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DYNAMIC INEQUALITIES OF STEFFENSEN'S TYPE FOR DIAMOND-α INTEGRABLE FUNCTIONS ON TIME SCALES

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Abstract. The current work explores some novel dynamic inequalities of the Steffensen type on time scales by using the diamond-α, which is characterized as a linear mixture of the delta and nabla integrals. The derived inequalities not only offer a generalization of some dynamic inequalities but also extend a few recognized continuous inequalities by establishing new discrete inequalities on time scales.

Keywords: Steffensen's inequality; dynamic inequality; diamond-α integral; time scales.

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1. INTRODUCTION

The Steffensen Integral Inequality is a powerful tool in the analysis of integrals, particularly in approximation theory and functional analysis. It is a generalization of several classical inequalities and is typically used to establish bounds for integrals involving convex functions.

The Steffensen Integral Inequality is an inequality that provides upper and lower bounds for the integral of a convex function under certain conditions. The original form of this inequality, developed by the Norwegian mathematician Karl Steffensen, is often applied to integrals

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involving the derivative of a convex function. It plays a crucial role in numerical integration, helping to estimate the error in various numerical methods.

The classical Steffensen's inequality [18] can be expressed as:
Let ϑ and $\mathfrak{S}$ be integrable functions on $[\mathfrak{y}_1, \mathfrak{y}_2]$ such that ϑ is non increasing and for every $\hat{\mathfrak{y}} \in [\mathfrak{y}_1, \mathfrak{y}_2]$, $0 \leq \mathfrak{S}(\hat{\mathfrak{y}}) \leq 1$. The inequality that follows is

$$\int_{\mathfrak{y}_2-\lambda}^{\mathfrak{y}_2} \vartheta(\hat{\mathfrak{y}})d\hat{\mathfrak{y}} \leq \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\hat{\mathfrak{y}})\mathfrak{S}(\hat{\mathfrak{y}})d\hat{\mathfrak{y}} \leq \int_{\mathfrak{y}_1}^{\mathfrak{y}_1+\lambda} \vartheta(\hat{\mathfrak{y}})d\hat{\mathfrak{y}} \tag{1.1}$$

holds, where $\lambda = \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{S}(\hat{\mathfrak{y}})d\hat{\mathfrak{y}}$. A key observation is that if ϑ is non-decreasing, the inequality (1.1) is reversed.

Also the discrete Steffensen's inequality [10] is written as:

$$\sum_{\hat{\mathfrak{y}}=n-\lambda_2+1}^{n} \vartheta(\hat{\mathfrak{y}}) \leq \sum_{\hat{\mathfrak{y}}=1}^{n} \vartheta(\hat{\mathfrak{y}})\mathfrak{S}(\hat{\mathfrak{y}}) \leq \sum_{\hat{\mathfrak{y}}=1}^{\lambda_1} \vartheta(\hat{\mathfrak{y}}) \tag{1.2}$$

such that $0 \leq \mathfrak{S}(\hat{\mathfrak{y}}) \leq 1$, $\lambda_1, \lambda_2 \in \{1, 2....n\}$ with $\lambda_2 \leq \sum_{\hat{\mathfrak{y}}=1}^{n} \mathfrak{S}(\hat{\mathfrak{y}}) \leq \lambda_1$. In [14, 15] an extensive survey of Steffensen's integral inequality is provided.

Karl Steffensen's extended the classical integral inequalities (such as the trapezoidal rule) and developed a more general version that could deal with convex and concave functions, enhancing its applicability. Inequality is particularly useful in the context of dynamic inequalities on time scales, which is a broader framework for analyzing both continuous and discrete phenomena.

Anderson [6] provides Steffensen's integral inequality in the context of time scales, which is expressed as:

$$\int_{\mathfrak{y}_2-\lambda}^{\mathfrak{y}_2} \vartheta(\Xi)\nabla\Xi \leq \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\mathfrak{S}(\Xi)\nabla\Xi \leq \int_{\mathfrak{y}_1}^{\mathfrak{y}_1+\lambda} \vartheta(\Xi)\nabla\Xi, \tag{1.3}$$

where ϑ is of one sign and non increasing, $0 \leq \mathfrak{S}(\Xi) \leq 1$ for every $\Xi \in [\mathfrak{y}_1, \mathfrak{y}_2]_{\top}$, $\lambda = \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{S}(\Xi)\nabla\Xi$, and $\mathfrak{y}_2 - \lambda$, $\mathfrak{y}_1 + \lambda \in [\mathfrak{y}_1, \mathfrak{y}_2]_{\top}$.

Some generalizations of inequality (1.3) are given by Özkan and Yildrim in [16] as follows. If we have the inequality

$$\int_{\ell}^{\mathfrak{y}_2} \chi(\Xi)\Diamond_{\alpha}(\Xi) \leq \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{S}(\Xi)\Diamond_{\alpha}(\Xi) \leq \int_{\mathfrak{y}_1}^{\gamma} \chi(\Xi)\Diamond_{\alpha}(\Xi),$$

for $\vartheta \geq 0, \Xi \in [\mathfrak{y}_1, \mathfrak{y}_2]_{\top}$, and the inequality

$$\int_{\mathfrak{y}_1}^{\gamma} \chi(\Xi)\Diamond_{\alpha}(\Xi) \leq \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathfrak{S}(\Xi)\Diamond_{\alpha}(\Xi) \leq \int_{\ell}^{\mathfrak{y}_2} \chi(\Xi)\Diamond_{\alpha}(\Xi),$$

for $\vartheta \leq 0, \Xi \in [\mathfrak{y}_1, \mathfrak{y}_2]_{\top}$, then

$$\int_{\ell}^{\mathfrak{y}_2} \vartheta(\Xi)\chi(\Xi)\diamondsuit_\alpha(\Xi) \leq \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha(\Xi) \leq \int_{\mathfrak{y}_1}^{\gamma} \vartheta(\Xi)\chi(\Xi)\diamondsuit_\alpha(\Xi),$$

where $0 \leq \mathfrak{S}(\Xi) \leq \chi(\Xi)$ for all $\Xi \in [\mathfrak{y}_1, \mathfrak{y}_2]$ with $\ell, \gamma \in [\mathfrak{y}_1, \mathfrak{y}_2]_{\top}$.

The foundation of time scale theory was laid by Steffen Hilger in his Ph.D. thesis in 1988 [11]. This approach was developed to unify the realms of discrete and continuous analysis (see [12]). This theory has since become a focal point of much attention. The books by Bohner and Peterson [7, 8] cover all the basic definitions related to time scaleS calculus.

Many authors have observed a wide range of dynamic inequalities on time scales throughout history, authors explored some novel nonlinear retarded integral inequalities of the Gronwall–Bellman–Pachpatte type in [1], authors proved some Steefensen-type inequalities on time scales in [2], authors discussed well-known inequalities Young's inequality, Jensen's inequality, Holder's inequality, Minkowski's inequality, Steffensen's inequality, Hermite-Hadamard inequality, Cebysv's inequality and Opial type on time scales and their extensions with weighted functions in [5], Authors discussed continuous and discrete analogous of certain inequalities which provides an explicit bound of some unknown functions in [13].

This article aims to establish new Steffensen-type inequalities on general time scales, as given in [9]. By employing diamond-α integrals, we obtain the one-of-a-kind Steffensen inequalities. When $\alpha = 1$, the diamond-α integrals become equivalent to delta integrals, and for $\alpha = 0$, they converge to the nabla integral. A valuable analysis of diamond-α calculus is presented in [17].

2. TIME SCALES ESSENTIALS

A nonempty closed subset of the real numbers $\mathfrak{R}$ is referred to as a time scale $\top$. If $\top$ has left-scattered maximum Q_1 then $\top^{\kappa} := \top - \{Q_1\}$; otherwise, $\top^{\kappa} = \top$. If $\top$ has right-scattered maximum Q_2 then $\top_{\kappa} := \top - \{Q_2\}$; otherwise, $\top_{\kappa} = \top$. Finally, we have $\top_{\kappa}^{\kappa} = \top_{\kappa} \cap \top^{\kappa}$.

Definition 2.1. [7, 8] *Let $\psi : \top \to \mathfrak{R}$ be a function, where $\Xi \in \top^{\kappa}$, and define $\psi^{\Delta}(\Xi)$ and $\psi^{\nabla}(\Xi)$ as the delta and nabla derivatives of ψ at Ξ, respectively. Given any $\varepsilon > 0$, there exist neighborhoods $\mathcal{V}_1$ and $\mathcal{V}_2$ of Ξ for every $s \in \mathcal{V}_1$ and $s \in \mathcal{V}_2$, we have:*

$$|[\psi(\sigma(\Xi)) - \psi(s)] - \psi^{\Delta}(\Xi)[\sigma(\Xi) - s]| \leq \varepsilon|\sigma(\Xi) - s|,$$

and

$$|[\psi(\rho(\Xi)) - \psi(s)] - \psi^{\nabla}(\Xi)[\rho(\Xi) - s]| \leq \varepsilon|\rho(\Xi) - s|.$$

The following is the delta and nabla integration by parts rule on time scales, as discussed in [7, 8]:

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta^{\Delta}(\Xi)\mathfrak{S}(\Xi)\Delta\Xi = \vartheta(\mathfrak{y}_2)\mathfrak{S}(\mathfrak{y}_2) - \vartheta(\mathfrak{y}_1)\mathfrak{S}(\mathfrak{y}_1) - \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta^{\sigma}(\Xi)\mathfrak{S}(\Xi)\Delta\Xi \tag{2.1}$$

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta^{\nabla}(\Xi)\mathfrak{S}(\Xi)\nabla\Xi = \vartheta(\mathfrak{y}_2)\mathfrak{S}(\mathfrak{y}_2) - \vartheta(\mathfrak{y}_1)\mathfrak{S}(\mathfrak{y}_1) - \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta^{\rho}(\Xi)\mathfrak{S}(\Xi)\nabla\Xi \tag{2.2}$$

The link between the time scale calculus $\top$ and the difference calculus $\mathbb{Z}$ or differential calculus $\Re$ will be employed as shown below.

(i)**:** If $\top = \Re$, then

$$\begin{aligned} &\sigma(\Xi) = \rho(\Xi) = \Xi, \qquad \mu(\Xi) = \nu(\Xi) = 0, \qquad \vartheta^{\Delta}(\Xi) = \vartheta^{\nabla}(\Xi) = \vartheta'(\Xi), \\ &\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\Delta\Xi = \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\nabla\Xi = \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)d\Xi. \end{aligned} \tag{2.3}$$

(ii)**:** If $\top = \mathbb{Z}$, then

$$\begin{aligned} &\sigma(\Xi) = \Xi + 1, \qquad \rho(\Xi) = \Xi - 1, \qquad \mu(\Xi) = \nu(\Xi) = 1, \\ &\vartheta^{\Delta}(\Xi) = \Delta\vartheta(\Xi), \qquad \vartheta^{\nabla}(\Xi) = \nabla\vartheta(\Xi), \\ &\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\Delta\Xi = \sum_{\Xi=\mathfrak{y}_1}^{\mathfrak{y}_2-1} \vartheta(\Xi), \qquad \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\nabla\Xi = \sum_{\Xi=\mathfrak{y}_1+1}^{\mathfrak{y}_2} \vartheta(\Xi). \end{aligned} \tag{2.4}$$

where Δ and ∇ are forward and backward difference operator.

For any $\Xi \in \top$, the diamond-α dynamic derivative of ϑ at Ξ is specified by

$$\vartheta^{\diamond_\alpha} = \alpha\vartheta^{\Delta}(\Xi) + (1-\alpha)\vartheta^{\nabla}(\Xi), \qquad 0 \leq \alpha \leq 1,$$

and is represented as $\vartheta^{\diamond_\alpha}$, where ϑ is a Delta and nabla differentiable function on $\top$, with $\top$ being a time scale. For further details on diamond-α calculus, we recommend the paper [17].

3. MAIN RESULTS

This section is dedicated to stating and proving the main results.

Lemma 3.1. *Consider that*

($\mathcal{A}_1$) $\mathring{\mathfrak{N}}$ *is positive* $\diamondsuit_\alpha$*-integrable function on* $[\mathfrak{y}_1,\mathfrak{y}_2]_\top$.

($\mathcal{A}_2$) $\vartheta,\mathfrak{S},\chi,\mathfrak{B}:[\mathfrak{y}_1,\mathfrak{y}_2]_\top\to\mathfrak{R}$ *are* $\diamondsuit_\alpha$*-integrable function on* $[\mathfrak{y}_1,\mathfrak{y}_2]_\top$.

($\mathcal{A}_3$) $[u,v]_\top\subseteq[c,d]_\top\subseteq[\mathfrak{y}_1,\mathfrak{y}_2]_\top$ *with*
$\int_u^v\chi(\Xi)\mathfrak{B}(\Xi)\diamondsuit_\alpha\Xi+\int_c^d\mathring{\mathfrak{N}}(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi=\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi.$

($\mathcal{A}_4$) $\theta\in[\mathfrak{y}_1,\mathfrak{y}_2]_\top$.

Then

$$\begin{aligned}&\int_u^v\vartheta(\Xi)\mathfrak{B}(\Xi)\diamondsuit_\alpha\Xi+\int_c^d\vartheta(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi-\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi\\&=\int_{\mathfrak{y}_1}^{c}\left(\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\diamondsuit_\alpha\Xi+\int_c^u\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi\\&+\int_u^v\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi\\&+\int_v^d\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi+\int_d^{\mathfrak{y}_2}\left(\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi.\end{aligned}\tag{3.1}$$

Proof. Using direct computation, we derive.

$$\begin{aligned}&\int_u^v\vartheta(\Xi)\mathfrak{B}(\Xi)\diamondsuit_\alpha\Xi+\int_c^d\vartheta(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi-\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi\\&=\int_u^v\vartheta(\Xi)\mathfrak{B}(\Xi)\diamondsuit_\alpha\Xi+\left[\int_c^u\vartheta(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi+\int_u^v\vartheta(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi+\int_v^d\vartheta(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi\right]\\&\quad-\left[\int_{\mathfrak{y}_1}^c\vartheta(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi+\int_c^u\vartheta(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi+\int_u^v\vartheta(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi+\int_v^d\vartheta(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi\right.\\&\quad\left.+\int_d^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi\right]\\&=\int_u^v\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\diamondsuit_\alpha\Xi+\int_c^u\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\diamondsuit_\alpha\Xi\\&\quad+\int_v^d\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\diamondsuit_\alpha\Xi-\int_{\mathfrak{y}_1}^c\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\diamondsuit_\alpha\Xi-\int_d^{\mathfrak{y}_2}\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\diamondsuit_\alpha\Xi\\&=\int_{\mathfrak{y}_1}^c\left(\frac{\vartheta(\theta)}{\vartheta(\Xi)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\diamondsuit_\alpha\Xi+\int_c^u\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi\\&\quad+\int_u^v\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi\\&\quad+\int_v^d\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi+\int_d^{\mathfrak{y}_2}\left(\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\diamondsuit_\alpha\Xi\end{aligned}$$

$$+\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\bigg[\int_u^v \mathring{\mathfrak{N}}(\Xi)\mathfrak{B}(\Xi)\diamondsuit_\alpha\Xi+\int_c^u \mathring{\mathfrak{N}}(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi+\int_c^u \mathring{\mathfrak{N}}(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi+\int_u^v \mathring{\mathfrak{N}}(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi$$

$$+\int_v^d \mathring{\mathfrak{N}}(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi-\int_{\mathfrak{y}_1}^c \mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi-\int_c^u \mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi-\int_u^v \mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi$$

(3.2)

$$-\int_v^d \mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi-\int_d^{\mathfrak{y}_2} \mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi\bigg].$$

From $(\mathscr{A}_3)$

$$\int_u^v \chi(\Xi)\mathfrak{B}(\Xi)\diamondsuit_\alpha\Xi+\int_c^d \mathring{\mathfrak{N}}(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi=\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi.$$

Therefore

$$\begin{aligned}
&\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\bigg[\int_u^v \mathring{\mathfrak{N}}(\Xi)\mathfrak{B}(\Xi)\diamondsuit_\alpha\Xi+\int_c^u \mathring{\mathfrak{N}}(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi+\int_u^v \mathring{\mathfrak{N}}(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi\\
&\quad+\int_v^d \mathring{\mathfrak{N}}(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi-\int_{\mathfrak{y}_1}^c \mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi-\int_c^u \mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi\\
&\quad-\int_u^v \mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi-\int_v^d \mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi-\int_d^{\mathfrak{y}_2} \mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi\bigg]=0.
\end{aligned}\tag{3.3}$$

Hence identity (3.1) is proved directly from (3.2) and (3.3). □

Corollary 3.1. *Delta version of Lemma 3.1 can be found by setting $\alpha=1$ in (3.1).*

(3.4)

$$\begin{aligned}
&\int_u^v \vartheta(\Xi)\mathfrak{B}(\Xi)\Delta\Xi+\int_c^d \vartheta(\Xi)\chi(\Xi)\Delta\Xi-\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\mathfrak{S}(\Xi)\Delta\Xi\\
&=\int_{\mathfrak{y}_1}^c \left(\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\Delta\Xi+\int_c^u \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\Delta\Xi\\
&+\int_u^v \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\Delta\Xi\\
&+\int_v^d \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\Delta\Xi+\int_d^{\mathfrak{y}_2} \left(\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\Delta\Xi.
\end{aligned}$$

Corollary 3.2. *When we set $\alpha=0$ in Lemma 3.1, we obtain the nabla form of (3.1), given by:*

$$
\begin{aligned}
&\int_u^v \vartheta(\Xi)\mathfrak{B}(\Xi)\nabla\Xi + \int_c^d \vartheta(\Xi)\chi(\Xi)\nabla\Xi - \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\mathfrak{S}(\Xi)\nabla\Xi \\
&= \int_{\mathfrak{y}_1}^{c} \left(\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\nabla\Xi + \int_c^u \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\nabla\Xi \\
&+ \int_u^v \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\nabla\Xi \\
&+ \int_v^d \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\nabla\Xi + \int_d^{\mathfrak{y}_2} \left(\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\nabla\Xi.
\end{aligned}
\tag{3.5}
$$

Corollary 3.3. *When $\top$ is equal to $\mathfrak{R}$ in Lemma 3.1, relation (2.3) allows us to deduce:*

$$
\begin{aligned}
&\int_u^v \vartheta(\Xi)\mathfrak{B}(\Xi)d\Xi + \int_c^d \vartheta(\Xi)\chi(\Xi)d\Xi - \int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\mathfrak{S}(\Xi)d\Xi \\
&= \int_{\mathfrak{y}_1}^{c} \left(\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)d\Xi + \int_c^u \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]d\Xi \\
&+ \int_u^v \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]d\Xi \\
&+ \int_v^d \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]d\Xi + \int_d^{\mathfrak{y}_2} \left(\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)d\Xi.
\end{aligned}
$$

Corollary 3.4. *If $\top = \mathbb{Z}$ in Lemma 3.1, and utilizing relation (2.3), the identity (3.1) becomes*

$$
\begin{aligned}
&\sum_{\Xi=u}^{v-1} \vartheta(\Xi)\mathfrak{B}(\Xi) + \sum_{\Xi=c}^{d-1} \vartheta(\Xi)\chi(\Xi) - \sum_{\Xi=\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\mathfrak{S}(\Xi) \\
&= \sum_{\Xi=\mathfrak{y}_1}^{c-1} \left(\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi) + \sum_{\Xi=c}^{d-1} \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)] \\
&+ \sum_{\Xi=u}^{v-1} \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)] \\
&+ \sum_{\Xi=v}^{d-1} \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)] + \sum_{\Xi=d}^{\mathfrak{y}_2-1} \left(\frac{\vartheta(\theta)}{\mathring{\mathfrak{N}}(\theta)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi).
\end{aligned}
$$

Theorem 3.1. *Let $(\mathcal{A}_1)-(\mathcal{A}_3)$ of Lemma 3.1,*

$(\mathcal{A}_5)$ $\vartheta/\mathring{\mathfrak{N}}$ *is non increasing, and*

$(\mathcal{A}_6)$ $0 \le \mathfrak{S}(\Xi) \le \chi(\Xi) \le \mathfrak{B}(\Xi)$ $\forall\Xi \in [\mathfrak{y}_1, \mathfrak{y}_2]_\top$.

be satisfied, then the following inequalities are hold:

$$\int_{\eta_1}^{\eta_2} \vartheta(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi \leq \mathfrak{L}, \tag{3.6}$$

$$\int_{\eta_1}^{\eta_2} \vartheta(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi \leq \mathfrak{L} - \int_v^d \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi \leq \mathfrak{L}, \tag{3.7}$$

$$\int_{\eta_1}^{\eta_2} \vartheta(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi \leq \mathfrak{L} - \int_d^{\eta_2} \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\diamondsuit_\alpha\Xi \leq \mathfrak{L}. \tag{3.8}$$

where

$$\begin{aligned}
\mathfrak{L} = &\int_u^v \vartheta(\Xi)\mathfrak{B}(\Xi)\diamondsuit_\alpha\Xi + \int_c^d \vartheta(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi \\
&+ \int_{\eta_1}^c \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\diamondsuit_\alpha\Xi \\
&+ \int_c^u \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi \\
&+ \int_u^v \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi.
\end{aligned}$$

If $\vartheta/\mathring{\mathfrak{N}}$ is non decreasing, then inequalities (3.6),(3.7) and (3.8) must be reversed.

Proof. Since $\vartheta/\mathring{\mathfrak{N}}$ is nonincreasing, $\mathring{\mathfrak{N}}$ is positive, and $0 \leq \mathfrak{S} \leq \chi \leq \mathfrak{B}$, we have

$$\int_v^d \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi \geq 0, \tag{3.9}$$

and

$$\int_d^{\eta_2} \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi \geq 0. \tag{3.10}$$

From (3.1),(3.9) and (3.10) with $\theta = d$, we obtain

$$\begin{aligned}
&\int_u^v \vartheta(\Xi)\mathfrak{B}(\Xi)\diamondsuit_\alpha\Xi + \int_c^d \vartheta(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi - \int_{\eta_1}^{\eta_2} \vartheta(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi \\
&+ \int_{\eta_1}^c \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\diamondsuit_\alpha\Xi \qquad (3.11) \\
&+ \int_c^u \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi
\end{aligned}$$

$$+\int_u^v \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi$$
$$=\int_v^d \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi$$
$$+\int_d^{\mathfrak{y}_2} \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi\geq 0.$$

Hence (3.6) is proved.

Also (3.7) is proved from (3.1) and (3.10), Similarly (3.8) is proved from (3.1) and (3.9). □

Corollary 3.5. *After setting $\alpha = 1$ in Theorem 3.1, we obtain the following delta version of inequalities* $(3.6),(3.7),(3.8)$*, respectively.*

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\Delta\Xi\leq\mathfrak{L}_1, \tag{3.12}$$

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\Delta\Xi\leq\mathfrak{L}_1-\int_v^d \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\Delta\Xi\leq\mathfrak{L}_1, \tag{3.13}$$

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\Delta\Xi\leq\mathfrak{L}_1-\int_d^{\mathfrak{y}_2} \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\Delta\Xi\leq\mathfrak{L}_1. \tag{3.14}$$

Where

$$\mathfrak{L}_1=\int_u^v \vartheta(\Xi)\mathfrak{B}(\Xi)\Delta\Xi+\int_c^d \vartheta(\Xi)\chi(\Xi)\Delta\Xi$$
$$+\int_{\mathfrak{y}_1}^{c} \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\Delta\Xi$$
$$+\int_c^u \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\Delta\Xi$$
$$+\int_u^v \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\Delta\Xi.$$

Corollary 3.6. *The nable version of inequalities* $(3.6),(3.7),(3.8)$ *can be found by setting* $\alpha=0$ *in Theorem 3.1 respectively.*

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\nabla\Xi\leq\mathfrak{L}_2, \tag{3.15}$$

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\nabla\Xi\leq\mathfrak{L}_2-\int_v^d \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\nabla\Xi\leq\mathfrak{L}_2, \tag{3.16}$$

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\mathfrak{S}(\Xi)\nabla\Xi \leq \mathfrak{L}_2 - \int_{d}^{\mathfrak{y}_2} \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right) \mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\nabla\Xi \leq \mathfrak{L}_2. \tag{3.17}$$

Where

$$\begin{aligned}
\mathfrak{L}_2 = &\int_u^v \vartheta(\Xi)\mathfrak{B}(\Xi)\nabla\Xi + \int_c^d \vartheta(\Xi)\chi(\Xi)\nabla\Xi \\
&+ \int_{\mathfrak{y}_1}^{c} \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right) \mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\nabla\Xi \\
&+ \int_{c}^{u} \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right) \mathring{\mathfrak{N}}(\Xi)[\chi(\Xi) - \mathfrak{S}(\Xi)]\nabla\Xi \\
&+ \int_{u}^{v} \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right) \mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi) + \chi(\Xi) - \mathfrak{S}(\Xi)]\nabla\Xi.
\end{aligned}$$

Corollary 3.7. *If $\top = \mathfrak{R}$ and using the relation (2.3) in Theorem 3.1, we obtain the following inequalities, respectively.*

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\mathfrak{S}(\Xi)d\Xi \leq \mathfrak{L}_3, \tag{3.18}$$

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\mathfrak{S}(\Xi)d\Xi \leq \mathfrak{L}_3 - \int_{v}^{d} \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right) \mathring{\mathfrak{N}}(\Xi)[\chi(\Xi) - \mathfrak{S}(\Xi)]d\Xi \leq \mathfrak{L}_3, \tag{3.19}$$

$$\int_{\mathfrak{y}_1}^{\mathfrak{y}_2} \vartheta(\Xi)\mathfrak{S}(\Xi)d\Xi \leq \mathfrak{L}_3 - \int_{d}^{\mathfrak{y}_2} \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right) \mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)d\Xi \leq \mathfrak{L}_3. \tag{3.20}$$

Where

$$\begin{aligned}
\mathfrak{L}_3 = &\int_u^v \vartheta(\Xi)\mathfrak{B}(\Xi)d\Xi + \int_c^d \vartheta(\Xi)\chi(\Xi)d\Xi \\
&+ \int_{\mathfrak{y}_1}^{c} \left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)} - \frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right) \mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)d\Xi \\
&+ \int_{c}^{u} \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right) \mathring{\mathfrak{N}}(\Xi)[\chi(\Xi) - \mathfrak{S}(\Xi)]d\Xi \\
&+ \int_{u}^{v} \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)} - \frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right) \mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi) + \chi(\Xi) - \mathfrak{S}(\Xi)]d\Xi.
\end{aligned}$$

Corollary 3.8. *If $\top = \mathbb{Z}$ and applying the relation (2.3) in Theorem 3.1, we get the following respectively.*

$$\sum_{\Xi=\mathfrak{y}_1}^{\mathfrak{y}_2-1} \vartheta(\Xi)\mathfrak{S}(\Xi) \leq \mathfrak{L}_4, \tag{3.21}$$

$$\sum_{\Xi=\mathfrak{y}_1}^{\mathfrak{y}_2-1}\vartheta(\Xi)\mathfrak{S}(\Xi)\leq\mathfrak{L}_4-\sum_{\Xi=v}^{d-1}\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\leq\mathfrak{L}_4, \tag{3.22}$$

$$\sum_{\Xi=\mathfrak{y}_1}^{\mathfrak{y}_2-1}\vartheta(\Xi)\mathfrak{S}(\Xi)\leq\mathfrak{L}_4-\sum_{\Xi=d}^{\mathfrak{y}_2-1}\left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\leq\mathfrak{L}_4. \tag{3.23}$$

Where

$$\begin{aligned}\mathfrak{L}_4=&\sum_{\Xi=u}^{v-1}\vartheta(\Xi)\mathfrak{B}(\Xi)+\sum_{\Xi=c}^{d-1}\vartheta(\Xi)\chi(\Xi)\\&+\sum_{\Xi=\mathfrak{y}_1}^{c-1}\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\\&+\sum_{\Xi=c}^{d-1}\left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\\&+\sum_{\Xi=u}^{v-1}\left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)].\end{aligned}$$

Theorem 3.2. *Let $(\mathcal{A}_1)-(\mathcal{A}_3)$ of Lemma 3.1 and*

$(\mathcal{A}_7)$ *$\vartheta/\mathring{\mathfrak{N}}$ is non increasing in the Δ and ∇ cense, be fulfilled.*

If

$$\begin{aligned}&\int_v^{\sigma(\Xi)}\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\Delta\Xi\leq\int_v^{\sigma(\Xi)}\mathring{\mathfrak{N}}(\Xi)\chi(\Xi)\Delta\Xi,\quad v\leq\Xi\leq d,\\&\int_c^{\rho(\Xi)}\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\nabla\Xi\leq\int_c^{\rho(\Xi)}\mathring{\mathfrak{N}}(\Xi)\chi(\Xi)\nabla\Xi,\quad v\leq\Xi\leq d,\\&\int_{\sigma(\Xi)}^{\mathfrak{y}_2}\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\Delta\Xi\geq 0,\quad d\leq\Xi\leq\mathfrak{y}_2,\\&\int_{\rho(\Xi)}^{\mathfrak{y}_2}\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\nabla\Xi\geq 0,\quad d\leq\Xi\leq\mathfrak{y}_2,\end{aligned}$$

then

$$\begin{aligned}&\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\Diamond_\alpha\Xi\\&\leq\int_u^v\vartheta(\Xi)\mathfrak{B}(\Xi)\Diamond_\alpha\Xi+\int_c^d\vartheta(\Xi)\chi(\Xi)\Diamond_\alpha\Xi\\&+\int_{\mathfrak{y}_1}^c\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\Diamond_\alpha\Xi\\&+\int_c^u\left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\Diamond_\alpha\Xi\end{aligned}\tag{3.24}$$

$$+\int_u^v \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_\alpha\Xi.$$

Proof. Using (3.4) with (2.1), we get

$$\begin{aligned}
&\int_u^v \vartheta(\Xi)\mathfrak{B}(\Xi)\Delta\Xi+\int_c^d \vartheta(\Xi)\chi(\Xi)\Delta\Xi-\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\Delta\Xi\\
&+\int_{\mathfrak{y}_1}^{c}\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\Delta\Xi\\
&+\int_c^u \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\Delta\Xi\\
&+\int_u^v \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\Delta\Xi\\
&=\left[-\int_v^d\left(\int_v^{\sigma(\Xi)}\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\Delta\Xi\right)\left(\frac{f(x)}{k(x)}\right)^{\Delta}\Delta\Xi\right]\\
&\quad\times\left[-\int_d^{\mathfrak{y}_2}\left(\int_{\sigma(\Xi)}^{b}\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\Delta\Xi\right)\left(\frac{f(x)}{k(x)}\right)^{\Delta}\Delta\Xi\right]\geq 0.
\end{aligned}$$

Similarly, taking (3.5) with (2.2), we have

$$\begin{aligned}
&\int_u^v \vartheta(\Xi)\mathfrak{B}(\Xi)\Delta\Xi+\int_c^d \vartheta(\Xi)\chi(\Xi)\Delta\Xi-\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\Delta\Xi\\
&+\int_{\mathfrak{y}_1}^{c}\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\Delta\Xi\\
&+\int_c^u \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\Delta\Xi\\
&+\int_u^v \left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\Delta\Xi\\
&=\left[-\int_v^d\left(\int_v^{\rho(\Xi)}\mathring{\mathfrak{N}}(\Xi)\,[\chi(\Xi)-\mathfrak{S}(\Xi)\nabla\Xi]\right)\left(\frac{f(x)}{k(x)}\right)^{\nabla}\nabla\Xi\right]\\
&\quad\times\left[-\int_d^{\mathfrak{y}_2}\left(\int_{\rho(\Xi)}^{b}\mathring{\mathfrak{N}}(\Xi)\mathfrak{S}(\Xi)\nabla\Xi\right)\left(\frac{f(x)}{k(x)}\right)^{\nabla}\nabla\Xi\right]\geq 0.
\end{aligned}$$

Therefore

$$\int_u^v \vartheta(\Xi)\mathfrak{B}(\Xi)\diamondsuit_\alpha\Xi+\int_c^d \vartheta(\Xi)\chi(\Xi)\diamondsuit_\alpha\Xi-\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\diamondsuit_\alpha\Xi$$

$$
\begin{aligned}
&+\int_{\mathfrak{y}_1}^{c}\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\diamondsuit_{\alpha}\Xi\\
&+\int_{c}^{u}\left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_{\alpha}\Xi\\
&+\int_{u}^{v}\left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\diamondsuit_{\alpha}\Xi\\
&=\alpha\int_{u}^{v}\vartheta(\Xi)\mathfrak{B}(\Xi)\Delta\Xi+(1+\alpha)\int_{u}^{v}\vartheta(\Xi)\mathfrak{B}(\Xi)\nabla\Xi\\
&\quad+\alpha\int_{c}^{d}\vartheta(\Xi)\chi(\Xi)\Delta\Xi+(1+\alpha)\int_{c}^{d}\vartheta(\Xi)\chi(\Xi)\nabla\Xi\\
&\quad-\alpha\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\Delta\Xi-(1+\alpha)\int_{\mathfrak{y}_1}^{\mathfrak{y}_2}\vartheta(\Xi)\mathfrak{S}(\Xi)\nabla\Xi\\
&\quad+\alpha\int_{\mathfrak{y}_1}^{c}\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\Delta\Xi\\
&\quad+(1+\alpha)\int_{\mathfrak{y}_1}^{c}\left(\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}-\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}\right)\mathfrak{S}(\Xi)\mathring{\mathfrak{N}}(\Xi)\nabla\Xi\\
&\quad+\alpha\int_{c}^{u}\left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\Delta\Xi\\
&\quad+(1+\alpha)\int_{c}^{u}\left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\chi(\Xi)-\mathfrak{S}(\Xi)]\nabla\Xi\\
&\quad+\alpha\int_{u}^{v}\left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\Delta\Xi\\
&\quad+(1+\alpha)\int_{u}^{v}\left(\frac{\vartheta(d)}{\mathring{\mathfrak{N}}(d)}-\frac{\vartheta(\Xi)}{\mathring{\mathfrak{N}}(\Xi)}\right)\mathring{\mathfrak{N}}(\Xi)[\mathfrak{B}(\Xi)+\chi(\Xi)-\mathfrak{S}(\Xi)]\nabla\Xi\geq 0.
\end{aligned}
$$

Hence, (3.24) proved. □

AUTHORS' CONTRIBUTIONS

All authors have read and finalized the manuscript with equal contribution.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests.

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