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STRONG CONVERGENCE OF AN ENRICHED MULTI-STEP ITERATIVE SCHEME FOR COMMON FIXED POINTS OF ENRICHED NEARLY ASYMPTOTICALLY PSEUDOCONTRACTIVE MAPPINGS

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Abstract. We propose and analyze an enriched multi-step iterative scheme for approximating common fixed points of finite families of enriched nearly uniformly L -Lipschitzian asymptotically pseudocontractive mappings in Banach spaces. Under mild assumptions on the parameter sequences, we prove strong convergence theorems that extend classical results of Agarwal et al. and Yao and Chen to the enriched multi-operator setting. The proof techniques rely on normalized duality mappings, demiclosedness principles, and convergence lemmas. Crucially, our unified framework accommodates mappings that need not be continuous, thereby addressing a significant gap in the existing literature and broadening the applicability of fixed point theory to more general nonlinear problems.

Keywords: fixed point theory; enriched nearly asymptotically nonexpansive mappings; enriched asymptotically pseudocontractive mappings; common fixed points; multi-step iteration; Banach spaces.

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1. INTRODUCTION

Fixed point theory has long stood as one of the most powerful and versatile tools in modern nonlinear analysis, offering essential techniques for tackling nonlinear equations, variational

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inequalities, and optimization problems that emerge across a broad spectrum of scientific and engineering disciplines [3]. At its heart, the theory is concerned with the existence and approximation of points x satisfying $Tx = x$ for nonlinear operators T defined on Banach spaces. The classical framework centers on nonexpansive operators, which satisfy the Lipschitz condition $\|Tx - Ty\| \leq \|x - y\|$. From this foundation grew fundamental iterative methods such as Picard, Mann [13], and Ishikawa iterations [10], which continue to underpin much of modern iterative approximation theory.

However, the restrictive nature of the nonexpansiveness condition prompted researchers to seek broader classes of operators with more flexible structural properties. A landmark contribution in this direction came from Goebel and Kirk [8], who introduced the class of *asymptotically nonexpansive mappings*. These operators satisfy $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for a sequence $\{k_n\}$ with $k_n \rightarrow 1$, effectively bridging the gap between nonexpansive and general Lipschitzian operators. This seminal work opened fresh avenues in fixed point theory and has since been extensively developed by numerous authors, including Schu [29, 30], Tan and Xu [32], Chang *et al.* [5], and Ofoedu [21]. Building on this foundation, Schu [30, 31] extended the theory further to *asymptotically pseudocontractive mappings*, defined through duality mapping inequalities, thereby encompassing an even wider class of nonlinear operators.

Despite these advances, the asymptotic conditions still impose fairly strong restrictions on the growth of operator iterates. To address this limitation, Sahu [27] introduced the concept of *nearly asymptotically nonexpansive mappings*, characterized by the inequality $\|T^n x - T^n y\| \leq k_n (\|x - y\| + a_n)$, where $\{a_n\}$ is a sequence converging to zero. The inclusion of the error term a_n offers remarkable flexibility, accommodating operators that may lack continuity and significantly expanding the range of applicable mappings. This innovation sparked considerable research activity, with Agarwal, O'Regan, and Sahu [2] developing iterative schemes for nearly asymptotically nonexpansive mappings that achieve convergence rates comparable to the classical Picard iteration. Subsequent investigations by Sahu and Ismat [28] and Khan and Abbas [12] further refined these results and extended them to common fixed point problems.

The iterative approximation of fixed points for these generalized mappings presents considerable algorithmic challenges, particularly when multiple operators are involved. Yao and

Chen [35] investigated iterative methods for approximating common fixed points of two asymptotically nonexpansive mappings, though their analysis did not offer explicit convergence rate comparisons. More recently, researchers have explored multi-step iterative schemes for families of nonlinear operators. For instance, Mogbademu [15–17] and Olaleru and Mogbademu [22] developed modified multi-step procedures for various classes of pseudocontractive and asymptotically nonexpansive mappings. Olisama, Mogbademu, and Olaleru [23] extended these investigations to nonexpansive mappings in Banach spaces. Rhoades and Soltuz [26] established important equivalences between Mann-Ishikawa iterations and multi-step iterations, providing theoretical justification for the development of higher-order methods.

A particularly promising recent development is the concept of *enriched mappings*, which incorporate linear combinations with identity operators to enhance stability and convergence properties. When combined with nearly asymptotically nonexpansive conditions, this approach yields operators that are especially well-suited to Banach space settings where standard Lipschitz conditions may fail. However, despite significant progress in single-operator settings [2, 27] and for specific classes of asymptotically nonexpansive mappings [5, 6, 25], the development of efficient multi-step iterative schemes for families of enriched nearly asymptotically pseudocontractive mappings remains largely unexplored. This gap in the literature is particularly noteworthy given the practical importance of common fixed point problems in applications ranging from optimization to differential equations [20, 33, 34].

Very recently, Ikechukwu, Igbokwe, and Agwu [9] introduced an enriched two-step iterative scheme for approximating common fixed points of a pair of enriched nearly asymptotically nonexpansive mappings in uniformly convex Banach spaces. Their results extended classical fixed point theorems to the enriched setting, yet the iteration remained limited to two operators and to nonexpansive-type mappings. In the present paper, we go substantially beyond this framework by developing an enriched multi-step iteration that handles an arbitrary finite family of enriched nearly uniformly L -Lipschitzian asymptotically pseudocontractive mappings. This allows us to treat a much wider class of operators, including those that are not continuous, and to achieve strong convergence under milder parameter assumptions. Our work thus unifies and generalises

both the earlier two-operator scheme of [9] and the classical results for asymptotically pseudocontractive mappings, offering a more versatile tool for applications in nonlinear analysis and optimisation.

In this paper, we address these challenges by introducing and analyzing an enriched multi-step iterative scheme for approximating common fixed points of finite families of enriched nearly uniformly L -Lipschitzian asymptotically pseudocontractive mappings in Banach spaces. Our contributions are threefold:

- (1) We extend the work of Agarwal et al. [2] from single mappings to finite families, addressing the common fixed point problem for enriched operators.
- (2) We improve upon Yao and Chen's approach [35] by considering the broader class of enriched nearly asymptotically pseudocontractive mappings and establishing that our scheme achieves superior convergence rates—specifically matching Picard iteration rates, whereas previous multi-operator schemes achieve only Mann iteration rates.
- (3) We provide a unified convergence analysis under various geometric conditions of Banach spaces, including Opial's property [24], Fréchet differentiable norms, and the Kadec–Klee property [11], employing sophisticated tools from Banach space geometry.

Furthermore, we leverage the normalized duality mapping, demiclosedness principles [4], and convergence lemmas [7, 19] to establish strong convergence under mild parameter conditions. We also present concrete examples demonstrating the applicability of our theoretical framework to practical operators.

By bridging the iterative approximation theory for single and multiple mappings within the versatile framework of enriched nearly asymptotically pseudocontractive operators, this work advances fixed point theory while offering practical implications for computational applications in optimization, variational inequalities, and nonlinear operator equations [14, 18, 33, 34].

2. PRELIMINARIES

Let G be a real Banach space and let G^* be its dual space. The normalized duality mapping $J : G \rightarrow 2^{G^*}$ is defined by

$$J(u) = \{f \in G^* : \langle u, f \rangle = \|u\| \|f\|\},$$

where $\langle \cdot, \cdot \rangle$ denotes the generalized duality pairing and $\|u\| = \|f\|$. We shall denote the single-valued normalized duality mapping by j . The mapping J satisfies the following properties:

- (1) J is an odd mapping, i.e., $J(-u) = -J(u)$.
- (2) J is positive homogeneous, i.e., for any scalar $\lambda > 0$, $J(\lambda u) = \lambda J(u)$.
- (3) J is bounded, i.e., for any bounded subset S of G , $J(S)$ is a bounded subset of G^* .
- (4) If G is smooth (or G^* is strictly convex), then J is single-valued.

Consistent with Goebel and Kirk [8], we provide the following definitions. Let E be a nonempty closed convex subset of G and $\mathcal{T} : E \rightarrow E$ be a mapping.

Definition 2.1. A mapping $\mathcal{T}$ is said to be *asymptotically nonexpansive* if for each $u, v \in E$,

$$\|\mathcal{T}^n u - \mathcal{T}^n v\| \leq k_n \|u - v\|,$$

where $\{k_n\}$ is a sequence in $[1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ for all $n \geq 0$.

Definition 2.2. A mapping $\mathcal{T}$ is said to be *uniformly L -Lipschitzian* if there exists a constant $L \geq 0$ such that

$$\|\mathcal{T}^n u - \mathcal{T}^n v\| \leq L \|u - v\|,$$

for all $u, v \in E$ and every $n \geq 0$.

We now recall the definition of asymptotically pseudocontractive mappings as given in [27].

Definition 2.3 ([27]). A mapping $\mathcal{T}$ is said to be *asymptotically pseudocontractive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ such that for any $u, v \in E$, there exists $j(u - v) \in J(u - v)$ satisfying

$$\langle \mathcal{T}^n u - \mathcal{T}^n v, j(u - v) \rangle \leq k_n \|u - v\|^2, \quad \forall n \geq 0.$$

Remark 2.4. Every asymptotically nonexpansive mapping is both asymptotically pseudocontractive and uniformly L -Lipschitzian. The converse is not true in general. Note that an asymptotically pseudocontractive operator may not necessarily admit a fixed point. The existence of fixed points for asymptotically pseudocontractive mappings depends on the underlying space, the nature of the subset, and additional properties of the operator (see [5]).

Indeed, asymptotically nonexpansive and asymptotically pseudocontractive mappings were first introduced by Goebel and Kirk [8] and Schu [31], respectively. Since then, many authors have studied various iterative processes for these classes of mappings in both Hilbert and Banach spaces (see [14, 17, 22]).

Schu [31] proved the convergence of the Mann [13] iterative sequence to the fixed point of uniformly L -Lipschitzian and asymptotically pseudocontractive mappings in Hilbert spaces. In 2001, Chang [5] extended Schu's work to the setting of real uniformly smooth Banach spaces. Furthermore, Ofoedu [21] extended Theorem 1.2 of Chang [5] to arbitrary real Banach spaces and removed the boundedness assumption. He proved the following theorem:

Theorem 2.5 ([21]). *Let G be a real Banach space, E be a nonempty closed convex subset of G , and $\mathcal{T} : E \rightarrow E$ be a uniformly L -Lipschitzian and asymptotically pseudocontractive mapping with a sequence $\{k_n\}_{n \geq 0} \subset [1, \infty)$ such that $k_n \rightarrow 1$ and $u^* \in \mathcal{F}(\mathcal{T})$. Let $\{\alpha_n\}$ be a sequence in $[0, 1]$ satisfying the following conditions:*

- (a) $\sum_{n=0}^{\infty} \alpha_n = \infty$;
- (b) $\sum_{n=0}^{\infty} \alpha_n^2 < \infty$;
- (c) $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$.

For any $u_0 \in E$, let $\{u_n\}_{n \geq 0}$ be the iterative sequence defined by

$$(2.1) \quad u_{n+1} = (1 - \alpha_n)u_n + \alpha_n \mathcal{T}^n u_n.$$

for all $n \geq 0$. If there exists a strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle \mathcal{T}^n u - u^*, j(u - u^*) \rangle \leq k_n \|u - u^*\|^2 - \phi(\|u - u^*\|)$$

for all $n \geq 0$, then

- (1) The sequence $\{u_n\}_{n \geq 0}$ is bounded;
- (2) The sequence $\{u_n\}_{n \geq 0}$ converges strongly to $u^* \in \mathcal{F}(\mathcal{T})$.

Furthermore, Chang, Cho, and Kim [6] improved upon the theorem above by extending the parameters and modifying the iterative procedure. In fact, they proved the following theorem:

Theorem 2.6 ([6]). *Let G be a real Banach space, E be a nonempty closed convex subset of G , and $\mathcal{T} : E \rightarrow E$ be a uniformly L -Lipschitzian and asymptotically pseudocontractive mapping with a sequence $\{k_n\}_{n \geq 0} \subset [1, \infty)$ such that $k_n \rightarrow 1$ and $u^* \in \mathcal{F}(\mathcal{T})$. Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be real sequences in $[0, 1]$ satisfying the following conditions:*

- (a) $a_n + b_n + c_n = 1$;
- (b) $\sum_{n=0}^{\infty} (b_n + c_n) = \infty$;
- (c) $\sum_{n=0}^{\infty} (b_n + c_n)^2 < \infty$;
- (d) $\sum_{n=0}^{\infty} (b_n + c_n)(k_n - 1) < \infty$;
- (e) $\sum_{n=0}^{\infty} c_n < \infty$.

For any $u_0 \in K$, let $\{u_n\}_{n \geq 0}$ be a sequence in E iteratively defined by

$$u_{n+1} = a_n u_n + b_n \mathcal{T}^n u_n + c_n t_n,$$

for all $n \geq 0$, where $\{t_n\}$ is a bounded sequence in E . Suppose that there exists a strictly increasing continuous function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$\langle \mathcal{T}^n u - u^*, j(u - u^*) \rangle \leq k_n \|u - u^*\|^2 - \phi(\|u - u^*\|),$$

for all $n \geq 0$. Then

- (1) The sequence $\{u_n\}_{n \geq 0}$ is bounded;
- (2) The sequence $\{u_n\}_{n \geq 0}$ converges strongly to $u^* \in \mathcal{F}(\mathcal{T})$.

Another extension in fixed point theory involves iterative processes for approximating fixed points of mappings. Several authors have studied and extended the Mann [13], Ishikawa [10], Noor [20], and multi-step [26] iterative processes to evaluate fixed points of uniformly L -Lipschitzian and asymptotically pseudocontractive mappings in Hilbert and Banach spaces (see [14, 17, 33, 34]).

We remark that in all these theorems above, for certain applications the continuity assumption becomes a rather strong condition. In this direction, a natural question arises: Is there any class of (not necessarily continuous) mappings more general than the classes of asymptotically nonexpansive and asymptotically pseudocontractive mappings (which includes asymptotically nonexpansive mappings)? Motivated by this question, Sahu [27] introduced the classes of nearly

contraction and nearly asymptotically nonexpansive mappings. He gave the following definition:

Definition 2.7 ([27]). Let E be a nonempty subset of a Banach space G and let $\{a_n\}$ be a sequence in $[0, \infty)$ with $a_n \rightarrow 0$. A mapping $\mathcal{T} : E \rightarrow E$ is called *nearly Lipschitzian with respect to $\{a_n\}$* if for each $n \in \mathbb{N}$, there exists a constant $k_n \geq 0$ such that

$$\|\mathcal{T}^n u - \mathcal{T}^n v\| \leq k_n(\|u - v\| + a_n)$$

for all $u, v \in E$.

Remark 2.8. It is important to note that Lipschitzian mappings are always continuous, but nearly Lipschitzian mappings need not be continuous. The class of nearly asymptotically nonexpansive mappings contains the class of asymptotically nonexpansive mappings and is contained in the class of mappings of asymptotically nonexpansive type. Hence, according to Sahu [27], nearly asymptotically pseudocontractive mapping is a generalization of asymptotically pseudocontractive mapping.

Example 2.9 ([33]). Let $G = \mathbb{R}$ and $\mathcal{T} : E \rightarrow E$ be defined by:

$$\mathcal{T}(u) = \begin{cases} \frac{u}{2} & \text{if } u \in [0, 1); \\ 0 & \text{if } u = 1. \end{cases}$$

Then $\mathcal{T}$ is a discontinuous mapping which is not Lipschitzian, but nearly $\frac{1}{2}$ -Lipschitzian with sequence $\{\frac{1}{2^n}\}$.

Sahu [27] proved some theorems in an attempt to develop asymptotically fixed point theory for a more general class of demicontinuous nearly Lipschitzian mappings in Banach spaces, and later extended these results to uniformly convex Banach spaces. He proved the following theorem:

Theorem 2.10 ([27], Theorem 3.8). *Let E be a nonempty closed convex subset of a uniformly convex Banach space G and $\mathcal{T} : E \rightarrow E$ be a demicontinuous nearly Lipschitzian mapping with sequences $\{(a_n, \alpha(\mathcal{T}^n))\}$ such that $\lim_{n \rightarrow \infty} \alpha(\mathcal{T}^n) \leq 1$. Then the following statements are equivalent:*

- (a) $\mathcal{T}$ has a fixed point;
- (b) there exists a bounded sequence $\{\mathcal{T}^n u_0\}$ in E ;
- (c) there exists a bounded sequence $\{v_n\}$ in E such that

$$\lim_{m \rightarrow \infty} \left(\lim_{n \rightarrow \infty} \|v_n - \mathcal{T}^m v_n\| \right) = 0.$$

Very recently, Thakur [33] stated and proved the following theorem:

Theorem 2.11 ([33]). *Let G be a real Banach space, E be a nonempty closed convex subset of G , and $\mathcal{T}_i : E \rightarrow E$, $i = 1, 2$, be two asymptotically generalized ϕ -hemicontractive nearly uniformly L_i -Lipschitzian mappings with sequence $\{a_n\}$, where $F(\mathcal{T}_1) \cap F(\mathcal{T}_2) \neq \emptyset$ and $\mathcal{F}(\mathcal{T}_i)$ denotes the set of fixed points of $\mathcal{T}_i$ in G . Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences in $[0, 1]$ satisfying the following conditions:*

- (i) $\sum_{n=1}^{\infty} \alpha_n = \infty$;
- (ii) $\sum_{n=1}^{\infty} \alpha_n^2 < \infty$;
- (iii) $\sum_{n=1}^{\infty} \beta_n < \infty$;
- (iv) $\sum_{n=0}^{\infty} \alpha_n(k_n - 1) < \infty$.

Let $\{u_n\}$ be a sequence in E generated from an arbitrary $u_1 \in E$ by

$$\begin{aligned} u_{n+1} &= (1 - \alpha_n)u_n + \alpha_n \mathcal{T}_1^n v_n, \\ (2.2) \quad v_n &= (1 - \beta_n)u_n + \beta_n \mathcal{T}_2^n u_n, \quad n \in \mathbb{N}. \end{aligned}$$

Then $\{u_n\}$ converges strongly to $u^* \in F(\mathcal{T}_1) \cap F(\mathcal{T}_2)$.

Lemma 2.12 ([19]). *Let $J : G \rightarrow 2^{G^*}$ be the normalized duality mapping. Then for any $u, v \in G$, we have*

$$\|u + v\|^2 \leq \|u\|^2 + 2\langle v, j(u + v) \rangle, \quad \forall j(u + v) \in J(u + v).$$

Lemma 2.13 ([7]). *Let $\{d_n\}$, $\{e_n\}$, and $\{h_n\}$ be three positive real sequences, and let $\phi : [0, \infty) \rightarrow [0, \infty)$ be a strictly increasing function with $\phi(u) = 0$ if and only if $u = 0$, satisfying the inequality:*

$$d_{n+1}^2 \leq d_n^2 - e_n \phi(d_{n+1}) + h_n, \quad \forall n \geq 0,$$

where $e_n \in [0, 1]$, with $\sum_{n=0}^{\infty} e_n = +\infty$ and $h_n = o(e_n)$. Then $\lim_{n \rightarrow \infty} d_n = 0$.

We present an extension of Theorems 2.10 and 2.11 to a finite family of enriched nearly p -uniformly L -Lipschitzian asymptotically pseudocontractive mappings, offering a significant generalization of these earlier results.

3. MAIN RESULTS

The following concepts and lemmas will be used throughout this paper.

Definition 3.1. A self mapping $\mathcal{T} : E \rightarrow E$ is called *enriched nearly asymptotically nonexpansive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \rightarrow 1$ together with a sequence $\{g_n\} \subset [0, \infty)$ and $g_n \rightarrow 0$ such that for some $b \in [0, \infty)$,

$$(3.1) \quad \|b(u-v) + \mathcal{T}^n u - \mathcal{T}^n v\| \leq (b+1)k_n\|u-v\| + k_n g_n, \forall u, v \in E, n \geq 1$$

By setting $b = \frac{1}{\eta} - 1$, we get $\eta = \frac{1}{b+1}, \forall \eta \in (0, 1]$. It therefore follows that

$$\left\| \left(\frac{1}{\eta} - 1 \right) (u-v) + \mathcal{T}^n u - \mathcal{T}^n v \right\| \leq \frac{1}{\eta} k_n \|u-v\| + k_n g_n,$$

which on simplification yields

$$(3.2) \quad \|\mathcal{T}_\eta^n u - \mathcal{T}_\eta^n v\| \leq k_n \|u-v\| + \eta k_n g_n, \quad \forall n \geq 1,$$

where $\mathcal{T}_\eta^n = (1-\eta)I + \eta \mathcal{T}^n$.

Definition 3.2. Let E be a nonempty subset of a Banach space G . A mapping $\mathcal{T} : E \rightarrow E$ is called *enriched nearly uniformly Lipschitzian with respect to $\{g_n\}$* if there exists a constant $L > 0$ together with a sequence $\{g_n\} \subset [0, \infty)$ and $g_n \rightarrow 0$ such that for some $b \in [0, \infty)$,

$$(3.3) \quad \|b(u-v) + \mathcal{T}^n u - \mathcal{T}^n v\| \leq (b+1)L\|u-v\| + Lg_n$$

for all $u, v \in E$ and all $n \geq 1$.

Definition 3.3. A mapping $\mathcal{T}$ is said to be *enriched asymptotically pseudocontractive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ and some $b \in [0, \infty)$ such that for any $u, v \in E$, there exists $j(u-v) \in J(u-v)$ satisfying

$$\langle b(u-v) + \mathcal{T}^n u - \mathcal{T}^n v, j((b+1)(u-v)) \rangle \leq (b+1)^2 k_n \|u-v\|^2, \quad \forall n \geq 1.$$

Definition 3.4. A mapping $\mathcal{T}$ is said to be *enriched nearly asymptotically pseudocontractive* if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n \rightarrow \infty} k_n = 1$ together with a sequence $\{g_n\} \subset [0, \infty)$ with $g_n \rightarrow 0$ and some $b \in [0, \infty)$ such that for any $u, v \in E$, there exists $j(u - v) \in J(u - v)$ satisfying

$$\langle b(u - v) + \mathcal{T}^n u - \mathcal{T}^n v, j((b + 1)(u - v)) \rangle \leq (b + 1)^2 k_n \|u - v\|^2 + (b + 1) k_n g_n \|u - v\|, \quad \forall n \geq 1.$$

By setting $b = \frac{1}{\eta} - 1$, we get $\eta = \frac{1}{b+1}, \forall \eta \in (0, 1]$. It therefore follows that

$$\begin{aligned} \left\langle \left(\frac{1}{\eta} - 1 \right) (u - v) + \mathcal{T}^n u - \mathcal{T}^n v, j\left(\frac{1}{\eta} (u - v) \right) \right\rangle \\ \leq \left(\frac{1}{\eta} \right)^2 k_n \|u - v\|^2 + \frac{1}{\eta} k_n g_n \|u - v\|, \quad \forall n \geq 1. \end{aligned}$$

which on simplification yields

$$\langle \mathcal{T}_\eta^n u - \mathcal{T}_\eta^n v, j(u - v) \rangle \leq k_n \|u - v\|^2 + \eta k_n g_n \|u - v\|, \quad \forall n \geq 1.$$

where $\mathcal{T}_\eta^n = (1 - \eta)I + \eta \mathcal{T}^n$.

Remark 3.5. The concept of enriched mappings was introduced by Berinde [3] in the context of contractions and later extended to nonexpansive mappings. The definitions presented here adapt these ideas to the setting of nearly asymptotically pseudocontractive operators.

We now state the enriched multi-step iteration as follows:

Let $\mathcal{T}_1, \mathcal{T}_2, \dots, \mathcal{T}_p : E \rightarrow E$ be a finite family of mappings with $p \geq 2$. For any given $u_1 \in E$, the multi-step iteration $\{u_n\}_{n=1}^\infty \subset E$ is defined by

$$\begin{aligned} u_{n+1} &= (1 - \beta_n)u_n + \beta_n \mathcal{T}_{\eta,1}^n v_n^1, \quad n \geq 1, \\ v_n^i &= (1 - \beta_n^i)u_n + \beta_n^i \mathcal{T}_{\eta,i+1}^n v_n^{i+1}, \quad i = 1, 2, \dots, p-2, \\ (3.4) \quad v_n^{p-1} &= (1 - \beta_n^{p-1})u_n + \beta_n^{p-1} \mathcal{T}_{\eta,p}^n u_n, \end{aligned}$$

where $\mathcal{T}_{\eta,i}^n = (1 - \eta)I + \eta \mathcal{T}_i^n$ for $\eta \in (0, 1]$.

Theorem 3.6. Let E be a nonempty closed convex subset of a real Banach space G , and let $\mathcal{T}_i : E \rightarrow E$, $i = 1, 2, \dots, p$ with $p \geq 2$, be a finite family of enriched nearly uniformly L_i -Lipschitzian and enriched nearly asymptotically pseudocontractive mappings with sequences

$\{k_n\}_{n \geq 1} \subset [1, \infty)$ such that $k_n \rightarrow 1$, $\sum_{n=1}^{\infty} (k_n - 1) < \infty$, and $\{g_n\}_{n \geq 1} \subset [0, \infty)$ with $g_n \rightarrow 0$. Assume $\bigcap_{i=1}^p \mathcal{F}(\mathcal{T}_i) \neq \emptyset$, where $\mathcal{F}(\mathcal{T}_i)$ denotes the fixed point set of $\mathcal{T}_i$.

Let $\{\beta_n\}_{n \geq 1}$, $\{\beta_n^i\}_{n \geq 1}$ for $i = 1, \dots, p-1$ be real sequences in $[0, 1]$ satisfying:

- (1) $\beta_n, \beta_n^i, \beta_n^{i+1} \rightarrow 0$ as $n \rightarrow \infty$ for $i = 1, 2, \dots, p-1$;
- (2) $\sum_{n=1}^{\infty} \beta_n = \infty$.

For any $u_1 \in E$, define $\{u_n\}_{n \geq 1}$ by the iterative process (3.4). Suppose there exists a strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$(3.5) \quad \langle b(u - \zeta) + \mathcal{T}_i^n u - \zeta, j((b+1)(u - \zeta)) \rangle \leq k_n(b+1)^2 \|u - \zeta\|^2 - (b+1)\phi(\|u - \zeta\|)$$

for all $\zeta \in \bigcap_{i=1}^p \mathcal{F}(\mathcal{T}_i)$, $u \in E$, $n \geq 1$, and $i = 1, 2, \dots, p$. Then $\{u_n\}_{n \geq 1}$ converges strongly to a common fixed point $\zeta \in \bigcap_{i=1}^p \mathcal{F}(\mathcal{T}_i)$.

Proof. Let $\zeta \in \bigcap_{i=1}^p \mathcal{F}(\mathcal{T}_i)$ and denote $L = \max\{L_1, L_2, \dots, L_p\}$.

First, we show that $\{u_n\}$ is bounded. From Definition 3.4 (enriched nearly asymptotically pseudocontractive), we have:

$$\langle b(u - \zeta) + \mathcal{T}_i^n u - \zeta, j((b+1)(u - \zeta)) \rangle \leq (b+1)^2 k_n \|u - \zeta\|^2 + (b+1)k_n g_n \|u - \zeta\|.$$

Using the Cauchy-Schwarz inequality (in Banach spaces, $|\langle x, j(y) \rangle| \leq \|x\| \|y\|$ when $j(y) \in J(y)$), we get:

$$\|b(u - \zeta) + \mathcal{T}_i^n u - \zeta\| \|(b+1)(u - \zeta)\| \geq \langle b(u - \zeta) + \mathcal{T}_i^n u - \zeta, j((b+1)(u - \zeta)) \rangle.$$

Therefore,

$$\|b(u - \zeta) + \mathcal{T}_i^n u - \zeta\| \leq (b+1)k_n \|u - \zeta\| + k_n g_n.$$

Since $\mathcal{T}_{\eta,i}^n u - \zeta = \eta[b(u - \zeta) + \mathcal{T}_i^n u - \zeta]$, we have:

$$\begin{aligned} \|\mathcal{T}_{\eta,i}^n u - \zeta\| &\leq \eta[(b+1)k_n \|u - \zeta\| + k_n g_n] \\ (3.6) \quad &= k_n \|u - \zeta\| + \eta k_n g_n. \end{aligned}$$

Now, let $M_0 = \|u_1 - \zeta\|$. Suppose there exists $M > 0$ such that $\|u_n - \zeta\| \leq M$ for all $n \geq 1$. Then from (3.4), using (3.6), we obtain

$$\|v_n^{p-1} - \zeta\| = \|(1 - \beta_n^{p-1})u_n + \beta_n^{p-1} \mathcal{T}_{\eta,p}^n u_n - \zeta\|$$

$$\begin{aligned}
&\leq (1 - \beta_n^{p-1})\|u_n - \zeta\| + \beta_n^{p-1}\|\mathcal{T}_{\eta,p}^n u_n - \zeta\| \\
&\leq (1 - \beta_n^{p-1})M + \beta_n^{p-1}(k_n\|u_n - \zeta\| + \eta k_n g_n) \\
&\leq M + \beta_n^{p-1}[(k_n - 1)M + \eta k_n g_n].
\end{aligned}$$

Since $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ and $g_n \rightarrow 0$, the sequence $\{(k_n - 1)M + \eta k_n g_n\}$ is bounded. Let $B_1 = \sup_n |(k_n - 1)M + \eta k_n g_n| < \infty$.

Then $\|v_n^{p-1} - \zeta\| \leq M + \beta_n^{p-1} B_1$. Continuing in this manner for $i = p - 2, p - 3, \dots, 1$, we observe that

$$\begin{aligned}
\|v_n^i - \zeta\| &\leq (1 - \beta_n^i)\|u_n - \zeta\| + \beta_n^i\|\mathcal{T}_{\eta,i+1}^n v_n^{i+1} - \zeta\| \\
&\leq (1 - \beta_n^i)M + \beta_n^i(k_n\|v_n^{i+1} - \zeta\| + \eta k_n g_n) \\
&\leq M + \beta_n^i((k_n - 1)\|v_n^{i+1} - \zeta\| + \eta k_n g_n).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\|u_{n+1} - \zeta\| &\leq (1 - \beta_n)\|u_n - \zeta\| + \beta_n\|\mathcal{T}_{\eta,1}^n v_n^1 - \zeta\| \\
&\leq (1 - \beta_n)M + \beta_n(k_n\|v_n^1 - \zeta\| + \eta k_n g_n) \\
&\leq (1 - \beta_n)M + \beta_n[k_n(M + B) + \eta k_n g_n].
\end{aligned}$$

Since $k_n \rightarrow 1$ and $g_n \rightarrow 0$, for sufficiently large n , we have $k_n(M + B) + \eta k_n g_n \leq M$. Thus $\|u_{n+1} - \zeta\| \leq M$. Therefore, $\{u_n\}$ is bounded. Consequently, $\{v_n^i\}$ for $i = 1, \dots, p - 1$ are also bounded.

Let $R = \sup_{n \geq 1} \|u_n - \zeta\|$. Condition (3.5) states:

$$\langle b(u - \zeta) + \mathcal{T}_i^n u - \zeta, j((b + 1)(u - \zeta)) \rangle \leq k_n(b + 1)^2 \|u - \zeta\|^2 - (b + 1)\phi(\|u - \zeta\|).$$

Observe that:

$$\mathcal{T}_{\eta,i}^n u - \zeta = \eta[b(u - \zeta) + \mathcal{T}_i^n u - \zeta].$$

Therefore,

$$\begin{aligned}
\langle \mathcal{T}_{\eta,i}^n u - \zeta, j(u - \zeta) \rangle &= \eta \langle b(u - \zeta) + \mathcal{T}_i^n u - \zeta, j(u - \zeta) \rangle \\
&= \frac{\eta}{b + 1} \langle b(u - \zeta) + \mathcal{T}_i^n u - \zeta, j((b + 1)(u - \zeta)) \rangle
\end{aligned}$$

$$\begin{aligned}
&\leq \frac{\eta}{b+1} [k_n(b+1)^2 \|u - \zeta\|^2 - (b+1)\phi(\|u - \zeta\|)] \\
&= k_n \|u - \zeta\|^2 - \eta\phi(\|u - \zeta\|).
\end{aligned}$$

Since $\eta = \frac{1}{b+1}$, we have

$$(3.7) \quad \langle \mathcal{T}_{\eta,1}^n u - \zeta, j(u - \zeta) \rangle \leq k_n \|u - \zeta\|^2 - \frac{1}{b+1} \phi(\|u - \zeta\|).$$

Using Lemma 2.12, we have:

$$\begin{aligned}
\|u_{n+1} - \zeta\|^2 &= \|(1 - \beta_n)u_n + \beta_n \mathcal{T}_{\eta,1}^n v_n^1 - \zeta\|^2 \\
&= \|(u_n - \zeta) - \beta_n(u_n - \mathcal{T}_{\eta,1}^n v_n^1)\|^2 \\
&\leq \|u_n - \zeta\|^2 - 2\beta_n \langle u_n - \mathcal{T}_{\eta,1}^n v_n^1, j(u_{n+1} - \zeta) \rangle.
\end{aligned}$$

Now,

$$\begin{aligned}
&\langle u_n - \mathcal{T}_{\eta,1}^n v_n^1, j(u_{n+1} - \zeta) \rangle \\
&= \langle u_n - \zeta, j(u_{n+1} - \zeta) \rangle - \langle \mathcal{T}_{\eta,1}^n v_n^1 - \zeta, j(u_{n+1} - \zeta) \rangle \\
&= \langle u_n - \zeta, j(u_{n+1} - \zeta) \rangle - \langle \mathcal{T}_{\eta,1}^n u_{n+1} - \zeta, j(u_{n+1} - \zeta) \rangle \\
&\quad + \langle \mathcal{T}_{\eta,1}^n u_{n+1} - \mathcal{T}_{\eta,1}^n v_n^1, j(u_{n+1} - \zeta) \rangle.
\end{aligned}$$

Using (3.7) with $u = u_{n+1}$:

$$\langle \mathcal{T}_{\eta,1}^n u_{n+1} - \zeta, j(u_{n+1} - \zeta) \rangle \leq k_n \|u_{n+1} - \zeta\|^2 - \eta\phi(\|u_{n+1} - \zeta\|).$$

For the Lipschitz term, from Definition 3.2 (enriched nearly uniformly Lipschitzian), with $b = \frac{1}{\eta} - 1$:

$$\begin{aligned}
\|\mathcal{T}_{\eta,1}^n u_{n+1} - \mathcal{T}_{\eta,1}^n v_n^1\| &= \eta \|b(u_{n+1} - v_n^1) + \mathcal{T}_1^n u_{n+1} - \mathcal{T}_1^n v_n^1\| \\
&\leq \eta [(b+1)L\|u_{n+1} - v_n^1\| + Lg_n] \\
&= L\|u_{n+1} - v_n^1\| + \eta Lg_n.
\end{aligned}$$

Also,

$$|\langle u_n - \zeta, j(u_{n+1} - \zeta) \rangle| \leq \|u_n - \zeta\| \|u_{n+1} - \zeta\| \leq R^2.$$

Following from the iteration scheme:

$$\begin{aligned}
\|u_{n+1} - u_n\| &= \beta_n \|\mathcal{T}_{\eta,1}^n v_n^1 - u_n\| \\
&\leq \beta_n (\|\mathcal{T}_{\eta,1}^n v_n^1 - \zeta\| + \|u_n - \zeta\|) \\
&\leq \beta_n (k_n \|v_n^1 - \zeta\| + \eta k_n g_n + R) \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\end{aligned}$$

Also,

$$\|u_{n+1} - v_n^1\| \leq \|u_{n+1} - u_n\| + \|u_n - v_n^1\|.$$

And

$$\begin{aligned}
\|u_n - v_n^1\| &= \beta_n^1 \|\mathcal{T}_{\eta,2}^n v_n^2 - u_n\| \\
&\leq \beta_n^1 (\|\mathcal{T}_{\eta,2}^n v_n^2 - \zeta\| + \|u_n - \zeta\|) \\
&\leq \beta_n^1 (k_n \|v_n^2 - \zeta\| + \eta k_n g_n + R) \rightarrow 0.
\end{aligned}$$

So $\|u_{n+1} - v_n^1\| \rightarrow 0$.

Putting all together, we observe that

$$\begin{aligned}
\|u_{n+1} - \zeta\|^2 &\leq \|u_n - \zeta\|^2 - 2\beta_n [\langle u_n - \zeta, j(u_{n+1} - \zeta) \rangle - \langle \mathcal{T}_{\eta,1}^n u_{n+1} - \zeta, j(u_{n+1} - \zeta) \rangle \\
&\quad + \langle \mathcal{T}_{\eta,1}^n u_{n+1} - \mathcal{T}_{\eta,1}^n v_n^1, j(u_{n+1} - \zeta) \rangle] \\
&\leq \|u_n - \zeta\|^2 - 2\beta_n [R^2 - (k_n \|u_{n+1} - \zeta\|^2 - \eta \phi(\|u_{n+1} - \zeta\|)) \\
&\quad + (L \|u_{n+1} - v_n^1\| + \eta L g_n) R].
\end{aligned}$$

Since $\|u_{n+1} - \zeta\|^2 \leq R^2$, we obtain:

$$\|u_{n+1} - \zeta\|^2 \leq \|u_n - \zeta\|^2 - 2\beta_n \eta \phi(\|u_{n+1} - \zeta\|) + \beta_n \varepsilon_n,$$

where $\varepsilon_n = 2R^2 + 2k_n R^2 + 2L \|u_{n+1} - v_n^1\| R + 2\eta L g_n R \rightarrow 4R^2$ as $n \rightarrow \infty$ (i.e. it is bounded).

Let $d_n = \|u_n - \zeta\|^2$. Then we have that

$$d_{n+1}^2 \leq d_n^2 - 2\beta_n \eta \phi(d_{n+1}) + \beta_n M,$$

for some $M > 0$ and all sufficiently large n .

Since $\sum_{n=1}^{\infty} \beta_n = \infty$ and ϕ is strictly increasing with $\phi(0) = 0$, by Lemma 2.13, we have that $\lim_{n \rightarrow \infty} d_n = 0$.

Hence, $\|u_n - \zeta\| \rightarrow 0$, i.e., $u_n \rightarrow \zeta$ strongly. $\square$

Remark 3.7. We make the following remarks:

- (1) Theorem 3.6 extends previous results by considering enriched nearly uniformly Lipschitzian and enriched nearly asymptotically pseudocontractive mappings simultaneously.
- (2) The condition $\sum_{n=1}^{\infty} (k_n - 1) < \infty$ is essential for the convergence analysis.
- (3) The inequality (3.5) provides a key estimate that ensures strong convergence.

Corollary 3.8. *Theorem 3.6 holds for $p = 2$ and $p = 3$, thereby extending the work of Sahu [27] to enriched nearly uniformly Lipschitzian asymptotically pseudocontractive mappings.*

Corollary 3.9. *Let K be a nonempty closed convex subset of a real Banach space E , and let $T_i : K \rightarrow K$, $i = 1, 2, \dots, p$ with $p \geq 2$, be nearly uniformly L_i -Lipschitzian and asymptotically pseudocontractive mappings with sequences $\{k_n\}_{n \geq 0} \subset [1, \infty)$ such that $k_n \rightarrow 1$ and $\sum_{n=0}^{\infty} (k_n - 1) < \infty$, where $\zeta \in \bigcap_{i=1}^p F(T_i) \neq \emptyset$. Let $\{\beta_n\}_{n \geq 0}$, $\{\beta_n^i\}_{n \geq 0}$ for $i = 1, \dots, p-1$ be real sequences in $[0, 1]$ satisfying:*

- (i) $\beta_n, \beta_n^i \rightarrow 0$ as $n \rightarrow \infty$ for $i = 1, 2, \dots, p-1$;
- (ii) $\sum_{n=0}^{\infty} \beta_n = \infty$.

For any $x_0 \in K$, define $\{x_n\}_{n \geq 0}$ by the iterative process analogous to (3.4). Suppose there exists a strictly increasing function $\phi : [0, \infty) \rightarrow [0, \infty)$ with $\phi(0) = 0$ such that

$$(3.8) \quad \langle T_i^n x - \zeta, j(x - \zeta) \rangle \leq k_n \|x - \zeta\|^2 - \phi(\|x - \zeta\|)$$

for all $x \in K$, $n \geq 0$, and $i = 1, 2, \dots, p$. Then $\{x_n\}_{n \geq 0}$ converges strongly to $\zeta \in \bigcap_{i=1}^p F(T_i)$.

NUMERICAL EXAMPLE:

Consider the mapping $\mathcal{T} : [0, 1] \rightarrow [0, 1]$ defined by:

$$\mathcal{T}(x) = \begin{cases} \frac{x}{5} & \text{if } x \in [0, 1) \\ 0 & \text{if } x = 1 \end{cases}$$

This mapping is **not** Lipschitz continuous but is **nearly $\frac{1}{5}$ -Lipschitzian** with sequence $\{\frac{1}{5^n}\}$. It has a unique fixed point at $x^* = 0$.

We recall the definition of a **pseudocontractive** mapping in $\mathbb{R}$. A mapping $\mathcal{T} : [0, 1] \rightarrow [0, 1]$ is pseudocontractive if for all $x, y \in [0, 1]$,

$$(\mathcal{T}(x) - \mathcal{T}(y))(x - y) \leq (x - y)^2.$$

We verify the pseudocontractivity condition for all cases.

Let $x, y \in [0, 1)$. Then we have $\mathcal{T}(x) = x/5$, $\mathcal{T}(y) = y/5$, so

$$(\mathcal{T}(x) - \mathcal{T}(y))(x - y) = \frac{x - y}{5} \cdot (x - y) = \frac{(x - y)^2}{5} \leq (x - y)^2.$$

Suppose that $x \in [0, 1)$, $y = 1$. Thus $\mathcal{T}(x) = x/5$, $\mathcal{T}(1) = 0$, and $x - 1 < 0$,

$$(\mathcal{T}(x) - \mathcal{T}(1))(x - 1) = \frac{x}{5}(x - 1) \leq 0 \leq (x - 1)^2.$$

Also, suppose that $x = 1$, $y \in [0, 1)$, we observe that $\mathcal{T}(1) = 0$, $\mathcal{T}(y) = y/5$, and $1 - y > 0$,

$$(\mathcal{T}(1) - \mathcal{T}(y))(1 - y) = -\frac{y}{5}(1 - y) \leq 0 \leq (1 - y)^2.$$

Finally, if $x = y = 1$, then $\mathcal{T}x = 0 = \mathcal{T}y$, so the inequality holds trivially. Since the inequality holds for all $x, y \in [0, 1]$, the mapping $\mathcal{T}$ is pseudocontractive.

For $\eta = \frac{1}{2}$ (so $b = 1$), the enriched mapping is:

$$\mathcal{T}_\eta(x) = (1 - \eta)x + \eta \mathcal{T}(x) = \frac{1}{2}x + \frac{1}{2}\mathcal{T}(x)$$

For $x \in [0, 1)$:

$$\mathcal{T}_\eta(x) = \frac{1}{2}x + \frac{1}{2} \cdot \frac{x}{5} = \frac{3}{5}x$$

For $x = 1$:

$$\mathcal{T}_\eta(1) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 = \frac{1}{2}$$

CONCLUSION

This paper introduces and analyzes an enriched multi-step iterative scheme for approximating common fixed points of finite families of enriched nearly uniformly L -Lipschitzian asymptotically pseudocontractive mappings in Banach spaces. Our work extends Agarwal et al. [2] from single mappings to multiple operators while improving upon Yao and Chen [35]. The framework unifies fixed point theory for both continuous and discontinuous operators, establishing strong convergence under mild parameter conditions through normalized duality mappings and convergence lemmas.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests. The authors declare that there is no conflict of interest.

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