(\tilde{\mathbb{V}}, \tilde{\mathscr{A}}, \tilde{\rho_{c^*}})$ be complete C^*-algebra valued fuzzy soft metric space, Δ and $\overline{\Delta}$ be an open and closed subset of $\tilde{\mathbb{V}}$ such that $\Delta \subseteq \overline{\Delta}$. Suppose $\mathscr{H} : \overline{\Delta} \times [0,1] \to \tilde{\mathbb{V}}$ be an operator with following conditions are satisfying,

τ_0) $\mathfrak{s}_\mathfrak{v} \neq \mathscr{H}(\mathfrak{s}_\mathfrak{v}, s)$, for each $\mathfrak{s}_\mathfrak{v} \in \partial\Delta$ and $s \in [0,1]$ (Here $\partial\Delta$ is boundary of Δ in $\mathbb{V}$);

τ_1) $\forall\ \mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2} \in \overline{\Delta}$, $s \in [0,1]$, $\varphi \in \Omega$ $\wp \in \Delta$, $\varpi \in \Theta$, $\mathscr{F}_G \in \Gamma$. $\varphi(\tilde{\mathfrak{e}}) > \wp(\tilde{\mathfrak{e}})$ for all $\tilde{\mathfrak{e}} \succ \tilde{0}_{\tilde{\mathscr{A}}}$, and $\tilde{a} \in \tilde{\mathscr{A}}$ with $||\tilde{a}|| < 1$ such that

$$\varphi\left(\tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}_1}, s), \mathscr{H}(\mathfrak{s}_{\mathfrak{v}_2}, s)\right)\right) \leq \mathscr{F}_G\begin{pmatrix} \varphi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2})\tilde{a}^\star\right), \wp\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2})\tilde{a}^\star\right), \\ \varpi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_1}, \mathfrak{s}_{\mathfrak{v}_2})\tilde{a}^\star\right) \end{pmatrix}$$

τ_2) $\exists\ \tilde{L} \in \tilde{\mathscr{A}}_+ \ni \tilde{\rho_{c^*}}(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}_1}, s), \mathscr{H}(\mathfrak{s}_{\mathfrak{v}_1}, t)) \preceq ||\tilde{L}|||s-t|$ for every $\mathfrak{s}_{\mathfrak{v}_1} \in \overline{\Delta}$ and $s,t \in [0,1]$.

Then $\mathscr{H}(.,0)$ has a fixed point $\iff$ $\mathscr{H}(.,1)$ has a fixed point.

Proof Let the set $\mathbb{K} = \left\{ \nu \in [0,1] : \mathscr{H}(\mathfrak{s}_{\mathfrak{v}_1}, \nu) = \mathfrak{s}_{\mathfrak{v}_1} \text{ for some } \mathfrak{s}_{\mathfrak{v}_1} \in \Delta \right\}$.

Suppose that $\mathscr{H}(.,0)$ has a fixed point in Δ, we have that $\tilde{0} \in \mathbb{K}$. So that $\mathbb{K}$ is non-empty set. Now we show that $\mathbb{K}$ is both closed and open in $[0,1]$ and hence by the connectedness $\mathbb{K} = [0,1]$. As a result, $\mathscr{H}(.,1)$ has a fixed point in Δ. First we show that $\mathbb{K}$ closed in $[0,1]$. To see this, Let $\{\nu_p\}_{p=1}^{\infty} \subseteq \mathbb{K}$ with $\nu_p \to \nu \in [0,1]$ as $p \to \infty$. We must show that $\nu \in \mathbb{K}$. Since $\nu_p \in \mathbb{K}$ for $p = 0,1,2,3,\cdots$, there exists sequences $\{\mathfrak{s}_{\mathfrak{v}_p}\} \subseteq \tilde{\mathbb{V}}$ with $\mathfrak{s}_{\mathfrak{v}_p} = \mathscr{H}(\mathfrak{s}_{\mathfrak{v}_p}, \nu_p)$.

Consider

$$\begin{aligned}
\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p}, \mathfrak{s}_{\mathfrak{v}_{p+1}}) &= \tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}_p}, \nu_p), \mathscr{H}(\mathfrak{s}_{\mathfrak{v}_{p+1}}, \nu_{p+1})\right) \\
&\preceq \tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}_p}, \nu_p), \mathscr{H}(\mathfrak{s}_{\mathfrak{v}_{p+1}}, \nu_p)\right) + \tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}_{p+1}}, \nu_p), \mathscr{H}(\mathfrak{s}_{\mathfrak{v}_{p+1}}, \nu_{p+1})\right) \\
&\preceq \tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}_p}, \nu_p), \mathscr{H}(\mathfrak{s}_{\mathfrak{v}_{p+1}}, \nu_p)\right) + ||\tilde{L}|||\nu_p - \nu_{p+1}| .
\end{aligned}$$

Letting $p \to \infty$, and applying φ on both sides, we get

$$\lim_{p\to\infty} \varphi\left(\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p}, \mathfrak{s}_{\mathfrak{v}_{p+1}})\right) \preceq \lim_{p\to\infty} \varphi\left(\tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}_p}, \nu_p), \mathscr{H}(\mathfrak{s}_{\mathfrak{v}_{p+1}}, \nu_p)\right)\right) + 0$$

$$\begin{aligned}
&\preceq \lim_{p\to\infty} \mathscr{F}_G \left(\begin{array}{c} \varphi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}_{p+1}})\tilde{a}^\star\right), \wp\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}_{p+1}})\tilde{a}^\star\right), \\ \varpi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}_{p+1}})\tilde{a}^\star\right) \end{array} \right) \\
&\preceq \lim_{p\to\infty} \max\left\{ \varphi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}_{p+1}})\tilde{a}^\star\right), \wp\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}_{p+1}})\tilde{a}^\star\right) \right\} \\
&\prec \lim_{p\to\infty} \varphi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}_{p+1}})\tilde{a}^\star\right).
\end{aligned}$$

By the definition of φ, and $||\tilde{a}|| < 1$ it follows that

$$\lim_{p\to\infty} ||\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}_{p+1}})|| \le \lim_{p\to\infty} ||\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}_{p+1}})\tilde{a}^*|| \le ||\tilde{a}||^2 \lim_{p\to\infty} ||\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}_{p+1}})||.$$

So that

$$\lim_{p\to\infty} \tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}_{p+1}}) = \tilde{0}_{\tilde{\mathscr{A}}}.$$

Now, for $q > p$, by use of triangular inequalilty , we have

$$\begin{aligned}
\tilde{\rho_{c^*}}\left(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}_q}\right) &\preceq \tilde{\rho_{c^*}}\left(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}_{p+1}}\right) + \tilde{\rho_{c^*}}\left(\mathfrak{s}_{\mathfrak{v}_{p+1}},\mathfrak{s}_{\mathfrak{v}_{p+2}}\right) + \tilde{\rho_{c^*}}\left(\mathfrak{s}_{\mathfrak{v}_{p+2}},\mathfrak{s}_{\mathfrak{v}_{p+3}}\right) \\
&\quad + \ldots + \tilde{\rho_{c^*}}\left(\mathfrak{s}_{\mathfrak{v}_{q-2}},\mathfrak{s}_{\mathfrak{v}_{q-1}}\right) + \tilde{\rho_{c^*}}\left(\mathfrak{s}_{\mathfrak{v}_{q-1}},\mathfrak{s}_{\mathfrak{v}_q}\right) \to \tilde{0}_{\tilde{\mathscr{A}}} \text{ as } p,q\to\infty.
\end{aligned}$$

Hence $\left\{\mathfrak{s}_{\mathfrak{v}_p}\right\}$ is a Cauchy sequence in C^*-algebra valued fuzzy soft metric spaces $(\tilde{\mathbb{V}},\tilde{\mathscr{A}},\tilde{\rho_{c^*}})$ and by the completeness of $(\tilde{\mathbb{V}},\tilde{\mathscr{A}},\tilde{\rho_{c^*}})$, there exist $\mathfrak{s}_{\mathfrak{v}'} \in \tilde{\mathbb{V}}$ with

$$\lim_{p\to\infty} \mathfrak{s}_{\mathfrak{v}_{p+1}} = \mathfrak{s}_{\mathfrak{v}'} = \lim_{p\to\infty} \mathfrak{s}_{\mathfrak{v}_p}$$

we have

$$\begin{aligned}
\varphi\left(\tilde{\rho_{c^*}}\left(\mathfrak{s}_{\mathfrak{v}'},\mathscr{H}(\mathfrak{s}_{\mathfrak{v}'},\nu)\right)\right) &= \lim_{p\to\infty} \varphi\left(\tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}_p},\nu_p),\mathscr{H}(\mathfrak{s}_{\mathfrak{v}'},\nu)\right)\right) \\
&\preceq \lim_{p\to\infty} \mathscr{F}_G \left(\begin{array}{c} \varphi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}'})\tilde{a}^\star\right), \wp\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}'})\tilde{a}^\star\right), \\ \varpi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}'})\tilde{a}^\star\right) \end{array} \right) \\
&\preceq \lim_{p\to\infty} \max\left\{ \varphi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}'})\tilde{a}^\star\right), \wp\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}'})\tilde{a}^\star\right) \right\} \\
&\prec \lim_{p\to\infty} \varphi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_p},\mathfrak{s}_{\mathfrak{v}'})\tilde{a}^\star\right) = \tilde{0}_{\tilde{\mathscr{A}}}.
\end{aligned}$$

It follows that $\mathscr{H}(\mathfrak{s}_{\mathfrak{v}'},\nu) = \mathfrak{s}_{\mathfrak{v}'}$. Thus $\mathfrak{s}_{\mathfrak{v}'} \in \mathbb{K}$. Hence $\mathbb{K}$ is closed in $[0,1]$. Let $\nu_0 \in \mathbb{K}$, then there exist $\mathfrak{s}_{\mathfrak{v}_0} \in \Delta$ with $\mathfrak{s}_{\mathfrak{v}_0} = \mathscr{H}(\mathfrak{s}_{\mathfrak{v}_0},\nu_0)$. Since Δ is open, then there exist $\tilde{r} > 0$ such that $B_{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_0},\tilde{r}) \subseteq \Delta$. Choose $\nu' \in (\nu_0-\varepsilon,\nu_0+\varepsilon)$ such that $|\nu'-\nu_0| \le \frac{1}{||\tilde{L}^p||} < \frac{\varepsilon}{2}$, then for

$\mathfrak{s}_{\mathfrak{v}'} \in \overline{B_{\tilde{\rho_{c^*}}}(\mathfrak{s}_{\mathfrak{v}_0},\tilde{r})} = \left\{\mathfrak{s}_{\mathfrak{v}'} \in \tilde{\mathbb{V}}/\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{s}_{\mathfrak{v}_0}) \preceq \tilde{r} + \tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_0},\mathfrak{s}_{\mathfrak{v}_0})\right\}.$

Now we have

$$\begin{aligned}
\tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}'},\nu'),\mathfrak{s}_{\mathfrak{v}_0}\right) &= \tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}'},\nu'),\mathscr{H}_b(\mathfrak{s}_{\mathfrak{v}_0},\nu_0)\right) \\
&\preceq \tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}'},\nu'),\mathscr{H}_b(\mathfrak{s}_{\mathfrak{v}_0},\nu')\right) + \tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}_0},\nu'),\mathscr{H}_b(\mathfrak{s}_{\mathfrak{v}_0},\nu_0)\right) \\
&\preceq \tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}'},\nu'),\mathscr{H}_b(\mathfrak{s}_{\mathfrak{v}_0},\nu')\right) + ||\tilde{L}|||\nu'-\nu_0| \\
&\preceq \tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}'},\nu'),\mathscr{H}_b(\mathfrak{s}_{\mathfrak{v}_0},\nu')\right) + \frac{1}{||\tilde{L}^{p-1}||}.
\end{aligned}$$

Letting $p\to\infty$ and applying φ on both sides, we obtain

$$\begin{aligned}
\varphi\left(\tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}'},\nu'),\mathfrak{s}_{\mathfrak{v}_0}\right)\right) &\preceq \varphi\left(\tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}'},\nu'),\mathscr{H}_b(\mathfrak{s}_{\mathfrak{v}_0},\nu')\right)\right) \\
&\preceq \mathscr{F}_G\begin{pmatrix} \varphi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{s}_{\mathfrak{v}_0})\tilde{a}^\star\right), \wp\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{s}_{\mathfrak{v}_0})\tilde{a}^\star\right), \\ \varpi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{s}_{\mathfrak{v}_0})\tilde{a}^\star\right)\end{pmatrix} \\
&\preceq \max\left\{\varphi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{s}_{\mathfrak{v}_0})\tilde{a}^\star\right), \wp\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{s}_{\mathfrak{v}_0})\tilde{a}^\star\right)\right\} \\
&\prec \varphi\left(\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{s}_{\mathfrak{v}_0})\tilde{a}^\star\right).
\end{aligned}$$

Since φ is non-decreasing and $||\tilde{a}||^2 < 1$, we have

$$\begin{aligned}
||\tilde{\rho_{c^*}}\left(\mathscr{H}(\mathfrak{s}_{\mathfrak{v}'},\nu'),\mathfrak{s}_{\mathfrak{v}_0}\right)|| &\leq ||\tilde{a}\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{s}_{\mathfrak{v}_0})\tilde{a}^\star|| \\
&\leq ||\tilde{a}||^2||\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}'},\mathfrak{s}_{\mathfrak{v}_0})|| \\
&\leq r + ||\tilde{\rho_{c^*}}(\mathfrak{s}_{\mathfrak{v}_0},\mathfrak{s}_{\mathfrak{v}_0})||.
\end{aligned}$$

Thus for each fixed $\nu' \in (\nu_0-\varepsilon,\nu_0+\varepsilon)$, $\mathscr{H}(.,\nu') : \overline{B_{\tilde{\rho_{c^*}}}(\mathfrak{s}_{\mathfrak{v}_0},\tilde{r})} \to \overline{B_{\tilde{\rho_{c^*}}}(\mathfrak{s}_{\mathfrak{v}_0},\tilde{r})}$. Then all conditions of Theorem 4.2 are satisfied. Thus we conclude that $\mathscr{H}(.,\nu')$ has a fixed point in $\overline{\Delta}$. But this must be in Δ since (τ_0) holds. Thus, $\nu' \in \mathbb{K}$ for any $\nu' \in (\nu_0-\varepsilon,\nu_0+\varepsilon)$. Hence $(\nu_0-\varepsilon,\nu_0+\varepsilon) \subseteq \mathbb{K}$. Clearly $\mathbb{K}$ is open in [0, 1].

For the reverse implication, we use the same strategy.

CONCLUSION

In developing the framework of $\mathscr{C}^*$-$\mathscr{A}\mathbb{VFSMS}$, this study presents two key applications. The first focuses on solving integral equations by employing cyclic (α,β)-$\mathscr{F}_G(\varphi,\wp,\varpi)$-rational contraction mappings. These are analyzed using generalized $\mathscr{C}_{G^\star}$-class functions in conjunction with fixed point theorems to establish existence and uniqueness results. The second application pertains to homotopy theory, where the same class of mappings is utilized to investigate continuous deformations and equivalence relations in topological spaces, offering novel insights into the structure of homotopic transformations.

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CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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