Available online at http://scik.org
Adv. Inequal. Appl. 2026, 2026:4
https://doi.org/10.28919/aia/9781
ISSN: 2050-7461

HERMITE HADAMARD TYPE INEQUALITIES FOR PRODUCT OF LOG CONVEX FUNCTION AND EXPONENTIAL CONVEX FUNCTIONS AND SOME GENERALIZED RESULTS

SHAMA FIRDOUS, LABEEB AHMAD*, WAQAS NAZEER

Department of Mathematics, Government College University, Katchery Road, Lahore 54000, Pakistan

Abstract. In this article, the author explores and establishes results related to Hermite-Hadamard type inequalities specifically for log-convex and exponentially convex functions. The study focuses on deriving and proving these inequalities, which provide bounds on the integral mean of a function in terms of its values at specific points. Additionally, the author extends these fundamental results by presenting generalized versions of the inequalities. This generalization is supported by concrete examples that illustrate how the inequalities apply in different cases. These examples help demonstrate the practical implications and validity of the derived inequalities in mathematical analysis and related fields.

Keywords: Hermite-Hadamard type inequality; convex functions; exponential convex functions; Logarithmic convex functions.

2010 AMS Subject Classification: 26D15, 26A51.

1. INTRODUCTION

Convex analysis, a field of mathematics having important applications in optimization theory, is concerned with the study of convex functions and convex sets. For example, it is essential for formulating and solving optimization problems, especially those involving linear programming

*Corresponding author
E-mail address: drlabeebahmad@gmail.com
Received January 17, 2026

and nonlinear optimization, and it offers the fundamental framework for convex optimization, which is widely used in many other domains; convex analysis is utilized in signal processing to enhance data interpretation by improving signal estimation and processing algorithms, and in economics to represent preferences and utility functions. For more recent applications in diverse disciplines of applied sciences, we refer to [1–3] and the references therein.

The study of inequalities involving convex functions and convex sets is known as the theory of convex inequalities. It focuses on determining the circumstances in which specific inequalities hold, generating new inequalities from existing ones, and using the outcomes to solve optimization problems and other mathematical difficulties. Researchers have developed various types of inequalities using various function classes, including Ostrowki [4], Jensen [5], Holder [6], Minkowski [7], Pachpatte [8], Newton [9], Simpson [10], Bullen [11], Young [12], Trapezoidal [13]. For more information, see [14–16] and references therein.

Among these inequalities, Hermite-Hadamard type inequality is a fundamental finding in convex analysis that limits the integral of a convex function over an interval and has numerous applications in convex optimization problems. The inequality is defined as follows:
Suppose $\Psi : I \subseteq \mathsf{R} \to \mathsf{R}$ be a convex mapping defined on the interval Ω with $\mu_1, \mu_2 \in I$. Then, the inequality stated below is true (see Ref. [17]):

$$\Psi\left(\frac{\mu_1+\mu_2}{2}\right) \leq \frac{1}{\mu_2-\mu_1}\int_{\mu_1}^{\mu_2} \Psi(x)\,\mathrm{d}x \leq \frac{\Psi(\mu_1)+\Psi(\mu_2)}{2}. \tag{1.1}$$

This double inequality has been generalized using several convex mappings, such as s-convex [18], p-convex [19], h-convex [20], log-convex [21], exponential convex [22], coordinated convex [23], α convex [24] (α, m) convex functions [25] and many more. For further information, see [26–28] and references therein.

In [29], B. G. Pachpatte used classical convex mappings to establish the following form of Hermite-Hadamard inequality.

Theorem 1.1 (see [29]). *We consider Φ and Ψ such that $\Phi, \Psi : I = [\hat{\Upsilon}_1, \hat{\Upsilon}_2] \subset \mathbb{R} \to \mathbb{R}$ to be convex real-valued and nonnegative on closed interval $[\hat{\Upsilon}_1, \hat{\Upsilon}_2]$. After using these assumptions the inequalities becomes,*

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx \leq \frac{1}{3}M(\hat{\Upsilon}_1,\hat{\Upsilon}_2)+\frac{1}{6}N(\hat{\Upsilon}_1,\hat{\Upsilon}_2),$$

$$\begin{aligned}2\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right) \leq & \frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx \\ & +\frac{1}{6}M(\hat{\Upsilon}_1,\hat{\Upsilon}_2)+\frac{1}{3}N(\hat{\Upsilon}_1,\hat{\Upsilon}_2),\end{aligned}$$

where $M(\hat{\Upsilon}_1,\hat{\Upsilon}_2)=\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)+\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2), N(\hat{\Upsilon}_1,\hat{\Upsilon}_2)=\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1), \overline{\Theta}\in [0,1]$.

In 2013, Feixiang Chen [30] developed Hermite-Hadamard type inequalities by using several product forms such as:

Theorem 1.2 (see [30])**.** *Let* $\Phi,\Psi : I=[\hat{\Upsilon}_1,\hat{\Upsilon}_2]\subset\mathbb{R}\to\mathbb{R}$ *are real-valued, nonnegative and convex functions on closed interval* $[\hat{\Upsilon}_1,\hat{\Upsilon}_2]$*. Then,*

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx \leq L(\overline{\Theta}) \leq \frac{1}{3}M(\hat{\Upsilon}_1,\hat{\Upsilon}_2)+\frac{1}{6}N(\hat{\Upsilon}_1,\hat{\Upsilon}_2),$$

and

$$\begin{aligned}L(\overline{\Theta}) =& \frac{\overline{\Theta}}{3}M(\hat{\Upsilon}_1,(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)+\frac{\overline{\Theta}}{6}N(\hat{\Upsilon}_1,(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\\ &+\frac{1-\overline{\Theta}}{3}M((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2,\hat{\Upsilon}_2)\\ &+\frac{1-\overline{\Theta}}{6}N((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2,\hat{\Upsilon}_2).\end{aligned}$$

In [22], Awan et al. used exponentially convex functions to create the following type of Hermite-Hadamard type inequality.

Theorem 1.3 (see [22])**.** *Assume that* $\Phi: I\subset\mathbb{R}\to\mathbb{R}$ *is an exponentially convex and integrable function. Then,*

$$\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right) \leq \frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\frac{\Phi(x)}{e^{\alpha x}}\,\mathrm{d}x \leq \frac{e^{-\alpha\hat{\Upsilon}_1}\Phi(\hat{\Upsilon}_1)+e^{-\alpha\hat{\Upsilon}_2}\Phi(\hat{\Upsilon}_2)}{2}.$$

Dragomir and Mond demonstrated in [31] that the Hermite-Hadamard inequalities for log-convex functions are as follows:

$$\begin{aligned}
\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right) &\leqslant \exp\left[\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1{}^{\hat{\Upsilon}_2}} \ln[\Phi(x)]dx\right] \\
&\leqslant \frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} \hat{G}(\Phi(x),\Phi(\hat{\Upsilon}_1+\hat{\Upsilon}_2-x))dx \\
&\leqslant \frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} \Phi(x)dx \leqslant L(\Phi(\hat{\Upsilon}_1),\Phi(\hat{\Upsilon}_2)) \\
&\leqslant \frac{\Phi(\hat{\Upsilon}_1)+\Phi(\hat{\Upsilon}_2)}{2}
\end{aligned}$$

where $\hat{G}(\rho,\nu) := \sqrt{\rho\nu}$ is the geometric mean and $L(\rho,\nu) := \frac{\rho-\nu}{\ln rho-\ln\nu}(\rho\neq\nu)$ is the logarithmic mean of the positive real numbers $\rho,\nu($ for $\rho=\nu$, we put $L(\rho,\rho)=\rho)$.

Novelty and Significance

This article develops various new types of Hermite-Hadamard type inequalities and their product forms by using two types of modified generalized convex mappings, namely log convex and exponential convex. In addition, several interesting and non-trivial examples were used to verify the development. Hopefully, readers will find the results presented in this article interesting, and they can work on this idea in the future using other types of generalized convex mappings such as Godunova-Levin, harmonic convex, p-convex, h-convex, and others.

Based on the extensive literature on developed results, and particularly the works of the following authors motivated us [21, 22, 30], to present new results in this paper. The format of this article is as follows. In Section 2, we will review some key terms related to developed results, as well as some auxiliary definitions. Section 3 develops a new product form of Hermite-Hadamard type inequalities by using two different forms of generalized convex mappings with examples. Lastly, this study is concluded with some recommendations for further research in Section 5.

2. PRELIMINIRIES

To begin, we will review some key terms related to developed results, as well as some auxiliary definitions.

Definition 2.1 (see [19–23]). A real-valued function $\Phi : I \to \mathbb{R}$ serves as a convex function for the interval I of real numbers when The set $I \subset \mathbb{R}$ satisfies the requirement for being a convex function if $\hat{\gamma}_1,\hat{\gamma}_2 \in I$ holds true and $\hat{\gamma}_1 < \hat{\gamma}_2$ and $\overline{\Theta} \in [0,1]$ for all ξ. Then we have,

$$\Phi\left(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\leq\overline{\Theta}\Phi(\hat{\Upsilon}_1)+(1-\overline{\Theta})\Phi(\hat{\Upsilon}_2). \tag{2.1}$$

Definition 2.2 (see [21])**.** A real-valued function that transforms values in I to range between $(0,\infty)$ is known as *multiplicative convex* or *logarithmically convex*.

$\log\Phi$ is convex when, for all pairs $\hat{\gamma}_1,\hat{\gamma}_2$ from I and value ξ in $[0,1]$, we obtain

$$log\Phi(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2)\leq\overline{\Theta}log\Phi(\hat{\Upsilon}_1)+(1-\overline{\Theta})log\Phi(\hat{\Upsilon}_2)$$

$$\Phi(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2)\leq(\Phi(\hat{\Upsilon}_1))^{\overline{\Theta}}(\Phi(\hat{\Upsilon}_2))^{1-\overline{\Theta}}.$$

In order to give another concept of composition of log-convex function,
we define log convexity for a function Ψ, that is

$$log\Psi(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2)\leq\overline{\Theta}log\Psi(\hat{\Upsilon}_1)+(1-\overline{\Theta})log\Psi(\hat{\Upsilon}_2)$$

$$\Psi(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2)\leq(\Psi(\hat{\Upsilon}_1))^{\overline{\Theta}}(\Psi(\hat{\Upsilon}_2))^{1-\overline{\Theta}}.$$

Since Φ and Ψ are non-decreasing and monotonic functions.Then $\Psi o\Phi$ is convex.
Every log convex is convex but converse is not true in general, which is stated as:

Proposition 2.3. *Are all log convex functions also convex functions, and vice versa?*

Proof. Suppose that $\Phi:(\hat{\Upsilon}_1,\hat{\Upsilon}_2)\to(0,1)$ such that $log\Phi$ is convex. We will prove that Φ is convex.
Taking into account the definition 2.2, we have,

$$\Phi\left(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\leq\overline{\Theta}\Phi(\hat{\Upsilon}_1)+(1-\overline{\Theta})\Phi(\hat{\Upsilon}_2)$$

for some $\overline{\Theta}\in[0,1]$.
Since we know that $\Psi(x)=log\Phi(x)$ is convex, therefore

$$\Psi\left(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\leq\overline{\Theta}\Psi(\hat{\Upsilon}_1)+(1-\overline{\Theta})\Psi(\hat{\Upsilon}_2)$$

$$log\Phi(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2)\leq\overline{\Theta}log\Phi(\hat{\Upsilon}_1)+(1-\overline{\Theta})log\Phi(\hat{\Upsilon}_2)$$

$$\Phi(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2)\leq(\Phi(\hat{\Upsilon}_1))^{\overline{\Theta}}(\Phi(\hat{\Upsilon}_2))^{1-\overline{\Theta}},$$

by arithmetic-geometric mean, we have

$$(\Phi(\hat{\Upsilon}_1))^{\overline{\Theta}}(\Phi(\hat{\Upsilon}_2))^{1-\overline{\Theta}}\leq\overline{\Theta}\Phi(\hat{\Upsilon}_1)+(1-\overline{\Theta})\Phi(\hat{\Upsilon}_2).$$

Shows that $log\Phi$ is convex, and hence Ψ is convex.

its converse is not true in general.

counter-example.

$$\Phi(x) = x^2$$

is convex but its log is not convex, as

$$log\Phi(x) = 2log|x|$$

is not convex.

Example 2.4. Assume that $\Phi : I \to \mathbb{R}$ is a log convex function that takes into consideration the constraints specified in definition2.1 such as,

$$\Phi(x) = e^{\Psi(x)},$$

where $\Psi(x)$ is any convex function. For instance, if $\Phi(x) = e^x$, then

$$\log(\Phi(x)) = x,$$

which is a convex function. Therefore, $\Phi(x) = e^x$ is log-convex.

Example 2.5. Let $\Phi : I \to \mathbb{R}$ be a log convex function taking into account the conditions defined as in definition2.1 such as,

$$\Phi(x) = x^a \quad \text{where} \quad a > 0.$$

Consequently,

$$\log(\Phi(x)) = a\log(x),$$

and additionally, $\log(x)$ is convex for $x > 0$, $\Phi(x)$ is log-convex.

Example 2.6. Since $\Phi, \Psi : I \to (0, \infty)$ are log convex functions as,

$$\Phi(x) = e^x \quad \text{and} \quad \Psi(x) = x^a \quad (\text{with } a > 0).$$

Their product is

$$h(x) = e^x \cdot x^a.$$

Since both e^x and x^a are log-convex, their product is also log-convex.

Example 2.7. Φ and Ψ are functions as defined above,

$$\Phi(x) = x^a \quad \text{and} \quad \Psi(x) = x^b \quad (\text{with } a > 0 \text{ and } b > 0).$$

The product is

$$h(x) = x^a \cdot x^b = x^{a+b}.$$

Since power functions with positive exponents are log-convex, So $h(x)$ is also log-convex.

3. The Main Results

Through the use of generalized log-convex type mappings, this section will examine Hermite-Hadamard inequality and its various product forms.

Theorem 3.1. *The real-valued functions Φ and Ψ defined on the interval $I = [\hat{\gamma}_1, \hat{\gamma}_2]$ produce positive real number, since $\Phi, \Psi : I = [\hat{\gamma}_1, \hat{\gamma}_2] \subset \mathbb{R} \to (0, \infty)$. Both functions are nonnegative while being log-convex on $[\hat{\gamma}_1, \hat{\gamma}_2]$.*

Then the following inequality holds:

$$\frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1} \int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} \Phi(x)\Psi(x)dx \leq L\big(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1), \Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\big) \tag{3.1}$$

where $L(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1), \Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)) = \frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1) - \Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{ln\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)\right) - ln\left(\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right)}$ is the logarithmic mean of nonnegative real numbers $\hat{\Upsilon}_1$ and $\hat{\Upsilon}_2$.

Proof. Since Φ and Ψ are log convex functions, it follows that,

$$\Phi(\overline{\Theta}\hat{\Upsilon}_1 + (1-\overline{\Theta})\hat{\Upsilon}_2) \leq (\Phi(\hat{\Upsilon}_1))^{\overline{\Theta}}(\Phi(\hat{\Upsilon}_2))^{1-\overline{\Theta}} \tag{3.2}$$

$$\Psi(\overline{\Theta}\hat{\Upsilon}_1 + (1-\overline{\Theta})\hat{\Upsilon}_2) \leq (\Psi(\hat{\Upsilon}_1))^{\overline{\Theta}}(\Psi(\hat{\Upsilon}_2))^{1-\overline{\Theta}} \tag{3.3}$$

Multiplying both inequalities we have,

$$\begin{aligned}
&\Phi(\overline{\Theta}\hat{\Upsilon}_1 + (1-\overline{\Theta})\hat{\Upsilon}_2)\Psi(\overline{\Theta}\hat{\Upsilon}_1 + (1-\overline{\Theta})\hat{\Upsilon}_2) \\
&\leq (\Phi(\hat{\Upsilon}_1))^{\overline{\Theta}}(\Phi(\hat{\Upsilon}_2))^{1-\overline{\Theta}}(\Psi(\hat{\Upsilon}_1))^{\overline{\Theta}}(\Psi(\hat{\Upsilon}_2))^{1-\overline{\Theta}} \\
&\leq \Phi(\hat{\Upsilon}_2)\left[\frac{\Phi(\hat{\Upsilon}_1)}{\Phi(\hat{\Upsilon}_2)}\right]^{\overline{\Theta}}\Psi(\hat{\Upsilon}_2)\left[\frac{\Psi(\hat{\Upsilon}_1)}{\Psi(\hat{\Upsilon}_2)}\right]^{\overline{\Theta}}
\end{aligned}$$

$$\leq \Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\left[\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}\right]^{\overline{\Theta}}.$$

Integrating both sides with regard to $\overline{\Theta}$ where ξ varies from 0 to 1 yields,

$$\begin{aligned}
&\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_0^1 \Phi(x)\Psi(x)dx \\
&\leq \int_0^1 \Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}]^{\overline{\Theta}}d\overline{\Theta} \\
&\leq \Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\frac{\left[\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}\right]^{\overline{\Theta}}}{ln\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}}\Bigg|_0^1 \\
&\leq \Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\frac{\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}}{ln\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)-ln\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}-\frac{1}{ln\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)-ln\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)} \\
&\leq \frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)-\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{ln\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)-ln\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)} \\
&\leq L\big(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\big).
\end{aligned}$$

this completes the proof.

Example 3.2. Assume that $\Phi,\Psi : I=[\hat{\Upsilon}_1,\hat{\Upsilon}_2]\subset\mathbb{R}\to(0,\infty)$ are real-valued, nonnegative and log-convex functions on closed interval $[\hat{\Upsilon}_1,\hat{\Upsilon}_2]$.

For instance

$$\Phi(x)=x^{\mu_1} \quad \text{and} \quad \Psi(x)=x^{\mu_2}\text{with} \quad \mu_1>0,\mu_2>0$$

$$\Phi(x)\Psi(x)=x^{\mu_1+\mu_2}.$$

We want to show

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} x^{\mu_1+\mu_2}\,dx \leq L(\hat{\Upsilon}_1^{\mu_1}\hat{\Upsilon}_1^{\mu_2},\hat{\Upsilon}_2^{\mu_1}\hat{\Upsilon}_2^{\mu_2}),$$

where

$$L(\hat{\Upsilon}_1^{\mu_1}\hat{\Upsilon}_1^{\mu_2},\hat{\Upsilon}_2^{\mu_1}\hat{\Upsilon}_2^{\mu_2})=\frac{\hat{\Upsilon}_1^{(\mu_1+\mu_2)}-\hat{\Upsilon}_2^{(\mu_1+\mu_2)}}{(\mu_1+\mu_2)(ln\hat{\Upsilon}_1-ln\hat{\Upsilon}_2)},$$

and

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} x^{(\mu_1+\mu_2)x}\,dx=\frac{\hat{\Upsilon}_2^{(\mu_1+\mu_2+1)}-\hat{\Upsilon}_1^{(\mu_1+\mu_2+1)}}{(\hat{\Upsilon}_2-\hat{\Upsilon}_1)(\mu_1+\mu_2+1)}.$$

It follows,

$$\frac{\hat{\Upsilon}_2^{(\mu_1+\mu_2+1)}-\hat{\gamma}_{21}^{(\mu_1+\mu_2+1)}}{(\hat{\Upsilon}_2-\hat{\Upsilon}_1)(\mu_1+\mu_2+1)}\leq\frac{\hat{\Upsilon}_1^{(\mu_1+\mu_2)}-\hat{\Upsilon}_2^{(\mu_1+\mu_2)}}{(\mu_1+\mu_2)(ln\hat{\Upsilon}_1-ln\hat{\Upsilon}_2)}$$

Hence,

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} x^{\mu_1+\mu_2}\,dx \le L\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right).$$

Theorem 3.3. *Assume that $\Phi,\Psi : I=[\hat{\Upsilon}_1,\hat{\Upsilon}_2]\subset\mathbb{R}\to(0,\infty)$ are real-valued, nonnegative and log-convex functions on closed interval $[\hat{\Upsilon}_1,\hat{\Upsilon}_2]$.*
Then the hollowing form of Hermite-Hadamard type inequality exists.

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx \le L(\overline{\Theta}) \le L(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)), \tag{3.4}$$

where

$$L(\overline{\Theta})=\overline{\Theta}\left[L(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\right]$$

$$+(1-\overline{\Theta})\left[L(\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right]. \tag{3.5}$$

Proof. The property of log-convex functions enables Φ and Ψ to produce convex diagram when taking logarithms over the interval $[\hat{\Upsilon}_1,\hat{\Upsilon}_2]$, from theorem (3.1) we have,

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx \le L\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right).$$

Firstly Φ and Ψ are log convex on $[\hat{\Upsilon}_1,(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2]$ and on $[(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2,\hat{\Upsilon}_2]$

$$\frac{1}{\overline{\Theta}(\hat{\Upsilon}_2-\hat{\Upsilon}_1)}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx \le L\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2\right), \tag{3.6}$$

similarly

$$\frac{1}{(1-\overline{\Theta})(\hat{\Upsilon}_2-\hat{\Upsilon}_1)}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx \le L\left(\Phi((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right). \tag{3.7}$$

Multiplying 3.6 by $\overline{\Theta}$ and 3.7 by $(1-\overline{\Theta})$ and adding the resultant inequalities, we get

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx \le \overline{\Theta}L\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2\right)$$
$$+(1-\overline{\Theta})L\left(\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right)$$

$$=\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx \le A+B$$

where

$$A=\overline{\Theta}\left[L(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\right],$$

and

$$B = (1-\overline{\Theta})\left[L(\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right].$$

Now solving A, by definition

$$\begin{aligned}
&\overline{\Theta}\left[L(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\right]\\
&=\overline{\Theta}\left[\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)-\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)}{ln(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1))-ln(\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)})\right]\\
&\leq\overline{\Theta}\left[\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)-[\Phi(\hat{\Upsilon}_1)]^{1-\overline{\Theta}}[\Phi(\hat{\Upsilon}_2)]^{\overline{\Theta}}[\Psi(\hat{\Upsilon}_1)]^{1-\overline{\Theta}}[\Psi(\hat{\Upsilon}_2)]^{\overline{\Theta}}}{ln\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)-ln[\Phi(\hat{\Upsilon}_1)]^{1-\overline{\Theta}}[\Phi(\hat{\Upsilon}_2)]^{\overline{\Theta}}[\Psi(\hat{\Upsilon}_1)]^{1-\overline{\Theta}}[\Psi(\hat{\Upsilon}_2)]^{\overline{\Theta}}}\right]\\
&\leq\overline{\Theta}\left[\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)-\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)\left[\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}\right]^{\overline{\Theta}}}{ln\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)-ln\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)\left[\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}\right]^{\overline{\Theta}}}\right]
\end{aligned}$$

$$\leq\overline{\Theta}L\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)\left[\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}\right]^{\overline{\Theta}}\right). \tag{3.8}$$

Let's consider inequality B,

$$\begin{aligned}
&(1-\overline{\Theta})\left[L(\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right]\\
&=(1-\overline{\Theta})\left[\frac{\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)-\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{ln\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)-ln\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}\right]\\
&\leq(1-\overline{\Theta})\left[\frac{\left[\Phi(\hat{\Upsilon}_1)\right]^{1-\overline{\Theta}}\left[\Phi(\hat{\Upsilon}_2)\right]^{\overline{\Theta}}\left[\Psi(\hat{\Upsilon}_1)\right]^{1-\overline{\Theta}}\left[\Psi(\hat{\Upsilon}_2)\right]^{\overline{\Theta}}-\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{ln\left[\Phi(\hat{\Upsilon}_1)\right]^{1-\overline{\Theta}}\left[\Phi(\hat{\Upsilon}_2)\right]^{\overline{\Theta}}\left[\Psi(\hat{\Upsilon}_1)\right]^{1-\overline{\Theta}}\left[\Psi(\hat{\Upsilon}_2)\right]^{\hat{\Upsilon}_2}-ln\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}\right]\\
&\leq(1-\overline{\Theta})\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)\left[\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}\right]^{\overline{\Theta}}-\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{ln\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)\left[\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}\right]^{\overline{\Theta}}-ln\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}
\end{aligned}$$

$$\leq(1-\overline{\Theta})L\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)\left[\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}\right]^{\overline{\Theta}},\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right). \tag{3.9}$$

Combining the results of both inequalities A and B, we have

$$L(\overline{\Theta})\leq\overline{\Theta}L\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)\left[\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}\right]^{\overline{\Theta}}\right)$$

$$+(1-\overline{\Theta})L\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)\left[\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}\right]^{\overline{\Theta}},\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right), \tag{3.10}$$

completes the proof.

Remark 3.4. The result in theorem (3.1) follows directly from applying theorem (3.3) using both for L(0) = L(1) = $L\big(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\big)$, our result indicate an enhancement of the findings presented in the theorem (3.1).

Example 3.5. Assume that $\Phi,\Psi : I=[\hat{\Upsilon}_1,\hat{\Upsilon}_2]\subset\mathbb{R}\to(0,\infty)$ are real-valued, nonnegative and log-convex functions on closed interval $[\hat{\Upsilon}_1,\hat{\Upsilon}_2]$.

For instance, we want to prove the following inequality for log-convex functions $\Phi(x)=x^{\mu_1}$ and $\Psi(x)=x^{\mu_2}$

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)\,dx\le L(\overline{\Theta})\le L(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)),$$

where $L(\overline{\Theta})$ is defined as,

$$L(\overline{\Theta})=\overline{\Theta}\left[L(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2))\right]$$

$$+(1-\overline{\Theta})\left[L(\Phi((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2))\right].$$

Since,

$$\Phi(x)=x^{\alpha},\quad \Psi(x)=x^{\beta}$$

To calculate right hand side, we have to find,

$$\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)=\hat{\Upsilon}_1^{\mu_1}\hat{\Upsilon}_1^{\mu_2}=\hat{\Upsilon}_1^{\mu_1+\mu_2}$$

$$\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)=\hat{\Upsilon}_2^{\mu_1}\hat{\Upsilon}_2^{\mu_2=\hat{\Upsilon}_2^{\mu_1+\mu_1}}$$

$\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)=\hat{\Upsilon}_1^{\mu_1+\mu_2}$ and $\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)=\hat{\Upsilon}_2^{\mu_1+\mu_2}$

$$L(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2))=\frac{\hat{\Upsilon}_2^{(\mu_1+\mu_2)}-\hat{\Upsilon}_1^{(\mu_1+\mu_2)}}{(\mu_1+\mu_1)(ln\hat{\Upsilon}_2-ln\hat{\Upsilon}_1)},$$

and

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)\,dx=\frac{1}{(\hat{\Upsilon}_2-\hat{\Upsilon}_1)(\mu_1+\mu_2+1)}\left(\hat{\Upsilon}_2^{\mu_1+\mu_2+1}-\hat{\Upsilon}_1^{\mu_1+\mu_2+1}\right).$$

We have to find,

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)\,dx \le L(\overline{\Theta}) \le L(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2))$$

$$\frac{1}{(\hat{\Upsilon}_2-\hat{\Upsilon}_1)(\mu_1+\mu_2+1)}\left(\hat{\Upsilon}_2{}^{\mu_1+\mu_2+1}-\hat{\Upsilon}_1{}^{\mu_1+\mu_2+1}\right) \le L(\overline{\Theta}) \le \frac{\hat{\Upsilon}_2{}^{(\mu_1+\mu_2)}-\hat{\Upsilon}_1{}^{(\mu_1+\mu_2)}}{(\mu_1+\mu_2)(ln\hat{\Upsilon}_2-ln\hat{\Upsilon}_1)}$$

Where

$$\begin{aligned}L(\overline{\Theta}) &= \overline{\Theta}L\left(\hat{\Upsilon}_1{}^{(\mu_1+\mu_2)},\hat{\Upsilon}_1{}^{(\mu_1+\mu_2)}\left(\frac{\hat{\Upsilon}_2{}^{\mu_1+\mu_2}}{\hat{\Upsilon}_1{}^{\mu_1+\mu_2}}\right)^{\overline{\Theta}}\right)\\ &+(1-\overline{\Theta})L\left(\hat{\Upsilon}_1{}^{\mu_1+\mu_2}\left(\frac{\hat{\Upsilon}_2{}^{\mu_1+\mu_2}}{\hat{\Upsilon}_1{}^{(\mu_1+\mu_2)}}\right)^{\overline{\Theta}},\hat{\Upsilon}_2{}^{\mu_1+\mu_2}\right)\end{aligned}$$

$L(\overline{\Theta})$ can be find for any value of $\hat{\Upsilon}_1,\hat{\Upsilon}_2$ of its domain and for $\overline{\Theta}\in(0,1)$ also for μ_1 and μ_2 of its domain.

Theorem 3.6. *Assume that $\Phi,\Psi : I=[\hat{\Upsilon}_1,\hat{\Upsilon}_2]\subset\mathbb{R}\to(0,\infty)$ are real-valued, nonnegative and log-convex functions on closed interval $[\hat{\Upsilon}_1,\hat{\Upsilon}_2]$. Then following version of Hermite-Hadamard type inequality for product of two log convex function exists, that is,*

$$\begin{aligned}&2\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right) \le \frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx\\ &+\frac{1}{2}L\left(\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right),\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)\right)\\ &+\frac{1}{2}L\left(\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right)\end{aligned} \tag{3.11}$$

Proof. Since Φ and Ψ are real-valued and log convex function on $[\hat{\Upsilon}_1,\hat{\Upsilon}_2]$

$$\begin{aligned}&2\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\\ &=2\Phi\left(\frac{\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2}{2}+\frac{\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_1}{2}\right)\\ &\times\Psi\left(\frac{\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2}{2}+\frac{\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_1}{2}\right)\\ &\le\frac{1}{2}\Phi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2\right)+\Phi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\end{aligned}$$

$$\times\Psi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2\right)+\Psi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)$$

$$\leq\frac{1}{2}\left[\Phi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\Psi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\right.$$

$$\left.+\Phi\left(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\Psi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\right]$$

$$+\frac{1}{2}\left[\Phi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\times\Psi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2)\right)\right.$$

$$\left.+\Phi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\times\Psi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\right]$$

$$\leq\frac{1}{2}\left[\Phi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\Psi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\right.$$

$$\left.+\Phi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\Psi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\right]$$

$$+\frac{1}{2}\left[\Phi\left[\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right]^{\overline{\Theta}}\left[\Phi(\hat{\Upsilon}_2)\right]^{1-\overline{\Theta}}\right)\left(\left[\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right]^{\overline{\Theta}}\left[\Psi(\hat{\Upsilon}_1)\right]^{1-\overline{\Theta}}\right)$$

$$+\left(\left[\Phi(\hat{\Upsilon}_1)\right]^{1-\overline{\Theta}}\left[\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right]^{\overline{\Theta}}\right)\left(\left[\Psi(\hat{\Upsilon}_2)\right]^{1-\overline{\Theta}}\left[\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right]^{\overline{\Theta}}\right)$$

$$\leq\frac{1}{2}\left[\Phi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\Psi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\right.$$

$$\left.+\Phi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\Psi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\right]$$

$$+\frac{1}{2}\left[\left(\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)^{\overline{\Theta}}\left(\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)\right)^{1-\overline{\Theta}}\right)$$

$$+\left[\left(\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)^{\overline{\Theta}}\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)\right)^{1-\overline{\Theta}}\right].$$

By integrating both sides of the above-mentioned inequality with respect to $\overline{\Theta}$ and changing variables, we get:

$$2\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)$$

$$\leq\frac{1}{2}\int_0^1\left[\Phi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\Psi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\right.$$

$$
\begin{aligned}
&+\Phi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\Psi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\Bigg]d\overline{\Theta}\\
&+\frac{1}{2}\left[\int_0^1\left(\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)^{\overline{\Theta}}\left(\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)\right)^{1-\overline{\Theta}}d\overline{\Theta}\right.\\
&\left.+\int_0^1\left(\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)^{\overline{\Theta}}\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)\right)^{1-\overline{\Theta}}d\overline{\Theta}\right]\\
&=\frac{1}{2}\left[\frac{2}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx+\frac{2}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}\Phi(x)\Psi(x)dx\right]\\
&+\frac{1}{2}\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)\int_0^1\left(\frac{\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)}\right)^{\overline{\Theta}}d\overline{\Theta}\\
&+\frac{1}{2}\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)\int_0^1\left(\frac{\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)}\right)^{\overline{\Theta}}d\overline{\Theta}\\
&=\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx\\
&+\frac{1}{2}L\left(\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right),\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)\right)\\
&+\frac{1}{2}L\left(\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right).
\end{aligned}
$$

completes the proof.

Example 3.7. Assume that $\Phi,\Psi : I=[\hat{\Upsilon}_1,\hat{\Upsilon}_2]\subset\mathbb{R}\to(0,\infty)$ are real-valued, nonnegative and log-convex functions on closed interval $[\hat{\Upsilon}_1,\hat{\Upsilon}_2]$. Then the following form of Hermite-Hadamard type inequality for product of two log convex functions exists, which is,

$$
\begin{aligned}
&2\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\le\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx\\
&+\frac{1}{2}L\left(\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right),\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)\right)\\
&+\frac{1}{2}L\left(\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right).
\end{aligned}
$$

We want to generalize this inequality for $\Phi(x) = x^{\mu_1}$ and $\Psi(x) = x^{\mu_2}$, $\Phi(x)\Psi(x) = x^{\mu_1+\mu_2}$

Let's consider left hand side of inequality

$$2\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)$$

Since

$$\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right) = \left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)^{\mu_1}, \quad \Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right) = \left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)^{\mu_2}$$

so,

$$2\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right) = 2\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)^{\mu_1+\mu_2}$$

Now consider other side of inequality,

$$\begin{aligned}&\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)dx\\&+\frac{1}{2}L\left(\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right),\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)\right)\\&+\frac{1}{2}L\left(\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right).\end{aligned}$$

Since,

$$\Phi(x)\Psi(x) = x^{(\mu_1+\mu_2)}$$

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\Phi(x)\Psi(x)\,dx = \frac{1}{(\hat{\Upsilon}_2-\hat{\Upsilon}_1)(\mu_1+\mu_2+1)}\left(\hat{\Upsilon}_2^{(\mu_1+\mu_2+1)}-\hat{\Upsilon}_1^{(\mu_1+\mu_2+1)}\right)$$

Also,

$$L\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1),\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right) = \frac{\hat{\Upsilon}_1^{\mu_1+\mu_2}-(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2})^{\mu_1+\mu_2}}{(\mu_1+\mu_2)(ln\hat{\Upsilon}_1-ln(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2})}$$

and

$$L\left(\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right),\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)\right) = \frac{(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2})^{\mu_1+\mu_2}-\hat{\Upsilon}_1^{\mu_1+\mu_2}}{(\mu_1+\mu_2)(ln(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2})-ln\hat{\Upsilon}_2)}.$$

Substituting these terms into the inequality, the above result becomes,

$$\begin{aligned}2\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)^{\mu_1+\mu_2} &\leq \frac{1}{(\hat{\Upsilon}_2-\hat{\Upsilon}_1)(\mu_1+\mu_2+1)}\left(\hat{\Upsilon}_2^{(\mu_1+\mu_2+1)}-\hat{\Upsilon}_1^{(\mu_1+\mu_2+1)}\right)\\&+\left(\frac{1}{2}\frac{\hat{\Upsilon}_1^{\mu_1+\mu_2}-(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2})^{\mu_1+\mu_2}}{(\mu_1+\mu_2)(ln\hat{\Upsilon}_1-ln(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2})}\right)+\frac{1}{2}\left(\frac{(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2})^{\mu_1+\mu_2}-\hat{\Upsilon}_1^{\mu_1+\mu_2}}{(\mu_1+\mu_2)(ln(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2})-ln\hat{\Upsilon}_2)}\right).\end{aligned}$$

, the expression can be calculated for any value of $\hat{\Upsilon}_1, \hat{\Upsilon}_2$ of its domain and for $\overline{\Theta} \in (0,1)$ also for μ_1 and μ_2 of its domain.

3.1. Hermite-Hadamard Inequality For Product of Exponential convex functions. In this section results for Hermite-Hadamard type inequalities for product of exponential convex functions are given.

Definition 3.8. A function $\Phi : I \subset \mathbb{R} \to \mathbb{R}$ gets classified as exponential convex when, it satisfies the relationship

$$\Phi(\overline{\Theta}\hat{\Upsilon}_1 + (1-\overline{\Theta})\hat{\Upsilon}_2) \leq \overline{\Theta}\frac{\Phi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}} + (1-\overline{\Theta})\frac{\Phi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_2}}, \tag{3.12}$$

for all $\hat{\Upsilon}_1, \hat{\Upsilon}_2 \in I$, $\overline{\Theta} \in [0,1]$ and $\alpha \in \mathbb{R}$.

Equation (3.12) maintains its validity for each pair $\hat{\gamma}_1, \hat{\gamma}_2 \in I$ and every real value of α, along with all values $\overline{\Theta}$ in the range $[0,1]$.

Under the condition that Equation (3.12) holds in the reverse direction, Φ qualifies as an *exponentially concave* function.

Furthermore, when $\alpha = 0$, the class of exponential convexity reduces to general convexity.

Counter example is given as,

Example 3.9. The following example demonstrates a function that possess convex properties but fails to meet exponential convex criteria.

$$\Phi(x) = \sqrt{x}$$

on the domain $x \geq 0$.

Thus the function $\Phi(x) = \sqrt{x}$ is convex on $x \geq 0$ because its second derivative

$$\Phi''(x) = -\frac{1}{4x^{3/2}}$$

is non-negative for $x > 0$ and goes to infinity as $x \to 0$.

Thus, $\Phi(x) = \sqrt{x}$ satisfies the convexity criterion on this domain.

For a function to be exponentially convex, $e^{\lambda\Phi(x)}$ must be convex for any real constant λ.

For $\Phi(x) = \sqrt{x}$, we have

$$e^{\lambda\Phi(x)} = e^{\lambda\sqrt{x}}.$$

However, $e^{\lambda\sqrt{x}}$ is not convex for all values of λ on $x \geq 0$.
For certain values of λ, this function has an inflection point, violating the condition for exponential convexity.
Thus, $f(x) = \sqrt{x}$ serves as an example of a function that is convex on its domain but not exponential convex.

4. Examples of Products of Exponential Convex Functions

Example 4.1. Consider two exponential functions $\Phi(x) = e^{\nu_1 x}$ and $\Psi(x) = e^{\nu_2 x}$, where $\nu_1 > 0$ and $\nu_2 > 0$. Their product is:

$$h(x) = \Phi(x) \cdot \Psi(x) = e^{\nu_1 x} \cdot e^{\nu_2 x} = e^{(\nu_1+\nu_2)x}.$$

Since $h(x) = e^{(\nu_1+\nu_2)x}$ is itself an exponential function.

Example 4.2. Let $\Phi(x) = e^{\gamma x^2}$ and $\Psi(x) = e^{\delta x^2}$, where $\gamma > 0$ and $\delta > 0$. Their product is,

$$h(x) = \Phi(x) \cdot \Psi(x) = e^{\gamma x^2} \cdot e^{\delta x^2} = e^{(\gamma+\delta)x^2}.$$

Since $h(x) = e^{(\gamma+\delta)x^2}$, which is exponential function.

Example 4.3. Let $\Phi(x) = x^a e^{\beta x}$ and $\Psi(x) = x^b e^{\gamma x}$, where $a \geq 0$ and $\beta, \gamma > 0$. Their product is,

$$h(x) = \Phi(x) \cdot \Psi(x) = (x^a e^{\beta x}) \cdot (x^b e^{\gamma x}) = x^{a+b} e^{(\beta+\gamma)x}$$

is exponentially convex.

Theorem 4.4. *Assume that $\Phi, \Psi : I \subset \mathbb{R} \to \mathbb{R}$ be two integrable exponentially convex functions on closed interval,$[\hat{\Upsilon}_1, \hat{\Upsilon}_2]$, then*

$$\frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1} \int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} \frac{\Phi(x)\Psi(x)}{e^{2\alpha x}} dx \leq \frac{1}{3} M(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e) + \frac{1}{6} N(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e),$$

where

$$M(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e) = \frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}} + \frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}},$$

and

$$N(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e) = \frac{1}{e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}} \left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2) + \Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)\right).$$

Proof. Exponential convexity defines functions Φ and Ψ as:

$$\Phi(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2)\leq\frac{\overline{\Theta}\Phi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}}+\frac{(1-\overline{\Theta})\Phi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_2}}\tag{4.-1}$$

$$\Psi(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2)\leq\frac{\overline{\Theta}\Psi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}}+\frac{(1-\overline{\Theta})\Psi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_2}}.\tag{4.0}$$

Multiplying equation (4.-1) and equation (4.0 we have,

$$\begin{aligned}
&\Phi(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2)\Psi(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2\\
&\leq\left(\frac{\overline{\Theta}\Phi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}}+\frac{(1-\overline{\Theta})\Phi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_2}}\right)\left(\frac{\overline{\Theta}\Psi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}}+\frac{(1-\overline{\Theta})\Psi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_2}}\right)\\
&\leq\frac{\overline{\Theta}^2\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\frac{\overline{\Theta}(1-\overline{\Theta})\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)}{e^{\alpha}(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}+\frac{\overline{\Theta}(1-\overline{\Theta})\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)}{e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}}+\frac{(1-\overline{\Theta})^2\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}\\
&\leq\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}\overline{\Theta}^2+\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)}{e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}}(\overline{\Theta}-\overline{\Theta}^2)+\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}(1-2\overline{\Theta}+\overline{\Theta}^2).
\end{aligned}$$

Integrating both sides with respect to $\overline{\Theta}$ and $\overline{\Theta}$ varies from 0 to 1, the resulting inequality becomes,

$$\begin{aligned}
&\int_0^1\Phi(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2)\Psi(\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2d\overline{\Theta}\\
&\leq\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}\int_0^1\overline{\Theta}^2+\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)}{e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}}\int_0^1(\overline{\Theta}-\overline{\Theta}^2)d\overline{\Theta}\\
&+\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}\int_0^1(1-2\overline{\Theta}+\overline{\Theta}^2)d\overline{\Theta}.
\end{aligned}$$

By changing variables $x=\overline{\Theta}\hat{\Upsilon}_1+(1-\overline{\Theta})\hat{\Upsilon}_2$ $dx=(\hat{\Upsilon}_1-\hat{\Upsilon}_2)d\overline{\Theta}$ and after adjusting the limits we have,

$$\begin{aligned}
&\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx\\
&\leq\frac{1}{3}\left(\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}\right)+\frac{1}{6e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}}\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)\right),\\
&=\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx\leq\frac{1}{3}M(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)+\frac{1}{6}N(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e),
\end{aligned}$$

this completes the proof.

Example 4.5. Assume that $\Phi,\Psi:I\subset\mathbb{R}\to\mathbb{R}$ be two integrable exponentially convex functions on closed interval,$[\hat{\Upsilon}_1,\hat{\Upsilon}_2]$ such as $\Phi(x)=e^{\nu_1 x}$, $\Psi(x)=e^{\nu_2 x}$ We will generalized the theorem

4.4.

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx \leq \frac{1}{3}M(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)+\frac{1}{6}N(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e),$$

where

$$M(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)=\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}},$$

and

$$N(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)=\frac{1}{e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}}\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)\right).$$

Let's consider

$$\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}=\frac{e^{\nu_1 x}\cdot e^{\nu_2 x}}{e^{2\alpha x}}=e^{(\nu_1+\nu_2)x}\cdot e^{-2\alpha x}=e^{(\nu_1+\nu_2-2\alpha)x}.$$

$$\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx=\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}e^{(\nu_1+\nu_2-2\alpha)x}dx$$

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}e^{(\nu_1+\nu_2-2\alpha)x}dx=\frac{1}{(\hat{\Upsilon}_2-\hat{\Upsilon}_1)}\frac{1}{(\nu_1+\nu_2-2\alpha)}\left(e^{(\nu_1+\nu_2-2\alpha)\hat{\Upsilon}_2}-e^{(\nu_1+\nu_2-2\alpha)\hat{\Upsilon}_1}\right).$$

Now consider,

$$\begin{aligned}M(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)&=\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}\\&=\frac{e^(\nu_1+\nu_2)\hat{\Upsilon}_1}{e^{\alpha\hat{\Upsilon}_1}}+\frac{e^(\nu_1+\nu_2)\hat{\Upsilon}_2}{e^{\alpha\hat{\Upsilon}_2}}\\&=e^{(\nu_1+\nu_2-2\alpha)\hat{\Upsilon}_1}+e^{(\nu_1+\nu_2-2\alpha)\hat{\Upsilon}_2}.\end{aligned}$$

$$\begin{aligned}N(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)&=\frac{1}{e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}}\left[\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Psi(\hat{\Upsilon}_1)\Phi(\hat{\Upsilon}_2)\right]\\&=\frac{1}{e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}}\left[e^{\nu_1\hat{\Upsilon}_1}e^{\nu_2\hat{\Upsilon}_2}+e^{\nu_1\hat{\Upsilon}_2}e^{\nu_2\hat{\Upsilon}_1}\right]\\&=e^{(\nu_1\hat{\Upsilon}_1+\nu_2\hat{\Upsilon}_2-\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}+e^{(\nu_1\hat{\Upsilon}_2+\nu_2\hat{\Upsilon}_1-\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}.\end{aligned}$$

Combining the both inequalities,

$$\begin{aligned}&\frac{1}{(\hat{\Upsilon}_2-\hat{\Upsilon}_1)}\frac{1}{(\nu_1+\nu_2-2\alpha)}\left(e^{(\nu_1+\nu_2-2\alpha)\hat{\Upsilon}_2}-e^{(\nu_1+\nu_2-2\alpha)\hat{\Upsilon}_1}\right)\\&\leq\frac{1}{3}e^{(\nu_1+\nu_2-2\alpha)\hat{\Upsilon}_1}+e^{(\nu_1+\nu_2-2\alpha)\hat{\Upsilon}_2}+\frac{1}{6}\left(e^{(\nu_1\hat{\Upsilon}_1+\nu_2\hat{\Upsilon}_2-\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}+e^{(\nu_1\hat{\Upsilon}_2+\nu_2\hat{\Upsilon}_1-\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}\right),\end{aligned}$$

the expression can be calculated for any value of $\hat{\Upsilon}_1,\hat{\Upsilon}_2$ of its domain and for $\overline{\Theta}\in(0,1)$ also for μ_1 and μ_2 of its domain.

Theorem 4.6. *Let Φ and Ψ be real-valued non-negative and exponentially convex functions on $[\hat{\Upsilon}_1, \hat{\Upsilon}_2]$. Then*

$$\frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1} \int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} \frac{\Phi(x)\Psi(x)}{e^{2\alpha x}} dx \leq \frac{1}{6} M^0(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e) + \frac{1}{3} N^0(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e).$$

Where

$$M^o(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e) = \frac{(\Phi(\hat{\Upsilon}_1))^2 + (\Psi(\hat{\Upsilon}_1))^2}{e^{2\alpha\hat{\Upsilon}_1}} + \frac{(\Phi(\hat{\Upsilon}_2))^2 + (\Psi(\hat{\Upsilon}_2)^2}{e^{2\alpha\hat{\Upsilon}_2}},$$

and

$$N^o(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e) = \frac{1}{e^{\alpha(\hat{\Upsilon}_1 + \hat{\Upsilon}_2)}} \left(\Phi(\hat{\Upsilon}_1)\Phi(\hat{\Upsilon}_2) + \Psi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2) \right).$$

Proof. By definition of exponential convexity, functions Φ and Ψ are defined as:

$$\Phi(\overline{\Theta}\hat{\Upsilon}_1 + (1-\overline{\Theta})\hat{\Upsilon}_2) \leq \frac{(\overline{\Theta})\Phi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}} + \frac{(1-\overline{\Theta})\Phi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_2}}$$
$$\Psi(\overline{\Theta}\hat{\Upsilon}_1 + (1-\overline{\Theta})\hat{\Upsilon}_2) \leq \frac{(\overline{\Theta})\Psi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}} + \frac{(1-\overline{\Theta})\Psi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_2}},$$

for all $\overline{\Theta} \in [0,1]$ it is easy to observe that

$$\frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1} \int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} \Phi(x)\Psi(x)dx = \int_0^1 \Phi(\overline{\Theta}\hat{\Upsilon}_1 + (1-\overline{\Theta})\hat{\Upsilon}_2)\Psi(\overline{\Theta}\hat{\Upsilon}_1 + (1-\overline{\Theta})\hat{\Upsilon}_2)d\overline{\Theta}.$$

Using the elementary inequality $cd \leq \frac{1}{2}[c^2 + d^2]$, $c, d \geq 0$ and after making change of variables, We have

$$\frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1} \int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} \Phi(x)\Psi(x)dx \leq \frac{1}{2} \int_0^1 \Phi(\overline{\Theta}\hat{\Upsilon}_1 + (1-\overline{\Theta})\hat{\Upsilon}_2)^2 + \Psi(\overline{\Theta}\hat{\Upsilon}_1 + (1-\overline{\Theta})\hat{\Upsilon}_2)^2 d\overline{\Theta}$$
$$\leq \frac{1}{2} \int_0^1 \left(\frac{\overline{\Theta}\Phi(\gamma_1)}{e^{\alpha\hat{\Upsilon}_1}} + \frac{(1-\overline{\Theta})\Phi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_2}} \right)^2 + \left(\frac{\overline{\Theta}\Psi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}} + \frac{(1-\overline{\Theta})\Psi(\hat{\Upsilon}_2))}{e^{\alpha\hat{\Upsilon}_2}} \right)^2 d\overline{\Theta}$$
$$\leq \frac{1}{2} \int_0^1 \frac{(\overline{\Theta}\Phi(\hat{\Upsilon}_1))^2}{e^{2\alpha\hat{\Upsilon}_1}} + \frac{\left((1-\overline{\Theta})\Phi(\hat{\Upsilon}_2)\right)^2}{e^{2\alpha\hat{\Upsilon}_2}} + 2\overline{\Theta}(1-\overline{\Theta}) \frac{\Phi(\hat{\Upsilon}_1)\Phi(\hat{\Upsilon}_2)}{e^{\alpha(\hat{\Upsilon}_1 + \hat{\Upsilon}_2)}} d\overline{\Theta}$$

(4.-1)

$$+ \frac{1}{2} \int_0^1 \frac{(\overline{\Theta}\Psi(\gamma_1))^2}{e^{2\alpha\hat{\Upsilon}_1}} + \frac{\left(1-\overline{\Theta})\Psi(\hat{\Upsilon}_2)\right)^2}{e^{2\alpha\hat{\Upsilon}_2}} + 2\overline{\Theta}(1-\overline{\Theta}) \frac{\Psi(\hat{\Upsilon}_1)\Phi(\hat{\Upsilon}_2)}{e^{\alpha(\hat{\Upsilon}_1 + \hat{\Upsilon}_2)}} d\overline{\Theta}.$$

Integrating and after simplification, we have

$$\frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1} \int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} \frac{\Phi(x)\Psi(x)}{e^{2\alpha x}} dx \leq \frac{1}{6} M^0(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e) + \frac{1}{3} N^0(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e).$$

Theorem 4.7. *Assume that $\Phi,\Psi : I \subset \mathbb{R} \to \mathbb{R}$ be two integrable exponentially convex functions on closed interval,$[\hat{\Upsilon}_1,\hat{\Upsilon}_2]$, then*

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx \le L(\overline{\Theta}) \le \frac{1}{3}\left[\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}\left(\overline{\Theta}+\frac{(1-\overline{\Theta})^2}{e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\right)\right.$$
$$+\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}\left((1-\overline{\Theta})+\frac{(\overline{\Theta})^2}{e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\right)\left.+N(\hat{\Upsilon}_1,\hat{\gamma_2;e})\left(\frac{\overline{\Theta}(1-\overline{\Theta})}{e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\right)\right]$$
$$+\frac{1}{e^{\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\left[\frac{1}{3}\overline{\Theta}(1-\overline{\Theta})M(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)+\frac{1}{6}N(\hat{\gamma}_1,\hat{\gamma}_2;e)(\overline{\Theta}^2+(1-\overline{\Theta})^2)\right].$$

Where

$$L(\overline{\Theta})=\frac{\overline{\Theta}}{3}M(\hat{\Upsilon}_1,(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2;e)+\frac{1-\overline{\Theta}}{3}M((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2,\hat{\Upsilon}_2;e)$$
$$+\frac{\overline{\Theta}}{6}N(\hat{\Upsilon}_1,(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2;e)+\frac{1-\overline{\Theta}}{6}N((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2,\hat{\Upsilon}_2;e).$$

Also

$$M(\hat{\Upsilon}_1,\hat{\Upsilon}_1;e)=\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}},$$

and

$$N(\gamma_1,\gamma_2;e)=\frac{1}{e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}}\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Psi(\hat{\Upsilon}_1)\Phi(\hat{\Upsilon}_2)\right).$$

Proof.

Since Φ and Ψ are exponential convex and non-negative on $[\hat{\Upsilon}_1,\hat{\Upsilon}_2]$, So by theorem (4.4) we have,

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx \le \frac{1}{3}M(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)+\frac{1}{6}N(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e). \tag{4.0}$$

Firstly Φ and Ψ are exponentially convex and non-negative on $[\hat{\Upsilon}_1,(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2]$ and on $[(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2,\hat{\Upsilon}_1]$, we get

(4.1)
$$\frac{1}{\overline{\Theta}(\hat{\Upsilon}_2-\hat{\Upsilon}_1)}\int_{\hat{\Upsilon}_1}^{(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx \le \frac{1}{3}M(\hat{\Upsilon}_1,(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2;e)+\frac{1}{6}N(\hat{\Upsilon}_1,(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2;e).$$

Similarly we can show that,

(4.2)
$$\frac{1}{(1-\overline{\Theta})(\hat{\Upsilon}_2-\hat{\Upsilon}_1)}\int_{(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}^{\hat{\Upsilon}_2}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx \le \frac{1}{3}M((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2,\hat{\Upsilon}_2;e)+\frac{1}{6}N((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2,\hat{\Upsilon}_2;e).$$

Multiplying (4.1) by $\overline{\Theta}$ and (4.2) by $(1-\overline{\Theta})$ and adding the resulting inequalities we get,

$$\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx \le \frac{\overline{\Theta}}{3}M\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2;e\right)+\frac{\overline{\Theta}}{6}N\left(\hat{\Upsilon}_1,(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2;e\right)$$

(4.3)

$$+\frac{1-\overline{\Theta}}{3}M\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2,\hat{\Upsilon}_2;e\right)+\frac{1-\overline{\Theta}}{6}N\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2,\hat{\Upsilon}_2;e\right)=L(\overline{\Theta}).$$

Using the fact that,

$$\begin{aligned}
&\frac{\overline{\Theta}}{3}M(\hat{\Upsilon}_1,(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2;e)+\frac{1-\overline{\Theta}}{3}M((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2,\hat{\Upsilon}_2;e)\\
&=\frac{\overline{\Theta}}{3}\left[\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\frac{\Phi((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)}{e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\right]\\
&+\frac{1-\overline{\Theta}}{3}\left[\frac{(\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)\Psi((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)}{e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}},\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}\right]\\
&=\frac{\overline{\Theta}}{3}\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\frac{1-\overline{\Theta}}{3}\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\frac{1}{3}\left[\frac{\Phi(\overline{\Theta}\hat{\Upsilon}_2+(1-\overline{\Theta})\hat{\Upsilon}_1)\Psi(\overline{\Theta}\hat{\Upsilon}_2+(1-\overline{\Theta})\hat{\Upsilon}_1}{e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\right]\\
&\le\frac{\overline{\Theta}}{3}\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\frac{1-\overline{\Theta}}{3}\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\frac{1}{e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\\
&\times\left[\frac{\overline{\Theta}\Phi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_2}}+\frac{(1-\overline{\Theta})\Phi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}})(\frac{\overline{\Theta}\Psi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_2}}+\frac{(1-\overline{\Theta})\Psi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\gamma}_1}}\right]\\
&\le\frac{\overline{\Theta}}{3}\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\frac{1-\overline{\Theta}}{3}\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}+\frac{1}{3e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\\
&\times\left[\frac{(1-\overline{\Theta})^2\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\overline{\Theta}(1-\overline{\Theta})\left(\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Psi(\hat{\Upsilon}_1)\Phi(\hat{\Upsilon}_2)}{e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}}\right)+\frac{\overline{\Theta}^2\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}\right]\\
&\le\frac{\overline{\Theta}}{3}\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\gamma_1}}+\frac{1-\overline{\Theta}}{3}\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\gamma_2}}+\frac{1}{3e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\\
&\times\left[\frac{(1-\overline{\Theta})^2\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\overline{\Theta}(1-\overline{\Theta})N(\gamma_1,\gamma_2,e))+\frac{\overline{\Theta}^2\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}\right]\\
&=\frac{1}{3}\left[\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}\left(\overline{\Theta}+\frac{(1-\overline{\Theta})^2}{e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\right)+\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}\left((1-\overline{\Theta})+\frac{\overline{\Theta}^2}{e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\right)\right.\\
&\left.+N(\hat{\Upsilon}_1,\hat{\Upsilon}_2,e)\left(\frac{\overline{\Theta}(1-\overline{\Theta})}{e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\right)\right].
\end{aligned}$$

Now consider,

$$\frac{\overline{\Theta}}{6}N(\hat{\Upsilon}_1,(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2;e)+\frac{1-\overline{\Theta}}{6}N((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2,\hat{\Upsilon}_2;e)$$

$$= \frac{\overline{\Theta}}{6}\left[\frac{\Phi(\hat{\Upsilon}_1)(\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_1}e^{\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}+\frac{\Psi(\hat{\Upsilon}_1)\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_1}e^{\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\right]$$
$$+\frac{1-\overline{\Theta}}{6}\left[\frac{\Phi(\hat{\Upsilon}_1)(\Psi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_1}e^{\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}+\frac{\Psi(\hat{\Upsilon}_1)\Phi(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_1}e^{\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\right]$$
$$\leq \frac{\overline{\Theta}}{6e^{\alpha\hat{\Upsilon}_1}e^{\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\left[2(1-\overline{\Theta})\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}}+\frac{\overline{\Theta}}{e^{\alpha\hat{\Upsilon}_2}}\left(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Psi(\hat{\Upsilon}_1)\Phi(\hat{\Upsilon}_2)\right)\right]$$
$$+\frac{1-\overline{\Theta}}{6e^{\alpha\hat{\Upsilon}_2}e^{\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\left[2\overline{\Theta}\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_2}}+(1-\overline{\Theta})\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}}\right]$$
$$\leq \frac{\overline{\Theta}}{6e^{\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\left[\frac{2(1-\overline{\Theta})\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\overline{\Theta}\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Psi(\hat{\Upsilon}_1)\Phi(\hat{\Upsilon}_2)}{e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2}}\right]$$
$$+\frac{1-\overline{\Theta}}{6e^{\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\left[2\overline{\Theta}\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}+(1-\overline{\Theta})\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)}{e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}}\right]$$
$$\leq \frac{\overline{\Theta}}{6e^{\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\left[\frac{2(1-\overline{\Theta})\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\overline{\Theta}N(\hat{\Upsilon}_1,\hat{\Upsilon}_2,e)\right]$$
$$+\frac{1-\overline{\Theta}}{6}\frac{1}{e^{\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\left[2\overline{\Theta}\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}+(1-\overline{\Theta})N(\hat{\Upsilon}_1,\hat{\Upsilon}_2,e)\right]$$
$$+\frac{1}{e^{\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\left[\frac{1}{3}\overline{\Theta}(1-\overline{\Theta})M(\hat{\Upsilon}_1,\hat{\Upsilon}_2,e)+\frac{1}{6}N(\hat{\Upsilon}_1,\hat{\Upsilon}_2,e)(\overline{\Theta}^2+(1-\overline{\Theta})^2)\right].$$

Hence, we have after combining above both inequalities,

$$L(\overline{\Theta}) \leq \frac{1}{3}\left[\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}\left(\overline{\Theta}+\frac{(1-\overline{\Theta})^2}{e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\right)\right.$$
$$\left.+\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}\left((1-\overline{\Theta})+\frac{(\overline{\Theta})^2}{e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\right)+N(\hat{\Upsilon}_1,\gamma_2,e)\left(\frac{\overline{\Theta}(1-\overline{\Theta})}{e^{2\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\right)\right].$$
$$+\frac{1}{e^{\alpha(1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\hat{\Upsilon}_2}}\left[\frac{1}{3}\overline{\Theta}(1-\overline{\Theta})M(\hat{\Upsilon}_1,\hat{\Upsilon}_2,e)+\frac{1}{6}N(\gamma_1,\gamma_2,e)(\overline{\Theta}^2+(1-\overline{\Theta})^2)\right],$$

which completes the proof.

Remark 4.8. In the theorem (4.7) for $\overline{\Theta}=0$ and $\overline{\Theta}=1$ we have

$$L(\overline{\Theta}) = L(0) = L(1) = \frac{1}{3}M(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)+\frac{1}{6}N(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e).$$

we get the result in theorem (4.4). So we conclude that our results are the improvement of theorem (4.4).

Remark 4.9. In theorem (4.7) when $\alpha = 0$

our result coincides with general convexity, that is

$$\frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} \Phi(x)\Psi(x)dx \le L(\overline{\Theta}) \le \frac{1}{3}M(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e) + \frac{1}{6}N(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e).$$

Corollary 4.10. *With notations above one has the following inequality:*

$$\frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} \frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx \le \inf_{0\le\xi\le1} L(\overline{\Theta}) \le \sup_{0\le\xi\le1} \le \frac{1}{3}M(\hat{\Upsilon}_1, \hat{\Upsilon}_2) + \frac{1}{6}N(\hat{\Upsilon}_1, \hat{\Upsilon}_2).$$

Example 4.11. Assume that $\Phi, \Psi : I \subset \mathbb{R} \to \mathbb{R}$ are two integrable exponentially convex functions on the closed interval $[\hat{\Upsilon}_1, \hat{\Upsilon}_2]$. Let

$$\Phi(x) = e^{\nu_1 x}, \qquad \Psi(x) = e^{\nu_2 x}.$$

Then Theorem (4.7) reduces to

$$\begin{aligned}
\frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} \frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx \le L(\overline{\Theta})& \\
&\le \frac{1}{3}\Bigg[\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}\left(\overline{\Theta} + \frac{(1-\overline{\Theta})^2}{e^{2\alpha((1-\overline{\Theta})\hat{\Upsilon}_1 + \overline{\Theta}\hat{\Upsilon}_2)}}\right) \\
&\quad + \frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}}\left((1-\overline{\Theta}) + \frac{\overline{\Theta}^2}{e^{2\alpha((1-\overline{\Theta})\hat{\Upsilon}_1 + \overline{\Theta}\hat{\Upsilon}_2)}}\right) \\
&\quad + N(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e)\frac{\overline{\Theta}(1-\overline{\Theta})}{e^{2\alpha((1-\overline{\Theta})\hat{\Upsilon}_1 + \overline{\Theta}\hat{\Upsilon}_2)}}\Bigg] \\
&\quad + \frac{1}{e^{\alpha((1-\overline{\Theta})\hat{\Upsilon}_1 + \overline{\Theta}\hat{\Upsilon}_2)}}\Bigg[\frac{1}{3}\overline{\Theta}(1-\overline{\Theta})M(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e) \\
&\qquad + \frac{1}{6}N(\hat{\Upsilon}_1, \hat{\Upsilon}_2; e)(\overline{\Theta}^2 + (1-\overline{\Theta})^2)\Bigg].
\end{aligned} \tag{4.4}$$

where

$$\begin{aligned}
L(\overline{\Theta}) = &\frac{\overline{\Theta}}{3}M\big(\hat{\Upsilon}_1, (1-\overline{\Theta})\hat{\Upsilon}_1 + \overline{\Theta}\hat{\Upsilon}_2; e\big) \\
&+ \frac{1-\overline{\Theta}}{3}M\big((1-\overline{\Theta})\hat{\Upsilon}_1 + \overline{\Theta}\hat{\Upsilon}_2, \hat{\Upsilon}_2; e\big) \\
&+ \frac{\overline{\Theta}}{6}N\big(\hat{\Upsilon}_1, (1-\overline{\Theta})\hat{\Upsilon}_1 + \overline{\Theta}\hat{\Upsilon}_2; e\big) \\
&+ \frac{1-\overline{\Theta}}{6}N\big((1-\overline{\Theta})\hat{\Upsilon}_1 + \overline{\Theta}\hat{\Upsilon}_2, \hat{\Upsilon}_2; e\big).
\end{aligned}$$

Also,

$$M(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)=\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_1)}{e^{2\alpha\hat{\Upsilon}_1}}+\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_2)}{e^{2\alpha\hat{\Upsilon}_2}},$$

and

$$N(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)=\frac{1}{e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}}\Big(\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)\Big).$$

Since $\Phi(x)=e^{\nu_1 x}$ and $\Psi(x)=e^{\nu_2 x}$, we obtain

$$\begin{aligned}\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx&=\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}e^{(\nu_1+\nu_2-2\alpha)x}dx\\&=\frac{e^{(\nu_1+\nu_2-2\alpha)\hat{\Upsilon}_2}-e^{(\nu_1+\nu_2-2\alpha)\hat{\Upsilon}_1}}{(\hat{\Upsilon}_2-\hat{\Upsilon}_1)(\nu_1+\nu_2-2\alpha)}.\end{aligned}$$

Further,

$$M(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)=e^{(\nu_1+\nu_2-2\alpha)\hat{\Upsilon}_1}+e^{(\nu_1+\nu_2-2\alpha)\hat{\Upsilon}_2},$$

and

$$N(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)=e^{\nu_1\hat{\Upsilon}_1+\nu_2\hat{\Upsilon}_2-\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}+e^{\nu_1\hat{\Upsilon}_2+\nu_2\hat{\Upsilon}_1-\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}.$$

Substituting these into (4.4), we obtain the required result.

Theorem 4.12. *Assume that $\Phi,\Psi:I\subset\mathbb{R}\to\mathbb{R}$ be two integrable exponentially convex functions on closed interval$[\hat{\Upsilon}_1,\hat{\Upsilon}_2]$, then*

$$2\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)$$

$$\leq\frac{1}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx$$

$$+\frac{1}{12}N(\hat{\Upsilon}_1,\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2},e)+\frac{1}{6}N\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2},\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2},e\right)+\frac{1}{6}N\left(\hat{\Upsilon}_1,\hat{\Upsilon}_2,e\right)+\frac{1}{12}N\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2},\hat{\Upsilon}_2,e\right).$$

Proof. Since Φ and Ψ are integrable exponentially convex on $[\hat{\Upsilon}_1,\hat{\Upsilon}_2]$ and non-negative for $\overline{\Theta}\in[0,1]$ we have,

$$\begin{aligned}&2\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\\&=2\Phi\Bigg[\left(\frac{\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta}\hat{\Upsilon}_2}{2})+\frac{\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_1}{2}\right)\end{aligned}$$

$$\times\Psi\left(\frac{\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2}{2}\right)+\left(\frac{\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_1}{2}\right)\Bigg]$$

$$\leq\frac{1}{2}\left[\Phi\left(\overline{\Theta}\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}+(1-\overline{\Theta})\hat{\Upsilon}_2\right)+\Phi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right]$$
$$\times\left[\Psi\left(\overline{\Theta}\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}+(1-\overline{\Theta})\hat{\Upsilon}_2\right)+\Psi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right]$$

$$\leq\frac{1}{2}\Phi\left(\overline{\Theta}\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\Psi\left(\overline{\Theta}\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}+(1-\overline{\Theta})\hat{\Upsilon}_2\right)$$
$$+\frac{1}{2}\Phi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\Psi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)$$
$$+\frac{1}{2}\left(\overline{\Theta}\left(\frac{\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}{e^{\alpha\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}}\right)+(1-\overline{\Theta})\frac{\Phi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_2}}\right)\times\left((1-\overline{\Theta})\frac{\Psi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}}\right)+\overline{\Theta}\left(\frac{\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}{e^{\alpha\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}}\right)$$
$$+\frac{1}{2}\left((1-\overline{\Theta})\frac{\Phi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}}\right)+\overline{\Theta}\left(\frac{\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}{e^{\alpha\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}}\right)\times\left(\overline{\Theta}\left(\frac{\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}{e^{\alpha\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}}\right)+(1-\overline{\Theta})\frac{\Psi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_2}}\right)$$

$$\leq\frac{1}{2}\Phi\left(\overline{\Theta}\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\Psi\left(\overline{\Theta}\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}+(1-\overline{\Theta})\hat{\Upsilon}_2\right)$$
$$+\frac{1}{2}\Phi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\Psi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)$$
$$+\frac{1}{2}[\overline{\Theta}(1-\overline{\Theta})\frac{\Phi(\hat{\Upsilon}_1)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}{e^{\alpha\hat{\Upsilon}_1}\cdot e^{\alpha\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}}+(1-\overline{\Theta})^2\frac{\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)}{e^{\alpha\hat{\Upsilon}_1}\cdot e^{\alpha\hat{\Upsilon}_2}}+\overline{\Theta}^2\frac{\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}{e^{\alpha\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}\cdot e^{\alpha\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}}$$
$$+\overline{\Theta}(1-\overline{\Theta})\frac{\Psi(\hat{\Upsilon}_2)\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}{e^{\alpha\hat{\Upsilon}_2}\cdot e^{\alpha\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}}+\overline{\Theta}(1-\overline{\Theta})\frac{\Psi(\hat{\Upsilon}_1)\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}{e^{\alpha\hat{\Upsilon}_1}\cdot e^{\alpha\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}}+\overline{\Theta}(1-\overline{\Theta})\frac{\Phi(\hat{\Upsilon}_2)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}{e^{\alpha\hat{\Upsilon}_2}\cdot e^{\alpha\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}}$$
$$+(1-\overline{\Theta})^2\frac{\Phi(\hat{\Upsilon}_2)\Psi(\hat{\Upsilon}_1)}{e^{\alpha\hat{\Upsilon}_1}\cdot e^{\alpha\hat{\Upsilon}_2}}+\overline{\Theta}^2\frac{\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}{e^{\alpha\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}\cdot e^{\alpha\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}}\Bigg].$$

Integrating with respect to $\overline{\Theta}$ and ξ varies from 0 to 1 we have,

$$\leq\frac{1}{2}\left[\int_0^1\Phi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\Psi\left(\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+(1-\overline{\Theta})\hat{\Upsilon}_2\right)\right.$$
$$\left.+\int_0^1\Phi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\Psi\left((1-\overline{\Theta})\hat{\Upsilon}_1+\overline{\Theta}\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right)\right]$$
$$+\frac{1}{2}\left[\frac{1}{6}N\left(\hat{\Upsilon}_1,\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2};e\right)+\frac{1}{3}N\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2},\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2};e\right)\right.$$

$$+\frac{1}{3}N\left(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e\right)+\frac{1}{6}N\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2},\hat{\Upsilon}_2;e\right)\Bigg].$$

By using change of variables and changing limits, we obtain

$$=\frac{1}{2}\left[\frac{2}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}^{\hat{\Upsilon}_2}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx+\frac{2}{\hat{\Upsilon}_2-\hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx\right]$$
$$+\frac{1}{2}\left[\frac{1}{6}N\left(\hat{\Upsilon}_1,\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2};e\right)+\frac{1}{3}N\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2},\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2};e\right)+\frac{1}{3}N(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)+\frac{1}{6}N\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_1}{2},\hat{\Upsilon}_2;e\right)\right]$$
$$=\frac{1}{\hat{\Upsilon}_1-\hat{\Upsilon}_2}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}\frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx$$
$$+\frac{1}{2}\left[\frac{1}{6}N\left(\hat{\Upsilon}_1,\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2};e\right)+\frac{1}{3}N\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2},\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2};e\right)+\frac{1}{3}N(\hat{\Upsilon}_1,\hat{\Upsilon}_2;e)+\frac{1}{6}N\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2},\hat{\Upsilon}_2;e\right)\right].$$

From above result, we have

$$N\left(\hat{\Upsilon}_1,\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2};e\right)$$
$$=\frac{1}{e^{\alpha\hat{\Upsilon}_1}e^{\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}}\left[\Phi(\hat{\Upsilon}_1)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+\Psi(\hat{\Upsilon}_1)\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right]$$
$$\leq\frac{1}{2}\frac{1}{e^{\alpha\hat{\Upsilon}_1}e^{\alpha\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}}}\left[\Phi(\hat{\Upsilon}_1)\left(\Psi(\hat{\Upsilon}_1)+\Psi(\hat{\Upsilon}_2)\right)+\Psi(\hat{\Upsilon}_1)\left(\Phi(\hat{\Upsilon}_1)+\Phi(\hat{\Upsilon}_2)\right)\right]$$

On the same way we have,

$$N\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2},\hat{\Upsilon}_2;e\right)$$
$$=\frac{1}{e^{\alpha\hat{\Upsilon}_1}e^{\alpha\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}}\left[\Phi(\hat{\Upsilon}_2)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+\Psi(\hat{\Upsilon}_2)\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right].$$

And,

$$N(\hat{\Upsilon}_1,\hat{\Upsilon}_2,e)$$
$$=\frac{1}{e^{\alpha(\hat{\Upsilon}_1+\hat{\Upsilon}_2)}}\left[\Phi(\hat{\Upsilon}_1)\Psi(\hat{\Upsilon}_2)+\Psi(\hat{\Upsilon}_1)+\Phi(\hat{\Upsilon}_2)\right].$$

Also,

$$N\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2},\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2};e\right)$$
$$=\frac{1}{e^{\alpha\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)}}\left[\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)+\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right]$$
$$=\frac{2}{e^{2\alpha(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2})}}\left[\Phi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1+\hat{\Upsilon}_2}{2}\right)\right].$$

Example 4.13. Taking into account the assumptions of theorem (4.12). Let Φ and Ψ be real-valued, non-negative and integrable exponentially convex functions on $[\hat{\Upsilon}_1, \hat{\Upsilon}_2]$, also $\overline{\Theta} \in [0,1]$ and $\nu_1, \nu_2 \geq 0$. we generalize the theorem (4.12) by using functions as $\Phi(x) = e^{\nu_1 x}$ and $\Psi(x) = e^{\nu_2 x}$.

Consider one side of the inequality

$$2\Phi\left(\frac{\hat{\Upsilon}_1 + \hat{\Upsilon}_2}{2}\right)\Psi\left(\frac{\hat{\Upsilon}_1 + \hat{\Upsilon}_2}{2}\right)$$

Using $\Phi(x) = e^{\nu_1 x}$ and $\Psi(x) = e^{\nu_2 x}$, we have

$$2e^{\nu_1 \frac{\hat{\Upsilon}_1 + \hat{\Upsilon}_2}{2}} \cdot e^{\nu_2 \frac{\hat{\Upsilon}_1 + \hat{\Upsilon}_2}{2}}$$

$$= 2e^{(\nu_1 + \nu_2)\frac{\hat{\Upsilon}_1 + \hat{\Upsilon}_2}{2}},$$

Now consider

$$\frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} \frac{\Phi(x)\Psi(x)}{e^{2\alpha x}}dx$$

Using $\Phi(x) = e^{\nu_1 x}$ and $\Psi(x) = e^{\nu_2 x}$

$$\frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} \frac{e^{\nu_1 x}e^{\nu_2 x}}{e^{2\alpha x}}dx$$

$$= \frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1}\int_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2} e^{(\nu_1 + \nu_2 - 2\alpha)x}dx$$

$$= \frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1}\frac{e^{(\nu_1 + \nu_2 - 2\alpha)x}}{\nu_1 + \nu_2 - 2\alpha}\Bigg|_{\hat{\Upsilon}_1}^{\hat{\Upsilon}_2}$$

$$= \frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1}\frac{e^{(\nu_1 + \nu_2 - 2\alpha)\hat{\Upsilon}_2} - e^{(\nu_1 + \nu_2 - 2\alpha)\hat{\Upsilon}_1}}{\nu_1 + \nu_2 - 2\alpha}.$$

Hence our inequality becomes,

$$2e^{(\nu_1 + \nu_2)\frac{\hat{\Upsilon}_1 + \hat{\Upsilon}_2}{2}} \leq \frac{1}{\hat{\Upsilon}_2 - \hat{\Upsilon}_1}\frac{e^{(\nu_1 + \nu_2 - 2\alpha)\hat{\Upsilon}_2} - e^{(\nu_1 + \nu_2 - 2\alpha)\hat{\Upsilon}_1}}{\nu_1 + \nu_2 - 2\alpha}$$

$$+\frac{1}{12}N\left(\hat{\Upsilon}_1, \frac{\hat{\Upsilon}_1 + \hat{\Upsilon}_2}{2}; e\right) + \frac{1}{6}N\left(\frac{\hat{\Upsilon}_1 + \hat{\Upsilon}_2}{2}, \frac{\hat{\Upsilon}_1 + \hat{\Upsilon}_2}{2}; e\right) + \frac{1}{6}N(\hat{\Upsilon}_1, \hat{\Upsilon}_2, e) + \frac{1}{12}N\left(\frac{\hat{\Upsilon}_1 + \hat{\Upsilon}_2}{2}, \hat{\Upsilon}_2; e\right),$$

the expression can be calculated for any value of $\hat{\Upsilon}_1, \hat{\Upsilon}_2$ of its domain and for $\overline{\Theta} \in (0,1)$ also for μ_1 and μ_2 of its domain.

5. Conclusion and Future Directions

The study of inequalities using convex functions is known as the theory of convex inequalities. In this paper we find results of Hermite Hadamrd type inequalities by using product of log convex functions and product of exponential convex functions and produce examples of each result to improve the authenticity of results. Additionally, we give remarks to show the accuracy of our newly developed results.

In the future, it will be fascinating for readers to take motivation from these results and employ this approach with various forms of generalized convex mappings, including harmonic convex, p-convex, tgs-convex, and quasi-convex, as well as different types of fractional, quantum, stochastic, and fuzzy integral operators.

Authors Contributions:

All authors have reviewed and finalized the manuscript, making equal contributions.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] C. Liu, C.R. Traas, On convexity of planar curves and its application in CAGD, Comput. Aided Geom. Des. 14 (1997), 653–669.

[2] E. Tokgoz, H. Musafer, M. Faezipour, A. Mahmood, Incorporating derivative-free convexity with trigonometric simplex designs for learning-rate estimation of stochastic gradient-descent method, Electronics 12 (2023), 419.

[3] M.D. Fajardo, S.M. Grad, J. Vidal, New duality results for evenly convex optimization problems, Optimization 70 (2021), 1837–1858.

[4] Ç. Yildiz, J.E.N. Valdés, L.I. Cotîrlă, A note on the new Ostrowski and Hadamard type inequalities via the Hölder–İşcan inequality, Axioms 12 (2023), 931.

[5] H. Ullah, M.A. Adil Khan, T. Saeed, Z.M.M.M. Sayed, Some improvements of Jensen's inequality via 4-convexity and applications, J. Funct. Spaces 2022 (2022), 2157375.

[6] L. Chancel, T. Piketty, E. Saez, G. Zucman (eds.), World inequality report 2022, Harvard Univ. Press, 2022.

[7] U.S. Kırmacı, On generalizations of Hölder's and Minkowski's inequalities, Math. Sci. Appl. E-Notes 11 (2023), 213–225.

[8] M. Tariq, A.A. Shaikh, S.K. Ntouyas, Some new fractional Hadamard and Pachpatte-type inequalities with applications via generalized preinvexity, Symmetry 15 (2023), 1033.

[9] S.I. Ihsan Butt, H. Budak, K. Nonlaopon, New quantum Mercer estimates of Simpson–Newton-like inequalities via convexity, Symmetry 14 (2022), 1935.

[10] S.I. Butt, I. Javed, P. Agarwal, J.J. Nieto, Newton–Simpson-type inequalities via majorization, J. Inequal. Appl. 2023 (2023), 16.

[11] M. Vivas–Cortez, M.Z. Javed, M.U. Awan, M.A. Noor, S.S. Dragomir, Bullen-Mercer type inequalities with applications in numerical analysis, Alex. Eng. J. 96 (2024), 15–33.

[12] T. Gagné, A. Nandi, I. Schoon, Time trend analysis of social inequalities in psychological distress among young adults before and during the pandemic: evidence from the UK Household Longitudinal Study COVID-19 waves, J. Epidemiol. Community Health 76 (2022), 421–427.

[13] V. Stojiljković, N. Mirkov, S. Radenović, Variations in the tensorial trapezoid type inequalities for convex functions of self-adjoint operators in Hilbert spaces, Symmetry 16 (2024), 121.

[14] A. Guessab, G. Schmeisser, Sharp integral inequalities of the Hermite–Hadamard type, J. Approx. Theory 115 (2002), 260–288.

[15] S.S. Dragomir, On some new inequalities of Hermite-Hadamard type for m-convex functions, Tamkang J. Math. 33 (2002), 45–56.

[16] J. de la Cal, J. Cárcamo, Multidimensional Hermite–Hadamard inequalities and the convex order, J. Math. Anal. Appl. 324 (2006), 248–261.

[17] J. Wang, C. Zhu, Y. Zhou, New generalized Hermite-Hadamard type inequalities and applications to special means, J. Inequal. Appl. 2013 (2013), 1–15.

[18] M.E. Özdemir, Ç. Yıldız, A.O. Akdemir, E. Set, On some inequalities for s-convex functions and applications, J. Inequal. Appl. 2013 (2013), 1–11.

[19] S. Sezer, Z. Eken, G. Tınaztepe, G. Adilov, p-convex functions and some of their properties, Numer. Funct. Anal. Optim. 42 (2021), 443–459.

[20] M.Z. Sarikaya, A. Saglam, H. Yildirim, On some Hadamard-type inequalities for h-convex functions, J. Math. Inequal. 2 (2008), 335–341.

[21] S.S. Dragomir, New inequalities of Hermite-Hadamard type for log-convex functions, Khayyam J. Math. 3 (2017), 98–115.

[22] M.U. Awan, M.A. Noor, K.I. Noor, Hermite-Hadamard inequalities for exponentially convex functions, Appl. Math. Inf. Sci. 12 (2018), 405–409.

[23] M.E. Ozdemir, C. Yildiz, A.O. Akdemir, On the coordinated convex functions, arXiv:1107.4033.

[24] O. Almutairi, A. Kılıçman, A review of Hermite–Hadamard inequality for alpha-type real-valued convex functions, Symmetry 14 (2022), 840.

[25] B.Y. Xi, R.F. Bai, F. Qi, Hermite-Hadamard type inequalities for the m- and (alpha, m)-geometrically convex functions, Aequationes Math. 84 (2012), 261–269.

[26] H. Kadakal, (alpha, m1, m2)-convexity and some inequalities of Hermite-Hadamard type, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 68 (2019), 2128–2142.

[27] Y. Qi, Q. Wen, G. Li, K. Xiao, S. Wang, Discrete Hermite–Hadamard-type inequalities for (s, m)-convex function, Fractals 30 (2022), 2250160.

[28] H. Kalsoom, M.A. Latif, M. Idrees, M. Arif, Z. Salleh, Quantum Hermite-Hadamard type inequalities for generalized strongly preinvex functions, AIMS Math. 6 (2021), 13291–13310.

[29] B.G. Pachpatte, A note on integral inequalities involving two log-convex functions, Math. Inequal. Appl. 7 (2004), 511–516.

[30] F. Chen, A note on Hermite-Hadamard inequalities for products of convex functions, J. Appl. Math. 2013 (2013).

[31] S.S. Dragomir, B. Mond, Integral inequalities of Hadamard type for log-convex functions, Demonstr. Math. 31 (1998), 355–364.