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NOVEL STEFFENSEN-TYPE INEQUALITIES FOR DELTA-INTEGRABLE FUNCTIONS IN THE TIME-SCALE FRAMEWORK

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Abstract. By employing delta integrals, we establish new Steffensen-type dynamic inequalities on arbitrary time scales, thereby extending and unifying several known results of Steffensen-type inequalities within a broader and more flexible analytical framework.

Keywords: Steffensen's inequality; dynamic inequality; delta integral; time scales.

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1. INTRODUCTION

The well-known Steffensen's inequality [17] is written as:

Let f and g be integrable functions on $[a,b]$ such that f is nonincreasing and for every $y \in [a,b]$, $0 \le g(y) \le 1$. Then the following inequality

$$\int_{b-\lambda}^{b} f(y)dy \le \int_{a}^{b} f(y)g(y)dy \le \int_{a}^{a+\lambda} f(y)dy \tag{1.1}$$

holds, where $\lambda = \int_a^b g(y)dy$. A point to be noted if f is nondecreasing then the inequality (1.1) is reversed. In [13, 14] a comprehensive survey on steffensen's integral inequality can be found.

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Anderson, in [4], gave the the time scales version of Steffensen's integral inequality as follows:

$$\int_{b-\lambda}^{b} f(t)\nabla t \leq \int_{a}^{b} f(t)g(t)\nabla t \leq \int_{a}^{a+\lambda} f(t)\nabla t, \tag{1.2}$$

where f is of one sign and nonincreasing, $0 \leq g \leq 1$ for every $t \in [a,b]_{\mathbb{T}}$, $\lambda = \int_a^b g(t)\nabla t$, and $b-\lambda,\ a+\lambda \in [a,b]_{\mathbb{T}}$. We get delta version of Steffensen's integral inequality by replacing $\triangle$ in place of ∇ in (1.2).

$$\int_{b-\lambda}^{b} f(t)\triangle t \leq \int_{a}^{b} f(t)g(t)\triangle t \leq \int_{a}^{a+\lambda} f(t)\triangle t, \tag{1.3}$$

where f is of one sign and nonincreasing, $0 \leq g(t) \leq 1$ for every $t \in [a,b]_{\mathbb{T}}$, $\lambda = \int_a^b g(t)\triangle t$, and $b-\lambda,\ a+\lambda \in [a,b]_{\mathbb{T}}$.

The theory of time scales was first developed by Steffen Hilger in 1988 in his Ph.D. thesis [11], marking a significant milestone in the history of mathematical analysis. This theory aimed to unify discrete and continuous analysis (see [12]). Since then, it has attracted considerable attention. The works by Bohner and Peterson [5,6] have made substantial contributions to the study of time scales, offering a comprehensive approach to time scales calculus and resolving numerous related problems.

By seeing the past history, there are many reasonable number of dynamic inequalities on time scales are studied by many researchers. In [1,7,8], Authors investigate and proved some Steffensen type inequalities using diamond-α integral. In [2,3], Authors proved different kind of inequalities on time scales.

2. Time Scales Essentials

A time scales $\mathbb{T}$ is an arbitrary nonempty closed subset of a real numbers $\mathbb{R}$. The set of real numbers $\mathbb{R}$ and set of integers $\mathbb{Z}$ are two well-known examples of time scales. If $\mathbb{T}$ has left-scattered maximum $\mathcal{M}_1$ then $\mathbb{T}^{\kappa} := \mathbb{T} - \{\mathcal{M}_1\}$; otherwise, $\mathbb{T}^{\kappa} = \mathbb{T}$.

Let a function $f : \mathbb{T} \to \mathbb{R}$, $t \in \mathbb{T}^{\kappa}$, $f^{\triangle}(t) \in \mathbb{R}$, is said to be the delta derivative of f at t, if for any $\varepsilon > 0$ there exist a neighborhood $\mathcal{V}_1$ of t such that, for all $s \in \mathcal{V}_1$, we have

$$|[f(\sigma(t)) - f(s)] - f^{\triangle}(t)[\sigma(t) - s]| \leq \varepsilon |\sigma(t) - s|$$

Theorem 2.1. *[18] Assume ψ, ϕ : are differentiable at $\varpi \in \mathbb{T}^{\kappa}$,*

(1) *The sum $\psi+\phi:\mathbb{T}\to\mathbb{R}$ are also differentiable at ϖ with*

$$(\psi+\phi)^{\triangle}(\varpi)=\psi^{\triangle}(\varpi)+\phi^{\triangle}(\varpi). \tag{2.1}$$

(2) *For any constant k and $k\psi:\mathbb{T}\to\mathbb{R}$ is differentiable function at ϖ with*

$$(k\psi)^{\triangle}(\varpi)=k\psi^{\triangle}(\varpi). \tag{2.2}$$

(3) *The product $\psi\phi:\mathbb{T}\to\mathbb{R}$ is differentiable function at ϖ with*

$$(\psi\phi)^{\triangle}=\psi^{\triangle}(\varpi)\phi(\varpi)+\psi(\sigma(\varpi))\phi^{\triangle}(\varpi). \tag{2.3}$$

(4) *If $\psi(\varpi)\psi(\sigma(\varpi))\neq 0$, then $\frac{1}{\psi}$ is differentiable function at ϖ with*

$$\left(\frac{1}{\psi}\right)^{\triangle}(\varpi)=-\frac{\psi^{\triangle}(\varpi)}{\psi(\varpi)\psi(\sigma(\varpi))}. \tag{2.4}$$

(5) *If $\phi(\varpi)\phi(\sigma(\varpi))\neq 0$, then $\frac{\psi}{\phi}$ is differentiable at ϖ and*

$$\left(\frac{\psi}{\phi}\right)^{\triangle}(\varpi)=\frac{\psi^{\triangle}(\varpi)\phi(\varpi)-\psi(\varpi)\phi^{\triangle}(\varpi)}{\phi(\varpi)\phi(\sigma(\varpi))}. \tag{2.5}$$

For example, $[(\varpi)^2]^{\triangle}=\varpi+\sigma(\varpi)$ and $\left(\frac{1}{\varpi}\right)^{\triangle}=-\frac{1}{\varpi\sigma(\varpi)}$.

Proposition 2.2. *[10] Let $\mathfrak{a},\mathfrak{b},\mathfrak{c}\in\mathbb{T}$, $\beta\in\mathbb{R}$, and $\mathfrak{f},\mathfrak{g}$ be continuous functions on $I_{\mathbb{T}}$, then*

(1) $\int_{\mathfrak{a}}^{\mathfrak{b}}(\mathfrak{f}(\varpi)+\mathfrak{g}(\varpi))\triangle\varpi=\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{f}(\varpi)\triangle\varpi+\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{g}(\varpi)\triangle\varpi$

(2) $\int_{\mathfrak{a}}^{\mathfrak{b}}(\beta\mathfrak{f}(\varpi))\triangle\varpi=\beta\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{f}(\varpi)\triangle\varpi$

(3) $\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{f}(\varpi)\triangle\varpi=-\int_{\mathfrak{b}}^{\mathfrak{a}}\mathfrak{f}(\varpi)\triangle\varpi$

(4) $\int_{\mathfrak{a}}^{\mathfrak{b}}\mathfrak{f}(\varpi)\triangle\varpi=\int_{\mathfrak{a}}^{\mathfrak{c}}\mathfrak{f}(\varpi)\triangle\varpi+\int_{\mathfrak{c}}^{\mathfrak{b}}\mathfrak{f}(\varpi)\triangle\varpi$

(5) $\int_{\mathfrak{a}}^{\mathfrak{a}}\mathfrak{f}(\varpi)\triangle\varpi=0.$

3. Main Results

This section is dedicated to stating and proving the main results.

(1) $\mathscr{A}_1=a,b\in\mathbb{T}^{\kappa}$ *with* $a<b$.

(2) $\mathscr{A}_2=f_1,f_2,f_3:[a,b]\to\mathbb{R}$ are delta integrable functions on $[a,b]_{\mathbb{T}}$.

(3) $\mathscr{A}_3=\mathscr{Q}$ is of one sign and decreasing and $0\leq\mathfrak{L}(\mathscr{W})\leq\mathfrak{J}(\mathscr{W})$ on $[a,b]_{\mathbb{T}}$.

(4) $\mathscr{A}_4=f_4:[a,b]\to\mathbb{R}$ is delta integrable function on $[a,b]_{\mathbb{T}}$.

Lemma 3.1. *Let $\mathscr{A}_1, \mathscr{A}_2$ hold. Assume a sub-interval $[u,v]_{\mathbb{T}} \subseteq [a,\mathscr{b}]_{\mathbb{T}}$ with the condition $\int_u^v \mathscr{f}_3(t)\triangle t = \int_a^{\mathscr{b}} \mathscr{f}_2(t)\triangle t$.*

Then

$$\begin{aligned}
&\int_u^v \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle \mathscr{W} - \int_a^{\mathscr{b}} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} \\
&= \int_a^u \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v) - \mathscr{Q}(\mathscr{W})]\triangle \mathscr{W} \\
&+ \int_u^v [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v)]\triangle \mathscr{W} + \int_v^{\mathscr{b}} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v) - \mathscr{Q}(\mathscr{W})]\triangle \mathscr{W}
\end{aligned} \tag{3.1}$$

And

$$\begin{aligned}
&\int_a^{\mathscr{b}} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} - \int_u^v \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle \mathscr{W} \\
&= \int_a^u \mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(u)]\triangle \mathscr{W} \\
&+ \int_u^v [\mathscr{Q}(u) - \mathscr{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})]\triangle \mathscr{W} + \int_v^{\mathscr{b}} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(u)]\triangle \mathscr{W}
\end{aligned} \tag{3.2}$$

Proof. By straightforward calculation, we obtain

$$\begin{aligned}
&\int_u^v \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle \mathscr{W} - \int_a^{\mathscr{b}} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} \\
= &\int_u^v \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle \mathscr{W} - \int_a^u \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} - \int_u^v \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} - \int_v^{\mathscr{b}} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} \\
= &\int_u^v \mathscr{Q}(\mathscr{W})[\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})]\triangle \mathscr{W} - \int_a^u \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} - \int_v^{\mathscr{b}} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} \qquad (3.3) \\
= &\int_u^v [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})]\mathscr{Q}(\mathscr{W})\triangle \mathscr{W} - \int_u^v \mathscr{Q}(v)[\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})]\triangle \mathscr{W} \\
&+ \int_u^v \mathscr{Q}(\mathfrak{s})[\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})]\triangle \mathscr{W} \\
&- \int_a^u \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} - \int_a^u \mathscr{Q}(v)\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} + \int_a^u \mathscr{Q}(v)\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} \\
&- \int_v^{\mathscr{b}} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} - \int_v^{\mathscr{b}} \mathscr{Q}(v)\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} + \int_v^{\mathscr{b}} \mathscr{Q}(v)\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} \\
= &\int_u^v [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v)]\triangle \mathscr{W} + \int_u^v \mathscr{Q}(v)[\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})]\triangle \mathscr{W} \\
&+ \int_a^u [\mathscr{Q}(v) - \mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} - \int_a^u \mathscr{Q}(v)\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} \\
&+ \int_v^{\mathscr{b}} [\mathscr{Q}(v) - \mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} - \int_v^{\mathscr{b}} \mathscr{Q}(v)\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} \\
= &\int_a^u [\mathscr{Q}(v) - \mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle \mathscr{W} + \int_u^v [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v)]\triangle \mathscr{W}
\end{aligned}$$

$$
\begin{aligned}
&+\int_{v}^{b}[\mathscr{Q}(v)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\\
&+\mathscr{Q}(v)\left(\int_{u}^{v}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}-\int_{a}^{u}\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{v}^{b}\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\right)\\
=\;&\int_{a}^{u}[\mathscr{Q}(v)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_{u}^{v}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v)]\triangle\mathscr{W}\\
&+\int_{v}^{b}[\mathscr{Q}(v)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\\
&+\mathscr{Q}(v)\left(-\int_{a}^{u}\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{u}^{v}\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{v}^{b}\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_{u}^{v}\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}\right)\\
=\;&\int_{a}^{u}[\mathscr{Q}(v)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_{u}^{v}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v)]\triangle\mathscr{W}\\
&+\int_{v}^{b}[\mathscr{Q}(v)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\mathscr{Q}(v)\left(-\int_{a}^{b}\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_{u}^{v}\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}\right)
\end{aligned}
$$

By applying the assumption $\int_u^v \mathfrak{J}(\mathscr{W})\triangle\mathscr{W}=\int_a^b \mathfrak{L}(\mathscr{W})\triangle\mathscr{W}$ in above equation we have

$$
\begin{aligned}
=\;&\int_{a}^{u}[\mathscr{Q}(v)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_{u}^{v}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v)]\triangle\mathscr{W}\\
&+\int_{v}^{b}[\mathscr{Q}(v)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}
\end{aligned}
$$

which proves identity (3.1).

To prove the identity (3.2) again take equation (3.3)

$$
\begin{aligned}
=\;&\int_{u}^{v}\mathscr{Q}(\mathscr{W})[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}-\int_{a}^{u}\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{v}^{b}\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\\
=\;&\int_{u}^{v}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\mathscr{Q}(\mathscr{W})\triangle\mathscr{W}-\int_{u}^{v}\mathscr{Q}(u)[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}\\
&+\int_{u}^{v}\mathscr{Q}(u)[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}\\
&-\int_{\tilde{\alpha}}^{u}\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{a}^{u}\mathscr{Q}(u)\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_{a}^{u}\mathscr{Q}(u)\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\\
&-\int_{v}^{b}\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{v}^{b}\mathscr{Q}(u)\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_{v}^{b}\mathscr{Q}(u)\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\\
=\;&\int_{u}^{v}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u)]\triangle\mathscr{W}+\int_{u}^{v}\mathscr{Q}(u)[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}\\
&+\int_{a}^{u}[\mathscr{Q}(u)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{a}^{u}\mathscr{Q}(u)\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\\
&+\int_{v}^{b}[\mathscr{Q}(u)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{v}^{b}\mathscr{Q}(u)\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\\
=\;&\int_{a}^{u}[\mathscr{Q}(u)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_{u}^{v}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u)]\triangle\mathscr{W}
\end{aligned}
$$

$$\begin{aligned}
&+\int_{v}^{b}[\mathscr{Q}(u)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\\
&+\mathscr{Q}(u)\left(\int_{u}^{v}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}-\int_{a}^{u}\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{v}^{b}\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\right)\\
=&\int_{a}^{u}[\mathscr{Q}(u)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_{u}^{v}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u)]\triangle\mathscr{W}\\
&+\int_{v}^{b}[\mathscr{Q}(u)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\\
&+\mathscr{Q}(u)\left(-\int_{a}^{u}\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{u}^{v}\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{v}^{b}\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_{u}^{v}\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}\right)\\
=&\int_{a}^{u}[\mathscr{Q}(u)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_{u}^{v}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u)]\triangle\mathscr{W}\\
&+\int_{v}^{b}[\mathscr{Q}(u)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\mathscr{Q}(u)\left(-\int_{a}^{b}\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_{u}^{v}\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}\right)
\end{aligned}$$

Using assumption $\int_u^v \mathfrak{J}(\mathscr{W})\triangle\mathscr{W}=\int_a^b \mathfrak{L}(\mathscr{W})\triangle\mathscr{W}$ in above equation.

$$\begin{aligned}
=&\int_{a}^{u}[\mathscr{Q}(u)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_{u}^{v}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u)]\triangle\mathscr{W}\\
&+\int_{v}^{b}[\mathscr{Q}(u)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}
\end{aligned}$$

The identity (3.2) is thus completely proved.

Remark 3.2. By setting $u\to a, v\to \mathscr{W}$ in (3.1) and $u\to \ell, v\to b$ in (3.2) respectively,we get the same result when $\alpha=1$ in [15, Lemma 3.3].

Theorem 3.3. *Assume $\mathscr{A}_1,\mathscr{A}_2,\mathscr{A}_3$ hold. Further suppose that $\int_a^b \mathfrak{L}(\mathscr{W})\triangle\mathscr{W}=\int_{u_j}^{v_j}\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}$ where $[u_j,v_j]_{\mathbb{T}}\subseteq[a,b]_{\mathbb{T}}$ for $j=1,2$ and $v_1\le v_2$.*
Then

$$\int_{u_2}^{v_2}\mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}-s(b,u_2,v_2)\le\int_a^b\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\le\int_{u_1}^{v_1}\mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}+S(a,u_1,v_1)$$

where

$$\begin{aligned}
S(a,u_1,v_1)&=\int_a^{u_1}[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v_1)]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\ge 0\\
s(b,u_2,v_2)&=\int_{v_2}^{b}[\mathscr{Q}(u_2)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\ge 0
\end{aligned}$$

Proof. By setting $u=u_1$ *and* $v=v_1$ in identity (3.1), we have

$$\begin{aligned}
&\int_{u_1}^{v_1}\mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}-\int_a^{\mathscr{b}}\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\\
&=\int_a^{u_1}\mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_1)-\mathscr{Q}(\mathscr{W})]\triangle\mathscr{W}\\
&+\int_{u_1}^{v_1}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v_1)]\triangle\mathscr{W}+\int_{v_1}^{\mathscr{b}}\mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_1)-\mathscr{Q}(\mathscr{W})]\triangle\mathscr{W}
\end{aligned}\tag{3.4}$$

Since $\mathscr{Q}$ is decreasing on intervals $[u_1,v_1]$,$[v_1,\mathscr{b}]$ and from inequality $0\le\mathfrak{L}(\mathscr{W})\le\mathfrak{J}(\mathscr{W})$ we conclude that

$$\int_{v_1}^{\mathscr{b}}\mathfrak{L}(\mathscr{W})\,[\mathscr{Q}(v_1)-\mathscr{Q}(\mathscr{W})]\,\triangle\mathscr{W}\ge 0\quad\textit{and}\quad\int_{u_1}^{v_1}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\,[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v_1)]\,\triangle\mathscr{W}\ge 0$$

which implies that

$$\int_{u_1}^{v_1}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\,[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v_1)]\,\triangle\mathscr{W}+\int_{v_1}^{\mathscr{b}}\mathfrak{L}(\mathscr{W})\,[\mathscr{Q}(v_1)-\mathscr{Q}(\mathscr{W})]\,\triangle\mathscr{W}\ge 0$$

Therefore equation (3.4) becomes

$$\begin{aligned}
&\int_{u_1}^{v_1}\mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}-\int_a^{\mathscr{b}}\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}+\int_a^{u_1}\mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v_1)]\triangle\mathscr{W}\ge 0\\
&\int_{u_1}^{v_1}\mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}+S(a,u_1,v_1)\ge\int_a^{\mathscr{b}}\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}
\end{aligned}\tag{3.5}$$

where

$$S(a,u_1,v_1)=\int_a^{u_1}\mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v_1)]\triangle\mathscr{W}\ge 0.$$

Now by setting $u=u_2$ *and* $v=v_2$ in identity (3.2), we have

$$\begin{aligned}
&\int_a^{\mathscr{b}}\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{u_2}^{v_2}\mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}\\
&=\int_a^{u_2}\mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_2)]\triangle\mathscr{W}\\
&+\int_{u_2}^{v_2}[\mathscr{Q}(u_2)-\mathscr{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}+\int_{v_2}^{\mathscr{b}}\mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_2)]\triangle\mathscr{W}
\end{aligned}\tag{3.6}$$

By the assumption on intervals $[a,u_2]$,$[u_2,v_2]$ the function $\mathscr{Q}$ is decreasing and $\mathscr{Q}\le 0$ and also from inequality $0\le\mathfrak{L}(\mathscr{W})\le\mathfrak{J}(\mathscr{W})$ we have

$$\int_a^{u_2}\mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_2)]\triangle\mathscr{W}\ge 0\quad\textit{and}\quad\int_{u_2}^{v_2}[\mathscr{Q}(u_2)-\mathscr{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}\ge 0$$

which implies that

$$\int_{a}^{u_2} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_2)]\triangle\mathscr{W}+\int_{u_2}^{v_2}[\mathscr{Q}(u_2)-\mathscr{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}\geq 0$$

So equation (3.6) becomes

$$\begin{aligned} &\int_{a}^{b} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{u_2}^{v_2} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}\geq \int_{v_2}^{b} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_2)]\triangle\mathscr{W} \\ &\int_{a}^{b} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\geq \int_{u_2}^{v_2} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}-s(b,u_2,v_2) \end{aligned} \tag{3.7}$$

where

$$s(b,u_2,v_2)=\int_{v_2}^{b}[\mathscr{Q}(u_2)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\geq 0.$$

By the combination of the inequalities (3.5) and (3.7) the proof of inequality (3.4) is completed.

Remark 3.4. By setting $u_1\to\alpha, u_2\to\beta-\lambda, v_1\to\alpha+\lambda, v_2\to\beta$ and $\mathfrak{J}(\mathscr{W})=1$, the inequality (3.4) reduces to inequality (1.1).

Theorem 3.5. *Assume $\mathscr{A}_1,\mathscr{A}_2,\mathscr{A}_3$ hold. Further let $\int_a^b \mathfrak{L}(\mathscr{W})\triangle\mathscr{W}=\int_{u_j}^{v_j}\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}$ where $[u_j,v_j]_{\mathbb{T}}\subseteq[a,b]_{\mathbb{T}}$ for $j=1,2,3$ and $v_1\leq v_2\leq v_3$.*
Then

$$\begin{aligned} &\sum_{j=2}^{3}\int_{u_j}^{v_j} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}-y(b,u_2,v_2,u_3,v_3) \\ &\leq 2\int_{a}^{b} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \\ &\leq \sum_{j=1}^{2}\int_{u_j}^{v_j} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}+Y(a,u_1,v_1,u_2,v_2) \end{aligned} \tag{3.8}$$

where

$$\begin{aligned} y(b,u_2,v_2,u_3,v_3) &= \textstyle\sum_{j=2}^{3}\int_{v_j}^{b}[\mathscr{Q}(u_j)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\geq 0 \\ Y(a,u_1,v_1,u_2,v_2) &= \textstyle\sum_{j=1}^{2}\int_{a}^{u_j}[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v_j)]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\geq 0 \end{aligned}$$

Proof. By setting $u=u_j$ and $v=v_j$ for j=1,2 in identity (3.1), respectively.

$$\begin{aligned} &\int_{u_1}^{v_1} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}-\int_{a}^{b} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \\ &=\int_{a}^{u_1} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_1)-\mathscr{Q}(\mathscr{W})]\triangle\mathscr{W} \\ &+\int_{u_1}^{v_1}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v_1)]\triangle\mathscr{W}+\int_{v_1}^{b} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_1)-\mathscr{Q}(\mathscr{W})]\triangle\mathscr{W} \end{aligned} \tag{3.9}$$

$$\begin{aligned}
&\int_{u_2}^{v_2} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} - \int_a^{\mathscr{b}} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \\
&= \int_a^{u_2} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_2) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W} \qquad (3.10)\\
&+ \int_{u_2}^{v_2} [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_2)]\triangle\mathscr{W} + \int_{v_2}^{\mathscr{b}} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_2) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W}
\end{aligned}$$

Addition of equations (3.9) and (3.10) yields.

$$\begin{aligned}
&\sum_{j=1}^{2} \int_{u_j}^{v_j} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} - 2\int_a^{\mathscr{b}} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \\
&= \sum_{j=1}^{2} \int_a^{u_j} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_j) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W} \\
&+ \sum_{j=1}^{2} \int_{u_j}^{v_j} [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_j)]\triangle\mathscr{W} + \sum_{j=1}^{2} \int_{v_j}^{\mathscr{b}} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_j) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W}
\end{aligned} \qquad (3.11)$$

From inequality $0 \le \mathfrak{L}(\mathscr{W}) \le \mathfrak{J}(\mathscr{W})$ *and by the assumption* $\mathscr{Q}$ *is decreasing on intervals* $[u_1, v_1], [u_2, v_2], [v_1, \mathscr{b}], [v_2, \mathscr{b}]$, *we conclude that*

$$\int_{u_j}^{v_j} [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_j)]\triangle\mathscr{W} \ge 0$$

and $$\int_{v_j}^{\mathscr{b}} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_j) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W} \ge 0 \qquad for \quad j = 1,2$$

So

$$\sum_{j=1}^{2} \left[\int_{u_j}^{v_j} [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_j)]\triangle\mathscr{W} + \int_{v_j}^{\mathscr{b}} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_j) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W} \right] \ge 0 \qquad (3.12)$$

Equation (3.11) becomes together with (3.12).

$$\sum_{j=1}^{2} \int_{u_j}^{v_j} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} - 2\int_a^{\mathscr{b}} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \ge \sum_{j=1}^{2} \int_a^{u_j} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_j) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W}$$

$$\sum_{j=1}^{2} \int_{u_j}^{v_j} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} + Y(a, u_1, v_1, u_2, v_2) \ge 2\int_a^{\mathscr{b}} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \qquad (3.13)$$

where

$$Y(a, u_1, v_1, u_2, v_2) = \sum_{j=1}^{2} \int_a^{u_j} [\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_j)]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \ge 0.$$

Now by replacing $u = u_j$ and $v = v_j$ for j=2,3 in identity (3.2), respectively.

$$\begin{aligned}
&\int_a^b \mathcal{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} - \int_{u_2}^{v_2} \mathcal{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} \\
&= \int_a^{u_2} \mathfrak{L}(\mathscr{W})[\mathcal{Q}(\mathscr{W}) - \mathcal{Q}(u_2)\triangle\mathscr{W} \\
&+ \int_{u_2}^{v_2} [\mathcal{Q}(u_2) - \mathcal{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})]\triangle\mathscr{W} + \int_{v_2}^{b} \mathfrak{L}(\mathscr{W})[\mathcal{Q}(\mathscr{W}) - \mathcal{Q}(u_2)]\triangle\mathscr{W}
\end{aligned} \tag{3.14}$$

$$\begin{aligned}
&\int_a^b \mathcal{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} - \int_{u_3}^{v_3} \mathcal{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} \\
&= \int_a^{u_3} \mathfrak{L}(\mathscr{W})[\mathcal{Q}(\mathscr{W}) - \mathcal{Q}(u_3)\triangle\mathscr{W} \\
&+ \int_{u_3}^{v_3} [\mathcal{Q}(u_3) - \mathcal{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})]\triangle\mathscr{W} + \int_{v_3}^{b} \mathfrak{L}(\mathscr{W})[\mathcal{Q}(\mathscr{W}) - \mathcal{Q}(u_3)]\triangle\mathscr{W}
\end{aligned} \tag{3.15}$$

Addition of equations (3.14) *and* (3.15) *yields.*

$$\begin{aligned}
&2\int_a^b \mathcal{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} - \sum_{j=2}^{3}\int_{u_j}^{v_j} \mathcal{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} \\
&= \sum_{j=2}^{3}\int_a^{u_j} \mathfrak{L}(\mathscr{W})[\mathcal{Q}(\mathscr{W}) - \mathcal{Q}(u_j)\triangle\mathscr{W} \\
&+ \sum_{j=2}^{3}\int_{u_j}^{v_j} [\mathcal{Q}(u_j) - \mathcal{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})]\triangle\mathscr{W} + \sum_{j=2}^{3}\int_{v_j}^{b} \mathfrak{L}(\mathscr{W})[\mathcal{Q}(\mathscr{W}) - \mathcal{Q}(u_j)]\triangle\mathscr{W}
\end{aligned} \tag{3.16}$$

In the view of assumption $\mathcal{Q}$ is decreasing on intervals $[a,u_2],[a,u_3],[u_2,v_2],[u_3,v_3]$ and $\mathcal{Q} \le 0$ and also from inequality $0 \le \mathfrak{L}(\mathscr{W}) \le \mathfrak{J}(\mathscr{W})$. we have,

$$\int_a^{u_j} \mathfrak{L}(\mathscr{W})[\mathcal{Q}(\mathscr{W}) - \mathcal{Q}(u_j)\triangle\mathscr{W} \ge 0$$

$$\text{and} \qquad \int_{u_j}^{v_j} [\mathcal{Q}(u_j) - \mathcal{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})]\triangle\mathscr{W} \ge 0 \qquad for \quad j = 2,3$$

So

$$\sum_{j=2}^{3}\left[\int_a^{u_j} \mathfrak{L}(\mathscr{W})[\mathcal{Q}(\mathscr{W}) - \mathcal{Q}(u_j)\triangle\mathscr{W} \ge 0 + \int_{u_j}^{v_j} [\mathcal{Q}(u_j) - \mathcal{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}\right] \ge 0 \tag{3.17}$$

Therefore equation (3.16) *becomes, together with equation* (3.17).

$$2\int_a^{b}\mathcal{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\sum_{j=2}^{3}\int_{u_j}^{v_j}\mathcal{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}\geq\sum_{j=2}^{3}\int_{v_j}^{b}\mathfrak{L}(\mathscr{W})[\mathcal{Q}(\mathscr{W})-\mathcal{Q}(u_j)]\triangle\mathscr{W}$$

$$2\int_a^{b}\mathcal{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\geq\sum_{j=2}^{3}\int_{u_j}^{v_j}\mathcal{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}-y(b,u_2,v_2,u_3,v_3) \tag{3.18}$$

where

$$y(b,u_2,v_2,u_3,v_3)=\sum_{j=2}^{3}\int_{v_j}^{b}[\mathcal{Q}(u_j)-\mathcal{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\geq 0$$

Combining the inequalities (3.13) *and* (3.18) *we have the required result(3.8).*

Theorem 3.6. *Assume* $\mathscr{A}_1,\mathscr{A}_2,\mathscr{A}_3$ *hold. Further let* $\int_a^{b}\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}=\int_{u_j}^{v_j}\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}$ *where* $[u_j,v_j]_{\mathbb{T}}\subseteq[a,b]_{\mathbb{T}}$ *for* $j=1,2,3,\ldots,n$ *and also* $v_1\leq v_2\leq v_3\ldots.\leq v_n$.
Then

$$\begin{aligned}&\sum_{j=2}^{n}\int_{u_j}^{v_j}\mathcal{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}-r(u_n,v_n)\\&\leq(n-1)\int_a^{b}\mathcal{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\\&\leq\sum_{j=1}^{n-1}\int_{u_j}^{v_j}\mathcal{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}+R(u_{n-1},v_{n-1}).\end{aligned} \tag{3.19}$$

where

$$\begin{aligned}r(u_n,v_n)&=\textstyle\int_{v_j}^{b}[\mathcal{Q}(u_j)-\mathcal{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\geq 0\\R(u_{n-1},v_{n-1})&=\textstyle\int_a^{u_j}[\mathcal{Q}(\mathscr{W})-\mathcal{Q}(v_j)]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\geq 0\end{aligned}$$

Proof. *By replacing* $u=u_j$ *and* $v=v_j$ *for j=1,2,...,n-1 in identity (3.1), respectively.*

$$\begin{aligned}&\int_{u_1}^{v_1}\mathcal{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}-\int_a^{b}\mathcal{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\\&=\int_a^{u_1}\mathfrak{L}(\mathscr{W})[\mathcal{Q}(v_1)-\mathcal{Q}(\mathscr{W})]\triangle\mathscr{W}\\&+\int_{u_1}^{v_1}[\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})][\mathcal{Q}(\mathscr{W})-\mathcal{Q}(v_1)]\triangle\mathscr{W}+\int_{v_1}^{b}\mathfrak{L}(\mathscr{W})[\mathcal{Q}(v_1)-\mathcal{Q}(\mathscr{W})]\triangle\mathscr{W}\end{aligned} \tag{3.20}$$

$$\begin{aligned}
&\int_{u_2}^{v_2} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} - \int_a^{\mathscr{b}} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \\
&= \int_a^{u_2} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_2) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W} \\
&+ \int_{u_2}^{v_2} [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_2)]\triangle\mathscr{W} + \int_{v_2}^{\mathscr{b}} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_2) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W}
\end{aligned} \tag{3.21}$$

Similarly

$$\begin{aligned}
&\int_{u_{n-1}}^{v_{n-1}} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} - \int_a^{\mathscr{b}} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \\
&= \int_a^{u_{n-1}} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_{n-1}) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W} \\
&+ \int_{u_{n-1}}^{v_{n-1}} [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_{n-1})]\triangle\mathscr{W} + \int_{v_{n-1}}^{\mathscr{b}} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_{n-1}) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W}
\end{aligned} \tag{3.22}$$

Addition of all above equations we have

$$\begin{aligned}
&\sum_{j=1}^{n-1} \int_{u_j}^{v_j} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} - (n-1)\int_a^{\mathscr{b}} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \\
&= \sum_{j=1}^{n-1} \int_a^{u_j} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_j) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W} \\
&+ \sum_{j=1}^{n-1} \int_{u_j}^{v_j} [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_j)]\triangle\mathscr{W} + \sum_{j=1}^{n-1} \int_{v_j}^{\mathscr{b}} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_j) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W}
\end{aligned} \tag{3.23}$$

From inequality $0 \leq \mathfrak{L}(\mathscr{W}) \leq \mathfrak{J}(\mathscr{W})$ *and by our assumption,* $\mathscr{Q}$ *is decreasing on intervals* $[u_1, v_1]$, $[u_2, v_2]$,$\ldots$,$[u_n, v_n]$. *we have*

$$\int_{u_j}^{v_j} [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_j)]\triangle\mathscr{W} \geq 0$$

and
$$\int_{v_j}^{\mathscr{b}} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_j) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W} \geq 0 \qquad for \quad j = 1, 2, ..., n-1$$

So

$$\sum_{j=1}^{n-1} \left[\int_{u_j}^{v_j} [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_j)]\triangle\mathscr{W} + \int_{v_j}^{\mathscr{b}} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(v_j) - \mathscr{Q}(\mathscr{W})]\triangle\mathscr{W} \right] \geq 0 \tag{3.24}$$

Finally Equation (3.23) becomes, together with equation (3.24).

$$\sum_{j=1}^{n-1}\int_{u_j}^{v_j}\mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}+R(u_{n-1},v_{n-1})\geq(n-1)\int_a^b\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \tag{3.25}$$

where

$$R(u_{n-1},v_{n-1})=\int_a^{u_j}[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v_j)]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}\geq 0$$

Now by setting $u=u_j$ *and* $v=v_j$ *for j=2,3,...,n in identity (3.2), respectively.*

$$\begin{aligned}&\int_a^b\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{u_2}^{v_2}\mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}\\&=\int_a^{u_2}\mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_2)\triangle\mathscr{W}\\&+\int_{u_2}^{v_2}[\mathscr{Q}(u_2)-\mathscr{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}+\int_{v_2}^b\mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_2)]\triangle\mathscr{W}\end{aligned} \tag{3.26}$$

$$\begin{aligned}&\int_a^b\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{u_3}^{v_3}\mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}\\&=\int_a^{u_3}\mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_3)\triangle\mathscr{W}\\&+\int_{u_3}^{v_3}[\mathscr{Q}(u_3)-\mathscr{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}+\int_{v_3}^b\mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_3)]\triangle\mathscr{W}\end{aligned} \tag{3.27}$$

Similarly

$$\begin{aligned}&\int_a^b\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\int_{u_n}^{v_n}\mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}\\&=\int_a^{u_n}\mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_n)\triangle\mathscr{W}\\&+\int_{u_n}^{v_n}[\mathscr{Q}(u_n)-\mathscr{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}+\int_{v_n}^b\mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_n)]\triangle\mathscr{W}\end{aligned} \tag{3.28}$$

Addition of all above equations yields.

$$\begin{aligned}&(n-1)\int_a^b\mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W}-\sum_{j=2}^{n}\int_{u_j}^{v_j}\mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W}\\&=\sum_{j=2}^{n}\int_a^{u_j}\mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_j)]\triangle\mathscr{W}\\&+\sum_{j=2}^{n}\int_{u_j}^{v_j}[\mathscr{Q}(u_j)-\mathscr{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}+\sum_{j=2}^{n}\int_{v_j}^b\mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_j)]\triangle\mathscr{W}\end{aligned} \tag{3.29}$$

By our assumption, $\mathscr{Q}$ is decreasing on intervals $[a,u_2]$, $[a,u_3]$,...,$[a,u_n]$ and from inequality $0 \leq \mathfrak{L}(\mathscr{W}) \leq \mathfrak{J}(\mathscr{W})$. we conclude that,

$$\int_a^{u_j} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_j)]\triangle\mathscr{W} \geq 0$$

$$\text{and} \qquad \int_{u_j}^{v_j} [\mathscr{Q}(u_j)-\mathscr{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W} \geq 0 \qquad for\ \ j=2,3,...,n$$

So

$$\sum_{j=2}^{n}\left[\int_a^{u_j} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_j)]\triangle\mathscr{W} + \int_{u_j}^{v_j} [\mathscr{Q}(u_j)-\mathscr{Q}(\mathscr{W})][\mathfrak{J}(\mathscr{W})-\mathfrak{L}(\mathscr{W})]\triangle\mathscr{W}\right] \geq 0 \tag{3.30}$$

Finally Equation (3.29) becomes, together with equation (3.30).

$$(n-1)\int_a^{b} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \geq \sum_{j=2}^{n}\int_{u_j}^{v_j} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} - r(u_n,v_n) \tag{3.31}$$

where

$$r(u_n,v_n) = \int_{v_j}^{b} [\mathscr{Q}(u_j)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \geq 0$$

which yields the conclusion of the theorem 3.6 by combining inequalities (3.25) and (3.31).

Theorem 3.7. *Suppose $\mathscr{A}_1,\mathscr{A}_2,\mathscr{A}_4$ hold. with $\mathscr{Q}$ is of one sign and decreasing and $0 \leq \mathfrak{N}(\mathscr{W}) \leq \mathfrak{L}(\mathscr{W}) \leq \mathfrak{J}(\mathscr{W}) - \mathfrak{N}(\mathscr{W})$ on $[a,b]_{\mathbb{T}}$. Further suppose $\int_a^b \mathfrak{L}(\mathscr{W})\triangle\mathscr{W} = \int_{u_j}^{v_j} \mathfrak{J}(\mathscr{W})\triangle\mathscr{W}$*
where $[u_j,v_j]_{\mathbb{T}} \subseteq [a,b]_{\mathbb{T}}$ for $j=1,2$ and $v_1 \leq v_2$.
Then

$$\begin{aligned}
&\int_{u_2}^{v_2} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} + \int_a^{v_2} |[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(u_2)]\mathfrak{N}(\mathscr{W})|\triangle\mathscr{W} - \int_{v_2}^{b}[\mathscr{Q}(u_2)-\mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \\
&\leq \int_a^{b} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \\
&\leq \int_{u_1}^{v_1} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} + \int_a^{u_1} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v_1)]\triangle\mathscr{W} \\
&-\int_{u_1}^{b} |[\mathscr{Q}(\mathscr{W})-\mathscr{Q}(v_1)]\mathfrak{N}(\mathscr{W})|\triangle\mathscr{W}
\end{aligned} \tag{3.32}$$

Proof. *By the assumption that function $\mathscr{Q}$ one sign and nonincreasing function on the interval $[v_1,b]$ and* $\quad 0 \leq \mathfrak{N}(\mathscr{W}) \leq \mathfrak{L}(\mathscr{W}) \leq \mathfrak{J}(\mathscr{W}) - \mathfrak{N}(\mathscr{W})$

it follows that

$$\begin{aligned}
&\int_{u_1}^{v_1} |\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_1)|[\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})]\triangle\mathscr{W} + \int_{v_1}^{b} |\mathscr{Q}(v_1) - \mathscr{Q}(\mathscr{W})|\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \\
&\geq \int_{u_1}^{v_1} |\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_1)|\mathfrak{N}(\mathscr{W})\triangle\mathscr{W} + \int_{v_1}^{b} |\mathscr{Q}(v_1) - \mathscr{Q}(\mathscr{W})|\mathfrak{N}(\mathscr{W})\triangle\mathscr{W} \qquad (3.33)\\
&= \int_{u_1}^{b} |[\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_1)]|\mathfrak{N}(\mathscr{W})\triangle\mathscr{W}
\end{aligned}$$

Also since $\mathscr{Q}$ one sign and nonincreasing function on intervals $[a,u_2]$ and $[u_2,v_2]$, it follows that

$$\begin{aligned}
&\int_{a}^{u_2} |\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(u_2)|\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} + \int_{u_2}^{v_2} |\mathscr{Q}(u_2) - \mathscr{Q}(\mathscr{W})|[\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})]\triangle\mathscr{W} \\
&\geq \int_{a}^{u_2} |\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(u_2)|\mathfrak{N}(\mathscr{W})\triangle\mathscr{W} + \int_{u_2}^{v_2} |\mathscr{Q}(u_2) - \mathscr{Q}(\mathscr{W})|\mathfrak{N}(\mathscr{W})\triangle\mathscr{W} \qquad (3.34)\\
&= \int_{a}^{v_2} |\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(u_2)|\mathfrak{N}(\mathscr{W})\triangle\mathscr{W}
\end{aligned}$$

using inequality (3.33) equation (3.4) can be written as

$$\begin{aligned}
&\int_{u_1}^{v_1} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} - \int_{a}^{b} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} + \int_{a}^{u_1} [\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_1)]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \\
&\geq \int_{u_1}^{b} |[\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_1)]\mathfrak{N}(\mathscr{W})|\triangle\mathscr{W}
\end{aligned} \qquad (3.35)$$

and using inequality (3.34) equation (3.6) can be written as

$$\begin{aligned}
&\int_{a}^{b} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} - \int_{u_2}^{v_2} \mathscr{Q}(\mathscr{W})\mathfrak{J}(\mathscr{W})\triangle\mathscr{W} - \int_{u_2}^{v_2} [\mathfrak{J}(\mathscr{W}) - \mathfrak{L}(\mathscr{W})][\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(u_2)]\triangle\mathscr{W} \\
&\geq \int_{a}^{v_2} |[\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(u_2)]\mathfrak{N}(\mathscr{W})|\triangle\mathscr{W}
\end{aligned} \qquad (3.36)$$

By the combination of the inequalities (3.35) and (3.36) the inequality (3.32) is proved.

Remark 3.8. Inequality (3.32) reduces to inequality (3.4), when $\mathfrak{N}(\mathscr{W}) = 0$.

Remark 3.9. A special case of inequality (3.32) can be found, if we put $\mathfrak{N} = \Im$ and $\mathfrak{J} = 1$, so

$$\begin{aligned}
&\int_{u_2}^{v_2} \mathscr{Q}(\mathscr{W})\triangle\mathscr{W} - \int_{v_2}^{b} [\mathscr{Q}(u_2) - \mathscr{Q}(\mathscr{W})]\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} + \Im\int_{a}^{v_2} |[\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(u_2)]|\triangle\mathscr{W} \\
&\leq \int_{a}^{b} \mathscr{Q}(\mathscr{W})\mathfrak{L}(\mathscr{W})\triangle\mathscr{W} \\
&\leq \int_{u_1}^{v_1} \mathscr{Q}(\mathscr{W})\triangle\mathscr{W} + \int_{a}^{u_1} \mathfrak{L}(\mathscr{W})[\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_1)]\triangle\mathscr{W} - \Im\int_{u_1}^{b} |[\mathscr{Q}(\mathscr{W}) - \mathscr{Q}(v_1)]|\triangle\mathscr{W}
\end{aligned}$$

where $u_1, u_2, v_1, v_2 \in [a, b]_{\mathbb{T}}$ and $0 \leq \mathfrak{I} \leq \mathfrak{L}(\mathscr{W}) \leq 1 - \mathfrak{I}$ for all $\mathscr{W} \in [a, b]_{\mathbb{T}}$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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