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STRENGTHENED HARDY-TYPE INEQUALITIES

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Abstract. In this paper, the results in [11] have been improved upon and a new simpler proof is given in dimensional form.

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1. INTRODUCTION

The following interesting classical theorem is well known(see [4]):

Theorem A. *Let $f(x)$ be a non-negative p-integrable function defined on $(0,\infty)$, and $p>1$. Then, f is integrable over the interval $(0,x)$ for each x and the following inequality is true:*

$$\int_0^\infty \left(\frac{1}{x}\left(\int_0^x f(y)\,dy\right)\right)^p dx \leq \left(\frac{p}{p-1}\right)^p \int_0^\infty f(x)^p dx \tag{1.1}$$

provided the right hand side of this inequality is finite.

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Some integral inequalities related to Hardy's inequality known as Hardy-type inequalities have been established by many authors [1, 2, 3, 5, 7, 8, 10, 13]. Recently, [11] gave a refined form of [1] and the following result was proved:

Theorem B. *Let $g(x)$ be continuous and non-decreasing on $[0,\infty]$ with $g(0)=0$, $g(x)>0$, $g(\infty)=\infty$. $\sum_{i=1}^{n} u_i = u$ and $\sum_{i=1}^{n} v_i = v$ where x,u and v are all positive. Also, let $f:[a,b]\to\mathbb{R}(a<b)$, be continuous and convex on the real interval $[a,b]$. Assume $\prod_{i=1}^{n}\alpha_i\beta_i \geq 0$ with $\sum_{i=1}^{n}(\alpha+\beta)>0$ for all $i\in\mathbb{N}$.*
Then, the following inequality holds:

$$\int_0^\infty g(x)^{-1}\Big[\int_a^b\int_a^b f(\sum_{i=1}^{n}(\alpha_i u_i+\beta_i v_i))u^{\alpha-1}dudv\Big]^p dg(x) \leq L^p\int_0^\infty G(x)dg(x) \tag{1.2}$$

where, $L=(\alpha^{-1}\beta)(b-a)(b^\alpha-a^\alpha)(k+1)$ and $G(x)=f(x)^p g(x)^{-1}$.
The left side of (1.1) and (1.2) exists when the right hands sides do.

This work is, therefore, devoted to Hardy-type inequalities and to some modifications and consequences contained in [11, 12]. The aim is to determine conditions on the data of our problem, i.e. on the domain and parameters, under which those inequalities hold on some classes of functions and to some new extension, generalization to multidimensional cases by making one of the weight functions a power function.

Throughout this paper, $p>1$ except otherwise stated, we shall use f to be integrable whenever f is measurable and $\int|f(x)|\,dx<\infty$. Hence, if f is an integrable function, then $\int f(x)\,dx$ exists whenever f is measurable and $\int|f(x)|\,dx<\infty$.

The multidimensional generalized Hardy-Polya type inequality described by convex functions would be discussed in the next section. Throughout the section, we use the notation

$$\int_{t_1}^{b_1}\cdots\int_{t_n}^{b_n} =: \int_{\mathbf{t}}^{\mathbf{b}}$$

and we have similar expressions for

$$\int_{\mathbf{0}}^{\mathbf{b}},\int_{\mathbf{0}}^{\mathbf{x}},\int_{\mathbf{0}}^{\infty} \text{ and } \int_{\mathbf{X}}$$

where b_i's, x_i's and v_i's are the components of $\mathbf{b}$, $\mathbf{x}$ and $\mathbf{v}$ for all $i=1,\dots,n\in\mathbb{Z}_+$ respectively. All functions are measurable except otherwise stated. Based on the methods in [11], [6] and [9], we further make some new generalizations of multidimensional Hardy-type integral inequalities by introducing real function $g(\mathbf{x})$. Some multidimensional Hardy-type integral inequalities are obtained. Some applications are also considered.

2. Multidimensional Hardy-Type Inequalities with Weights

In this section, we prove the following theorems which are more general than the results contained in [11]. We shall first give some lemmas which are crucial to prove certain inequalities in our context. This first and second of this are from [6] and [9].

Lemma C. *Let $0 < b_i \le \infty$, $i = 1, 2 \ldots, n \in \mathbb{Z}_+$, $-\infty \le a < c \le \infty$ and let Φ be a positive function $[a, c]$.*

If Φ is convex (respectively concave), then

$$\int_{\mathbf{0}}^{\mathbf{b}} \Phi\left(\frac{1}{x_1 \ldots x_n}\int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t})d\mathbf{t}\right)\frac{d\mathbf{x}}{x_1 \ldots x_n}$$

is less than or equal to (respectively greater than or equal to)

$$\int_{\mathbf{0}}^{\mathbf{b}} \Phi(f(\mathbf{x}))\left(1-\frac{x_1}{b_1}\right)\ldots\left(1-\frac{x_n}{b_n}\right)\frac{d\mathbf{x}}{x_1 \ldots x_n}$$

for every function f on $(\mathbf{0}, \mathbf{b})$ such that $a < f(\mathbf{x}) < c$.

Lemma D. *Let $\mathbf{b} \in (\mathbf{0}, \infty]$, $-\infty \le a < c \le \infty$ and Φ be a positive function on $[a, c]$. Suppose that the weight function u defined on $(\mathbf{0}, \mathbf{b})$ is nonnegative such that $\frac{u(\mathbf{x})}{x_1^2 \ldots x_n^2}$ is locally integrable on $(\mathbf{0}, \mathbf{b})$ and the weight function v is defined by*

$$v(\mathbf{t}) = t_1 \ldots t_n \int_{\mathbf{t}}^{\mathbf{b}} \frac{u(\mathbf{x})}{x_1^2 \ldots x_n^2} d\mathbf{x}, \mathbf{t} \in (\mathbf{0}, \mathbf{b}).$$

If Φ is convex (respectively concave), then

$$\int_{\mathbf{0}}^{\mathbf{b}} u(\mathbf{x})\Phi\left(\frac{1}{x_1 \ldots x_n}\int_{\mathbf{0}}^{\mathbf{x}} f(\mathbf{t})d\mathbf{t}\right)\frac{d\mathbf{x}}{x_1 \ldots x_n}$$

is less than or equal to (respectively greater than or equal to)

$$\int_{\mathbf{0}}^{\mathbf{b}} v(\mathbf{x})\Phi(f(\mathbf{x}))\frac{d\mathbf{x}}{x_1 \ldots x_n}$$

holds for every function f on $(\mathbf{0}, \mathbf{b})$ such that $a < f(x_1, \ldots, x_1) < c$.

Theorem 2.1. *If Φ is positive and continuous on $[\mathbf{0},\infty)$, f and h are a non-negative functions on $[\mathbf{0},\mathbf{b}]$, $0 < x_i < b_i \leq \infty$ $(i = 1\dots n \in \mathbb{Z}_+)$ and λ is non-decreasing on $[\mathbf{0},\infty]$, assume*

$$0 < \int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi(f(\mathbf{x}))dg(x_1)\dots dg(x_n) < \infty$$

for each continuous and non-decreasing function g on $[\mathbf{0},\infty)$ and $\mathbf{v}\in\mathbb{R}^n$ such that $0 < v_i < \infty$ with,

Φ convex (respectively concave), then

$$\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi\left(L^{-1}\int_{\mathbf{0}}^{\mathbf{x}}(f(v_1,\dots,v_n)h(v_1,\dots,v_n))d\lambda(v_1)\dots d\lambda(v_n)\right)dg(x_1)\dots dg(x_n)$$

is less than or equal to (respectively greater than or equal to)

$$\left(\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi(f(\mathbf{x}))dg(x_1)\dots dg(x_n)\right)\left(\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi(h(\mathbf{x}))dg(x_1)\dots dg(x_n)\right)$$

where $L = \int_{\mathbf{0}}^{\infty} d\lambda(v_1)\dots d\lambda(v_n)$

Proof. Applying the iterative integrals of functions h and f on measurable set X with measure λ and Y with measure μ (σ-finite) then,

$$\int_{\mathbf{X}} h(y_1,\dots,y_n)\left(\int_{\mathbf{Y}} f(\mathbf{x})d\lambda_1\dots d\lambda_n\right)d\mu_1\dots d\mu_n$$

$$= \int_{(X_1\dots X_n)\times(Y_1\dots Y_n)} f(\mathbf{x})\times h(y_1,\dots,y_n)(d\mu_1\dots d\mu_n\times d\lambda_1\dots d\lambda_n)$$

$$= \int_{(Y_1\dots Y_n)} h(\mathbf{x})\left(\int_{(X_1\dots X_n)} f(y_1,\dots,y_n)d\mu_1\dots d\mu_n\right)d\lambda_1\dots d\lambda_n$$

Since Φ is convex, $L = \int_{\mathbf{0}}^{\infty}(d\lambda(v_1)\dots d\lambda(v_n))$ and by imploring Fubini's theorem, then,

$$\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi\left(L^{-1}\int_{\mathbf{0}}^{\mathbf{x}} f(v_1,\dots,v_n)h(v_1,\dots,v_n)d\lambda(v_1)\dots d\lambda(v_n)\right)dg(x_1)\dots dg(x_n)$$

$$\leq L^{-1}\int_{\mathbf{0}}^{\infty} g(\mathbf{x})^{-p}dg(x_1)\dots dg(x_n)\int_{\mathbf{0}}^{\mathbf{x}}\Phi(f(v_1,\dots,v_n)h(v_1,\dots,v_n))d\lambda(v_1)\dots d\lambda(v_n)$$

$$\leq L^{-1}\int_{\mathbf{0}}^{\infty} d\lambda(v_1)\dots d\lambda(v_n)\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi(f(\mathbf{x})h(\mathbf{x}))dg(x_1)\dots dg(x_n)$$

$$= \int_{\mathbf{0}}^{\mathbf{b}}\Phi(f(\mathbf{x})h(\mathbf{x}))g(\mathbf{x})^{-p}dg(x_1)\dots dg(x_n)$$

$$\leq \left(\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi(f(\mathbf{x}))dg(x_1)\dots dg(x_n)\right)\left(\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi(h(\mathbf{x}))dg(x_1)\dots dg(x_n)\right)$$

The proof of Φ concave is now easily obtained from the above by reversing the inequalities. $\square$

Corollary 2.2. *If $p>1$, $f,h\geq 0$, $g(\mathbf{x})=\mathbf{x}$ is continuous, non-decreasing on $[\mathbf{0},\infty)$. Let Φ be positive and continuous on $[\mathbf{0},\infty)$, and define $d\lambda(v_1)\dots d\lambda(v_n)$ by $(v_1^{\alpha-1}\dots v_n^{\alpha-1})dv_1\dots dv_n$ on $[0,1]$ and 0 for $v>1$, $1<\alpha\leq n$ and $n\in\mathbb{Z}_+$. Assume*

$$\int_{\mathbf{0}}^{\mathbf{b}}\Phi((f(\mathbf{x})h(\mathbf{x}))^p)d\mathbf{x}<\infty$$

with,

Φ convex (respectively concave), then

$$\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi\left(\left(\int_{\mathbf{0}}^{\mathbf{1}} f(v_1,\dots,v_n)h(v_1,\dots,v_n)d\lambda(v_1)\dots d\lambda(v_n)\right)^p\right)d\mathbf{x}$$

is less than or equal to (respectively greater than or equal to)

$$\prod_{i=1}^{n}(\alpha_i-1)^{-p}\left(\int_{\mathbf{0}}^{\mathbf{b}}\Phi(f(\mathbf{x})^p)d\mathbf{x}\right)\left(\int_{\mathbf{0}}^{\mathbf{b}}\Phi(h(\mathbf{x})^p)d\mathbf{x}\right)$$

Proof. The integral of two or more variables of a summable functions can generally be obtained by successive integrations with respect to each variable separately or by pairs that is an iterative integral and with Φ convex, then

$$\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi\left(\left(\int_{\mathbf{0}}^{\mathbf{1}}(v_1^{\alpha-1}\dots v_n^{\alpha-1})f(v_1,\dots,v_n)h(v_1,\dots,v_n)dv_1\dots dv_n\right)^p\right)d\mathbf{x}$$

$$\leq\int_{\mathbf{0}}^{\mathbf{b}}\Phi((f(\mathbf{x})h(\mathbf{x}))^p)\left(\int_{\mathbf{0}}^{\mathbf{1}} g(v_1,\dots,v_n)^{-1}(v_1^{\alpha-1}\dots v_n^{\alpha-1})dv_1\dots dv_n\right)^p d\mathbf{x}$$

by substituting and integrating the inner integral on $[0,1]$ by single step of integration by part, we have

$$\int_{\mathbf{0}}^{\mathbf{b}}\Phi((f(\mathbf{x})h(\mathbf{x}))^p)\left(\int_{\mathbf{0}}^{\mathbf{1}} g(v_1,\dots,v_n)^{-1}(v_1^{\alpha-1}\dots v_n^{\alpha-1})dv_1\dots dv_n\right)^p d\mathbf{x}$$

$$\leq\prod_{i=1}^{n}(\alpha_i-1)^{-p}\left(\int_{\mathbf{0}}^{\mathbf{b}}\Phi(f(\mathbf{x})^p)d\mathbf{x}\right)\left(\int_{\mathbf{0}}^{\mathbf{b}}\Phi(h(\mathbf{x})^p)d\mathbf{x}\right)$$

Also, if there exist a continuous inverse which is necessarily concave on function Φ then the proof is easily obtained using similar method with inequalities reversed. The results also hold if we assume $g(\mathbf{x})=\mathbf{x}^k$ whenever $1<k<\alpha\leq n\in\mathbb{Z}_+$. $\square$

Corollary 2.3. *If $p>1$, f and h are continuous, non-decreasing on $[\mathbf{0},\mathbf{b}]$. Let Φ be positive and continuous on $[\mathbf{0},\infty)$, and define $d\lambda(v_1)\dots d\lambda(v_n)$ by $(v_1^{\alpha-1}\dots v_n^{\alpha-1})dv_1\dots dv_n$ on $[0,1]$, $\alpha\in\mathbb{R}$ and λ is non-decreasing on $[\mathbf{0},\mathbf{1}]$. Assume*

$$0<\int_{\mathbf{0}}^{\infty}\Phi((f(\mathbf{x})h(\mathbf{x}))^p)d\mathbf{x}<\infty$$

if Φ is convex, then,

$$\int_{\mathbf{0}}^{\mathbf{b}} g(\mathbf{x})^{-p}\Phi\left(\left(\int_{\mathbf{0}}^{\mathbf{1}} f(v_1,\dots,v_n)h(v_1,\dots,v_n)d\lambda(v_1)\dots d\lambda(v_n)\right)^p\right)d\mathbf{x}$$

$$\leq(\alpha(1-k))^{-np}\left(\int_{\mathbf{0}}^{\mathbf{b}}\Phi(f(\mathbf{x})^p)d\mathbf{x}\right)\left(\int_{\mathbf{0}}^{\mathbf{b}}\Phi(h(\mathbf{x})^p)d\mathbf{x}\right)$$

whenever $g(\mathbf{x})=\mathbf{x}^k$ is a decreasing function over $[0,1]$ and $1<k\in\mathbb{Z}_+$.

Proof. Since g is decreasing on $[0,1]$, then we obtain, by using Chebyshev's integral inequality on

$$\int_{\mathbf{0}}^{\mathbf{b}}\Phi((f(\mathbf{x})h(\mathbf{x}))^p)\left(\int_{\mathbf{0}}^{\mathbf{1}} g(v_1,\dots,v_n)^{-1}(v_1^{\alpha-1}\dots v_n^{\alpha-1})dv_1\dots dv_n\right)^p d\mathbf{x}$$

$$\leq\int_{\mathbf{0}}^{\mathbf{b}}\Phi((f(\mathbf{x})h(\mathbf{x}))^p)\left(\int_{\mathbf{0}}^{\mathbf{1}} g(v_1,\dots,v_n)^{-1}dv_1\dots dv_n\right)^p$$

$$\times\left(\int_{\mathbf{0}}^{\mathbf{1}}(v_1^{\alpha-1}\dots v_n^{\alpha-1})dv_1\dots dv_n\right)^p d\mathbf{x}$$

$$\leq(\alpha(1-k))^{-np}\left(\int_{\mathbf{0}}^{\mathbf{b}}\Phi(f(\mathbf{x})^p)d\mathbf{x}\right)\left(\int_{\mathbf{0}}^{\mathbf{b}}\Phi(h(\mathbf{x})^p)d\mathbf{x}\right)$$

since g and λ are not similarly ordered, otherwise the inequality is reversed. Also, the inequality is reversed if they are not similarly ordered and p lies between $0<p<1$. See ([4], Theorem 43, see also section 5.8 page 123).

We distingused some valid cases for this inequality as follows:

p	α,k,n		
$p<0$	$\alpha>0$	$k<1$	$n<0$
$p>0$	$\alpha>0$	$k>1$	$n=0$
$p=0$	$\alpha>0$	$k>0$	$n>0$
$p=0$	$\alpha>0$	$k<0$	$n>0$
$p=0$	$\alpha>0$	$k>0$	$n=0$
odd	$\alpha>0$	$k>1$	even
even	$\alpha>0$	$k>1$	odd

□

We obtain the corresponding reverse inequalities if Φ has a continuous inverse which is necessarily concave. Similar results were obtained for refined form of Theorem 1 in [11] and Theorem 3.1 of the current paper.

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