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APPROXIMATION OF COMMON FIXED POINTS OF FINITE FAMILY OF MIXED-TYPE TOTAL ASYMPTOTICALLY QUASI PSEUDOCONTRACTIVE-TYPE MAPPINGS IN BANACH SPACES

I. K. AGWU*, D. I. IGBOKWE

Department of Mathematics, Micheal Okpara University of Agriculture, Umudike, Umuahia Abia State Nigeria

Abstract. We propose an m-step hybrid-type iteration scheme for finite family of total asymptotically quasi pseudocontractive-type self mappings and finite family of total asymptotically quasi pseudocontractive-type nonself mappings. Under suitable conditions on the iteration parameters, we established strong convergence theorems of the scheme to the common fixed point of the mappings in real Banach spaces. Our results modify, improve and generalise numerous results currently in literature.

Keywords: total quasi asymptotically pseudoconyractive-type nonself mapping; hybrid mixed type iteration scheme; common fixed point; real banach space; finite family; strong convergence.

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1. INTRODUCTION

Let K be a nonempty, closed convex subset of a real Banach space E. If $E^\star$ is the dual of E, then the mapping $J : E \longrightarrow 2^{E^\star}$ defined by

$$J(x) = \{x^\star \in E : \langle x, x^\star \rangle = \|x\|\|x^\star\|, \|x\| = \|x^\star\|\}, \tag{1.1}$$

*Corresponding author
E-mail address: agwuimo@gmail.com
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is called normalised duality mapping.

Let $T : K \longrightarrow K$ be a nonlinear mapping, we denote the set of all fixed points of T by $F(T)$. The set of common fixed points of four mappings S_1, S_2, T_1 *and* T_2 will be denoted by $F = F(S_1) \cap F(S_2) \cap F(T_1) \cap F(T_2)$.

Definition 1.1. A mapping $T : C \longrightarrow C$ is said to be asymptotically nonexpansive if there exists a sequence $\{k_n\} \in [0, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that, for all $x, y \in C$,

$$\|T^n(x) - T^n(y)\| \leq k_n \|x - y\|, \forall n \in \mathbb{N}. \tag{1.2}$$

ln 1972, the class of asymptotically nonexpansive mapping was introduced as a generalization of the class of nonexpansive mapping by Goebel and Kirk [6]. They proved that if K is a nonempty closed convex subset of a uniformly convex Banach space and T is an asymptotically nonexpansive mapping of K, then T has a fixed point.

Since then, many results on asymptotically nonexpansive mappings have been obtained in literature (See [5], [6], [7], [9], etc for details).

Definition 1.2. A mapping T is said to be to uniformly Lipschitizian with the lipschitz constant $L > 0$ if

$$\|T^n(x) - T^n(y)\| \leq L\|x - y\|, \forall x, y \in K \text{ and } n \in \mathbb{N}. \tag{1.3}$$

Definition 1.3. T is said to be asymptotically pseudocontractive if there exists $k_n \subseteq [1, \infty)$ with $k_n \to 0$ as $n \to \infty$ and $j(x - y) \in J(x - y)$ such that

$$\langle T^n x - T^n y, j(x - y)\rangle \leq k_n \|x - y\|^2, \forall x \in K. \tag{1.4}$$

It is easy to see from (1.4) that every asymptotically nonexpansive mapping is a subclass of the class of asymptotically pseudocontractive mapping. This class of mapping was introduced by Schu [27] and has been extensively studied generalizesd by various authors in different spaces using modified Mann and Ishikawa iteration schemes.

Definition 1.4. T is said to be total asymptotically pseudocontractive mapping in the sense of Osilike and Chima [22] if for every $x, y \in K$, there exists a strictly increasing continous function

$\phi : [0,\infty] \longrightarrow [0,\infty)$ with $\phi(0) = 0$ and nonnegative sequences $\mu_n, \xi_n \in [0,\infty) : \mu_n, \xi_n \to 0$ as $n \to \infty$ such that

$$\langle T^n x - T^n y, j(x-y)\rangle \leq \|x-y\|^2 + \frac{\mu_n}{2}\phi(\|x-y\|) + \frac{\xi_n}{2}. \tag{1.5}$$

Observe that if $\phi(t) = t^2$ and $\xi_n = 0, \forall n \geq 1$, then (1.5) becomes

$$\langle T^n x - T^n y, j(x-y)\rangle \leq k_n\|x-y\|^2, \tag{1.6}$$

where $k_n = 1 + \frac{1}{2}\mu_n \subseteq [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$.

Definition 1.5. A subset K of a Banach space E is said to be a retract of E if there exists a continous mapping $P : E \longrightarrow K$ (*called retraction*) such that $P(x) = x$ for all $x \in K$. If, in addition P is nonexpansive, then P is said to be nonexpansive retraction of E.

If $P : E \longrightarrow K$ is a retraction, then $P^2 = P$. A retract of a Hausdorff space must be a closed subset. Every closed convex subset of a uniformly convex Banach space is a retract.

Definition 1.6. Let K be a nonempty, closed and convex subset of a Banach space E. A nonself mapping $T : K \to E$ is said to be total asymptotically pseudocontractive mapping if for every $x, y \in K$ and $j(x-y) \in J(x-y)$, there exist sequences $\mu_n, \xi_n \in [0,\infty) : \mu_n, \xi_n \to 0$ as $n \to \infty$ and a strictly increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\langle T(PT)^{n-1}(x) - T(PT)^{n-1}(y), j(x-y)\rangle \leq \|x-y\|^2 + \frac{\mu_n}{2}\phi(\|x-y\|) + \frac{\xi_n}{2}, \forall n \in \mathbb{N}. \tag{1.7}$$

From definitions (1.5) and (1.7), we see that the class of total asymptotically pseudocontractive mappings include the class of asymptotically pseudocontractive mappings as a special case; that is, each asymptotically psudocontractive mapping is total asymptotically psudocontractive mapping with $\xi_n = 0, \phi(t) = t^2, \mu_n = 2(k_n - 1), \forall n \geq 1, t \geq 1$. Also, if the retraction map $P : E \longrightarrow K$ is an identity, then (1.7) reduces to (1.5).

Definition 1.7. T is said to be asymptotically pseudocontractive-type mapping [26] if for every $x, y \in K$ and $j(x-y) \in J(x-y)$ there exists a sequence $k_n \in [1,\infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$(1.8)\limsup_{n\to\infty}\sup_{x,y\in K}\liminf_{j(x-y)\in J(x-y)}(\langle T(PT)^{n-1}x-T(PT)^{n-1}y, j(x-y)\rangle - k_n\|x-y\|^2)\leq 0.$$

Note that if $F(T)\neq\emptyset$ and $q\in F(T)$, then(1.8) becomes

$$\limsup_{n\to\infty}\sup_{x\in K}\liminf_{j(x-q)\in J(x-q)}(\langle T(PT)^{n-1}x-q, j(x-q)\rangle - k_n\|x-q\|^2)\leq 0. \tag{1.9}$$

and is called quasi asymptotically pseudocontractive-type mapping. This class of mapping was introduced by Wang and Shi [26] as a generalisation of the class of asymptotically pseudocontractive mapping, asymptotically pseudocontractive-type mapping and quasi asymptotically pseudocontractive-type mapping (see [26] for more details).

Chidume et al. [3] studied the following iterative scheme in 2003:

$$\begin{aligned} x_1 &= x\in K \\ x_{n+1} &= P(\alpha_n T(PT)^{n-1}x_n+(1-\alpha_n)x_n), n\geq 1 \end{aligned} \tag{1.10}$$

where α_n is a sequence in $(0,1)$ and K is a nonempty closed convex subset of of a real uniformly convex Banach space E, P is a nonexpansive retraction of E onto K, and proved some strong and weak convergence theorems for asymptotically nonexpansive nonself mappings in the intermediate sense in the framework of uniformly convex Banach spaces.

ln 2014, Wang and Shi [26] introduced the following modified lshikawa iteration process:

$$\left.\begin{aligned} &x_1\in K \\ &x_{n+1} = P((1-\alpha_n-\gamma_n)x_n+\alpha_n T_1(PT_1)^{n-1}\sigma_n+\gamma_n u_n) \\ &y_n = P((1-\alpha_n'-\gamma_n')x_n+\alpha_n' T_2(PT_2)^{n-1}\delta_n+\gamma_n' v_n) \end{aligned}\right\}, \tag{1.11}$$

where $\sigma_n = (1-\beta_n)y_n+\beta_n T_1(PT_1)^{n-1}y_n, \delta_n = (1-\beta_n')x_n+\beta_n' T_2(PT_2)^{n-1}x_n$ $\{\alpha_n\},\{\gamma_n\},\{\beta_n\},\{\alpha_n'\},\{\gamma_n'\},\{\beta_n'\}\in[0,1]$, T_1,T_2 are L-Lipschitizian and quasi asymptotically pseudocontractive-type mappings, $\{u_n\},\{v_n\}\subset K$ are two bounded sequences, and proved strong convergence theorems for the above mappings in the setting of real Banach spaces.

Hybrid Mixed-Type Iteration Scheme

Let E be a real Banach space, K a nonempty closed convex subset of E and $P : E \longrightarrow K$ a nonexpansive retraction of E onto K. Let $S_i : K \longrightarrow K$ be two L_i'-Lipschitizian and total quasi asymptotically psudocontractive-type self mappings and $T_i : K \longrightarrow E$ be two L_i''-Lipschitizian and total quasi asymptotically pseudocontractive-type nonself mappings, $i = 1, 2, \cdots N$. If $x_0 \in K$, then the new hybrid-type iteration scheme for the above mentioned mappings is as follows:

$$
\begin{aligned}
x_1 &= P((1-\alpha_0-\gamma_0)x_0 + \alpha_0 T_1(PT_1)^0\tau_1 + \gamma_0 u_0),\\
x_2 &= P((1-\alpha_1-\gamma_1)x_1 + \alpha_1 T_2(PT_2)^0\tau_2 + \gamma_1 u_1),\\
&\vdots\\
x_N &= P((1-\alpha_{N-1}-\gamma_{N-1})x_{N-1} + \alpha_{N-1} T_N(PT_N)^0\tau_{N-1} + \gamma_{N-1} u_{N-1}),\\
x_{N+1} &= P((1-\alpha_N-\gamma_N)x_N + \alpha_N T_1(PT_1)^1\tau_N + \gamma_N u_N),\\
x_{N+2} &= P((1-\alpha_{N+1}-\gamma_{N+1})x_{N+1} + \alpha_{N+1} T_2(PT_2)^1\tau_{N+1} + \gamma_{N+1} u_{N+1}),\\
&\vdots\\
x_{2N} &= P((1-\alpha_{2N-1}-\gamma_{2N-1})x_{2N-1} + \alpha_{2N-1} T_N(PT_N)^2\tau_{2N-1} + \gamma_{2N-1} u_{2N-1}),\\
x_{2N+1} &= P((1-\alpha_{2N}-\gamma_{2N})x_{2N} + \alpha_{2N} T_1(PT_1)^3\tau_{2N} + \gamma_{2N} u_{2N}),\\
x_{2N+2} &= P((1-\alpha_{2N+1}-\gamma_{2N+1})x_{2N+1} + \alpha_{2N+1} T_2(PT_2)^3\tau_{2N+1} + \gamma_{2N+1} u_{2N+1}),\\
&\vdots
\end{aligned}
$$

with

$$
\begin{aligned}
\tau_1 &= (1-\beta_0)S_1 y_1 + \beta_0 T_1(PT_1)^0 y_1,\\
\tau_2 &= (1-\beta_1)S_2 y_2 + \beta_1 T_2(PT_2)^0 y_2,\\
&\vdots\\
\tau_N &= (1-\beta_N)S_N^2 y_N + \beta_N T_1(PT_1)^1 y_N,\\
\tau_{N+1} &= (1-\beta_N)S_{N+1}^2 y_{N+1} + \beta_N T_2(PT_2)^1 y_{N+1},\\
&\vdots
\end{aligned}
$$

where

$$\begin{aligned}
y_1 &= P((1-\alpha_0'-\gamma_0')x_0+\alpha_0'T_1(PT_1)^0\rho_1+\gamma_0'v_0),\\
y_2 &= P((1-\alpha_1'-\gamma_1')x_1+\alpha_1'T_2(PT_2)^0\rho_2+\gamma_1'v_1),\\
&\vdots\\
y_N &= P((1-\alpha_{N-1}'-\gamma_{N-1}')x_{N-1}+\alpha_{N-1}'T_N(PT_N)^0\rho_{N-1}+\gamma_{N-1}'v_{N-1}),\\
y_{N+1} &= P((1-\alpha_N'-\gamma_N')x_N+\alpha_N'T_1(PT_1)^1\rho_N+\gamma_N'u_N),\\
y_{N+2} &= P((1-\alpha_{N+1}'-\gamma_{N+1}')x_{N+1}+\alpha_{N+1}'T_2(PT_2)^1\rho_{N+1}+\gamma_{N+1}'v_{N+1}),\\
&\vdots\\
y_{2N} &= P((1-\alpha_{2N-1}'-\gamma_{2N-1}')x_{2N-1}+\alpha_{2N-1}'T_N(PT_N)^2\rho_{2N-1}+\gamma_{2N-1}'v_{2N-1}),\\
y_{2N+1} &= P((1-\alpha_{2N}'-\gamma_{2N}')x_{2N}+\alpha_{2N}'T_1(PT_1)^3\rho_{2N}+\gamma_{2N}'v_{2N}),\\
y_{2N+2} &= P((1-\alpha_{2N+1}'-\gamma_{2N+1}')x_{2N+1}+\alpha_{2N+1}'T_2(PT_2)^3\rho_{2N+1}+\gamma_{2N+1}'v_{2N+1}),\\
&\vdots
\end{aligned}$$

with

$$\begin{aligned}
\rho_1 &= (1-\beta_0')S_1x_0+\beta_0'T_1(PT_1)^0x_0,\\
\rho_2 &= (1-\beta_1')S_2x_1+\beta_1'T_2(PT_2)^0x_1,\\
&\vdots\\
\rho_N &= (1-\beta_N')S_N^2x_N+\beta_N'T_1(PT_1)^1x_N,\\
\rho_{N+1} &= (1-\beta_{N+1}')S_{N+1}^2x_{N+1}+\beta\prime_{N+1}T_2(PT_2)^1x_{N+1},\\
&\vdots
\end{aligned}$$

The above Hybrid-type iteration sequence can be written in compact form as

$$(1.12)\qquad \begin{cases} x_1\in K \\ \qquad x_{n+1}=P((1-\alpha_n-\gamma_n)x_n+\alpha_nT_i(PT_i)^{k-1}\tau_{n+1}+\gamma_nu_n) \\ \qquad y_{n+1}=P((1-\alpha_n'-\gamma_n')x_n+\alpha_n'T_i(PT_i)^{k-1}\rho_{n+1}+\gamma_n'v_n) \end{cases},$$

where $k = \{\frac{n-i}{N}\} + 1$, $\tau_{n+1} = (1-\beta_n)S_i^k y_{n+1} + \beta_n T_i(PT_i)^{k-1} y_{n+1}, \rho_{n+1} = (1-\beta_n')S_i^k x_n + \beta_n' T_i(PT_i)^{k-1} x_n$ $\{\alpha_n\}, \{\gamma_n\}, \{\beta_n\}, \{\alpha_n'\}, \{\gamma_n'\}, \{\beta_n'\} \in [0,1]$ and $\{u_n\}, \{v_n\} \subset K$ are two bounded sequences.

Moltivated and inspired by the works of Wang and Shi [26], in this paper, we study this new hybrid mixed-type iteration scheme (1.12) and then establish some convergence theorems for mixed-type mappings in the setting of real Banach spaces.

2. Preliminary

For the sake of convenience, we restate the following concepts and results:

Lemma 2.1. (see [26]) Let E be a real Banach space. Then, for all $x, y \in E, j(x-y) \in J(x-y)$, the following inequality holds:

$$\|x+y\|^2 \leq \|x\|^2 + 2\langle y, j(x+y)\rangle. \tag{2.1}$$

Lemma 2.2. (see [26]) Suppose that $\phi : [0,\infty) \longrightarrow [0,\infty)$ is a strictly increasing function with $\phi(0) = 0$. Let $\{a_n\}, \{b_n\}, \{c_n\}, \{\lambda_n\}(0 \leq \lambda_n \leq 1)$ e four sequences of nonnegative numbers satisfying the recursive inequality:

$$a_{n+1} \leq (1+b_n)a_n - \lambda_n \phi(a_{n+1}) + c_n, \forall n \geq n_0, \tag{2.2}$$

where n_o is some nonnegative integer. If $\sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} \lambda_n = \infty$, then $\lim_{n\to\infty} a_n = 0$.

3. Main Results

Definition 3.1. Let $T : K \longrightarrow E$ be a nonself mapping. Then T is said to be total quasi asymptotically pseudocontractive-type mapping in the sense of Osilike and Chima [22] if $F(T) \neq \emptyset$ and for every $x \in K, q \in F(T)$ and $j(x-q) \in J(x-q)$, there exist a strictly increasing continous function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ and sequences $\mu_n, \xi_n \in [0,\infty)$ with $\mu_n, \xi_n \to 0$ as

$n \to \infty$ such that

$$0 \geq \limsup_{n\to\infty} \sup_{x\in K} \liminf_{j(x-q)\in J(x-y)} (\langle T(PT)^{n-1}x - q, j(x-q)\rangle - \|x-q\|^2$$

$$- \frac{\mu_n}{2}\phi(\|x-q\|) - \frac{\xi_n}{2}). \tag{3.1}$$

Observe that if $\phi(t) = t^2$ and $\xi_n = 0$, then (3.1) reduces to (1.9) with $k_n = 1 - \frac{\mu_n}{2}$. The class of total quasi asymptotically pseudocontractive-type mapping is introduced as a generalisation of the following classes of mappings: asymptotically pseudocontractive mapping, asymptotically pseudocontractive-type mapping and quasi asymptotically pseudocontractive-type mapping.

Theorem 3.1. Let E be a real Banach space and K a nonexpansive retract of E. Let $S_i : K \longrightarrow K$ $(i = 1,2,\cdots,N)$ be N-uniformly L_i'-Lipschitizian self mappings, $T_i : K \longrightarrow E$ $(i = 1,2,\cdots,N)$ be N-uniformly L_i''-Lipschitizian nonself mappings and each T_i be total asymptotically quasi pseudocontractive-type nonself mapping with the sequence $\mu_{in} \in [1,\infty) : \mu_{in} \to 1$ $(i = 1,2,\cdots,N)$ as $n \to \infty$. Let $\{x_n\}$ be a sequence defined by

$$\begin{cases} x_1 \in K \\ \qquad x_{n+1} = P((1-\alpha_n-\gamma_n)x_n + \alpha_n T_i(PT_i)^{k-1}\tau_{n+1} + \gamma_n u_n) \\ \qquad y_{n+1} = P((1-\alpha_n'-\gamma_n')x_n + \alpha_n' T_i(PT_i)^{k-1}\rho_{n+1} + \gamma_n' v_n) \end{cases}, \tag{3.2}$$

where $\tau_{n+1} = (1-\beta_n)S_i^k y_n + \beta_n T_i(PT_i)^{k-1}y_{n+1}, \rho_{n+1} = (1-\beta_n')S_i^k x_n + \beta_n' T_i(PT_i)^{k-1}x_n$ $\{\alpha_n\},\{\gamma_n\},\{\beta_n\},\{\alpha_n'\},\{\gamma_n'\},\{\beta_n'\} \in [0,1]$ and $\{u_n\},\{v_n\} \subset K$ are two bounded sequences. Suppose $F = \cap_{i=1}^N (F(S_i) \cap F(T_i)) \neq \emptyset$. If the following conditions hold:

i. $\sum_{n=1}^\infty (\mu_n^2 - 1) < \infty, 0 < \beta < \alpha_n < 1, \sum_{n=1}^\infty \alpha_n = \infty, \sum_{n=1}^\infty \alpha_n^2 < \infty, \sum_{n=1}^\infty \gamma_n < \infty, \sum_{n=1}^\infty \gamma_n' < \infty$;

ii. $\gamma_n + \gamma_n' \geq 1, \alpha_n + \gamma_n \geq 1, \alpha_n' + \gamma_n' \geq 1$;

iii. $\sum_{n=1}^\infty \alpha_n\beta_n < \infty, \sum_{n=1}^\infty \alpha_n\alpha_n' < \infty, \sum_{n=1}^\infty \alpha_n\gamma_n' < \infty, \alpha_n \geq \gamma_n'$.

Then, the sequence defined by (3.2) converges strongly to the fixed point $q \in F$ if and only if there exists a strictly increasing function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ such that

$$\begin{aligned} 0 \quad \geq \quad & \limsup_{n\to\infty} \inf_{j(x_{n+1}-q)\in J(x_{n+1}-q)} (\langle T_i(PT_i)^{k-1}x_{n+1} - q, j(x-q)\rangle - \|x_{n+1} - q\| \\ & -\frac{\mu_{in}}{2}\phi(\|x_{n+1} - q\|) - \frac{\xi_n}{2}) + \phi(\|x_{n+1} - q\|)). \end{aligned}$$

Proof. (Adequacy)

For $i = 1, 2, \cdots, N$, let

$$\begin{aligned} \sigma_n^\star \quad = \quad & \inf_{j(x_{n+1}-q)\in j(x_{n+1}-q)} [\langle T_i(PT_i)^{k-1}x_{n+1} - q, j(x_{n+1} - q)\rangle - \|x_{n+1} - q\| \\ & -\frac{\mu\phi}{2}(\|x_{n+1} - q\|) - \frac{\xi_n}{2} + \phi(\|x_{n+1} - q\|)], \end{aligned}$$

$$\sigma_n = \max\{\sigma_n^\star, 0\} + \frac{1}{n},$$

$$M = \sup\{\|u_n - q\|, |v_n - q\|\},$$

$$\mu_n = \max\{\mu_i\}$$

and

$$L = \max\{L_i', L_i''\}$$

Then, there exists $j(x_{n+1} - q) \in j(x_{n+1} - q)$ such that

$$\begin{aligned} \sigma_n \quad \geq \quad & \langle T_i(PT_i)^{k-1}x_{n+1} - q, j(x_{n+1} - q)\rangle - \|x_{n+1} - q\| \\ & -\frac{\mu\phi}{2}(\|x_{n+1} - q\|) - \frac{\xi_n}{2} + \phi(\|x_{n+1} - q\|) \end{aligned} \tag{3.3}$$

By hypothesis, it is clear that $\limsup_{n\to\infty} \sigma_n^\star \leq 0$. Thus, $\lim_{n\to\infty} \sigma_n = 0$.

Again, since $\tau_{n+1} = (1-\beta_n)S_i^k y_n + \beta_n T_i(PT_i)^{k-1}y_{n+1}$ and $\rho_{n+1} = (1-\beta_n')S_i^k x_n + \beta_n' T_i(PT_i)^{k-1}x_n$, we have ,$\forall n \geq 0$,

$$\begin{aligned} \|\rho_{n+1} - q\| \quad &= \quad |(1-\beta_n')S_i^k x_n + \beta_n' T_i(PT_i)^{k-1}x_n) - q\| \\ &\leq \quad (1-\beta_n')\|S_i^k x_n - \alpha q\| + \beta_n'\|T_i(PT_i)^{k-1}x_n - q\| \\ &\leq \quad (1-\beta_n')L_i'\|x_n - \alpha q\| + \beta_n' L_i''\|x_n - q\| \\ &\leq \quad (1-\beta_n')L\|x_n - q\| + \beta_n' L\|x_n - q\| \\ &= \quad L\|x_n - q\|; \end{aligned} \tag{3.4}$$

$$
\begin{aligned}
\|y_{n+1}-q\| &= \|P((1-\alpha_n'-\gamma_n')x_n+\alpha_n'T_i(PT_i)^{k-1}\rho_{n+1}+\gamma_n'v_n)-q\| \\
&\leq \|(1-\alpha_n'-\gamma_n')x_n+\alpha_n'T_i(PT_i)^{k-1}\rho_{n+1}+\gamma_n'v_n-q\| \\
&= \|(1-\alpha_n'-\gamma_n')(x_n-q)+\alpha_n'(T_i(PT_i)^{n-1}\rho_{n+1}-\alpha q)+\gamma_n'(v_n-q)\| \\
&\leq (1-\alpha_n'-\gamma_n')\|x_n-q\|+\alpha_n'\|T_i(PT_i)^{k-1}\rho_{n+1}-q\|+\gamma_n'\|v_n-q\| \\
&\leq (1-\alpha_n'-\gamma_n')\|x_n-q\|+\alpha_n'L''\|\rho_{n+1}-q\|+\gamma_n'\|v_n-q\| \\
&\leq (1-\alpha_n')\|x_n-q\|+\alpha_n'L\|\rho_{n+1}-q\|+\gamma_n'M \\
&= (1-\alpha_n')\|x_n-q\|+\alpha_n'L^2\|x_n-q\|+\gamma_n'M \qquad (\ by\ (3.4)) \\
&\leq (1+\alpha_n'L^2)\|x_n-q\|+\gamma_n'M; \qquad (3.5)
\end{aligned}
$$

$$
\begin{aligned}
\|\rho_{n+1}-y_{n+1}\| &\leq \|\rho_{n+1}-q\|+\|q-y_{n+1}\| \\
&\leq L\|x_n-q\|+(1+\alpha_n'L^2)\|x_n-q\|+\gamma_n'M \qquad (\ by\ (3.4)\ and\ (3.5)) \\
&= (1+L+\alpha_n'L^2)\|x_n-q\|+\gamma_n'M; \qquad (3.6)
\end{aligned}
$$

$$
\begin{aligned}
\|\tau_{n+1}-q\| &= \|(1-\beta_n)S_i^k y_{n+1}+\beta_n T_i(PT_i)^{k-1}y_{n+1}-q\| \\
&\leq (1-\beta_n)\|S_i^k y_{n+1}-q\|+\beta_n\|T_i(PT_i)^{k-1}y_{n+1}-q\| \\
&\leq (1-\beta_n)L'\|y_{n+1}-\alpha q\|+\beta_n L''\|y_{n+1}-q\| \\
&\leq (L'+L'')\|y_{n+1}-q\| \\
&\leq 2L\|y_{n+1}-q\| \\
&\leq 2L[(1+\alpha_n'L^2)\|x_n-q\|+\gamma_n'M] \qquad (\ by\ (3.5)) \\
&= 2L(1+\alpha_n'L^2)\|x_n-q\|+2L\gamma_n'M; \qquad (3.7)
\end{aligned}
$$

$$
\begin{aligned}
\|y_{n+1}-x_{n+1}\| &= \|P((1-\alpha_n'-\gamma_n')x_n+\alpha_n'T_i(PT_i)^{k-1}\rho_{n+1}+\gamma_n'v_n)-P((1-\alpha_n-\gamma_n)x_n \\
&\quad +\alpha_nT_i(PT_i)^{k-1}\tau_n+\gamma_nu_n)\| \\
&\leq \|(1-\alpha_n'-\gamma_n')x_n+\alpha_n'T_i(PT_i)^{k-1}\rho_{n+1}+\gamma_n'v_n-[(1-\alpha_n-\gamma_n)x_n \\
&\quad +\alpha_nT_i(PT_i)^{k-1}\tau_{n+1}+\gamma_nu_n]\| \\
&= \|(1-\alpha_n'-\gamma_n')(x_n-q)+\alpha_n'(T_i(PT_i)^{k-1}\rho_{n+1}-q)+\gamma_n'(v_n-q) \\
&\quad -[(1-\alpha_n-\gamma_n)(x_n-q)+\alpha_n(T_i(PT_i)^{k-1}\tau_{n+1}-q)+\gamma_n(u_n-q)]\| \\
&= \|-(\alpha_n'+\gamma_n')(x_n-q)+\alpha_n'(T_i(PT_i)^{k-1}\rho_{n+1}-q)+\gamma_n'(v_n-q) \\
&\quad -[-(\alpha_n+\gamma_n)(x_n-q)+\alpha_n(T_i(PT_i)^{k-1}\tau_{n+1}-q)+\gamma_n(u_n-q)]\| \\
&\leq \|\alpha_n'(T_i(PT_i)^{k-1}\rho_{n+1}-q)+\gamma_n'(v_n-q)-\alpha_n(T_i(PT_i)^{k-1}\tau_{n+1}-q) \\
&\quad -\gamma_n(u_n-q)\| \\
&\leq \alpha_n'\|T_i(PT_i)^{k-1}\rho_{n+1}-q\|+\gamma_n'\|v_n-q\|+\alpha_n\|T_i(PT_i)^{k-1}\tau_{n+1}-q\| \\
&\quad +\gamma_n\|u_n-q\|\|\| \\
&\leq \alpha_n'L_i''\|\rho_{n+1}-q\|+\gamma_n'M+\alpha_nL_i''\|\tau_{n+1}-q\|+\gamma_nM \\
&\leq \alpha_n'L\|\rho_{n+1}-q\|+\alpha_nL\|\tau_{n+1}-q\|+(\gamma_n+\gamma_n')M \\
&\leq \alpha_n'L^2\|x_n-q\|+2\alpha_nL^2(1+\alpha_n'L^2)\|x_n-q\|+2\alpha\gamma_n'L^2M \\
&\quad +(\gamma_n+\gamma_n')M \qquad (\ by\ (3.4)\ and\ (3.7)) \\
&= [\alpha_n'+2\alpha_n(1+\alpha_n'L^2)]L^2\|x_n-q\|+(\gamma_n+\gamma_n'+2\alpha\gamma_n'L^2)M \qquad (3.8)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\tau_{n+1}-x_{n+1}\| &= \|(1-\beta_n)S_i^k y_{n+1}+\beta_n T_i(PT_i)^{k-1}y_{n+1}-x_{n+1}\| \\
&\le \|S_i^k y_{n+1}-y_{n+1}\|+\|y_{n+1}-x_{n+1}\|+\beta_n\|T_i(PT_i)^{k-1}y_{n+1}-\alpha q\|+\beta_n\|\alpha q-S_i^k y_{n+1}\| \\
&\le \|S_i^k y_{n+1}-q\|+\|y_{n+1}-q\|+\|y_{n+1}-x_{n+1}\|+\beta_n\|T_i(PT_i)^{k-1}y_{n+1}-q\| \\
&\quad +\beta_n\|q-S_i^k y_{n+1}\| \\
&\le L'\|y_{n+1}-q\|+\|y_{n+1}-q\|+\|y_{n+1}-x_{n+1}\|+\beta_n L_i''\|y_{n+1}-q\|+\beta_n L_i'\|q-y_{n+1}\| \\
&\le L\|y_{n+1}-\alpha q\|+\|y_{n+1}-q\|+\|y_{n+1}-x_{n+1}\|+\beta_n L\|y_{n+1}-q\|+\beta_n L\|q-y_{n+1}\| \\
&= (1+L+2\beta_n L)\|y_{n+1}-q\|+\|y_{n+1}-x_{n+1}\| \\
&\le (1+L+2\beta_n L)[(1+\alpha_n')L^2\|x_n-q\|+\gamma_n' M]+[\alpha_n'+2\alpha_n(1+\alpha_n' L^2)]L^2 \\
&\quad \times\|x_n-q\|+(\gamma_n+\gamma_n'+2\alpha_n\gamma_n' L^2)M] \qquad (\textit{ by } (3.5) \textit{ and } (3.8)) \\
&= [(1+L+2\beta_n L)(1+\alpha_n' L^2)+(\alpha_n'+2\alpha_n(1+\alpha_n')L^2))L^2]\|x_n-q\|] \\
&\quad +[(2+L+2\beta_n L+2\alpha_n L^2)\gamma_n'+\gamma_n]M. \qquad (3.9)
\end{aligned}
$$

Now, using Lemma 2.1 and condition (ii), and the fact that each $T_i : K \longrightarrow E$ is total asymptotically quasi pseudocontractive-type mapping in the intermediate sense, we get

$$
\begin{aligned}
\|x_{n+1}-q\|^2 &= \|P((1-\alpha_n-\gamma_n)x_n+\alpha_n T_i(PT_i)^{k-1}\tau_n+\gamma_n u_n)-q\|^2 \\
&\le \|(1-\alpha_n-\gamma_n)x_n+\alpha_n T_i(PT_i)^{k-1}\tau_{n+1}+\gamma_n u_n-q\|^2 \\
&= \|(1-\alpha_n-\gamma_n)(x_n-q)+\alpha_n(T_i(PT_i)^{k-1}\tau_{n+1}-q)+\gamma_n(u_n-q)\|^2 \\
&\le (1-\alpha_n-\gamma_n)^2\|x_n-q\|^2+2\alpha_n\langle(T_i(PT_i)^{k-1}\tau_{n+1}-q) \\
&\quad +\gamma_n(u_n-q), j(x_{n+1}-q)\rangle \\
&= (1-\alpha_n-\gamma_n)^2\|x_n-q\|^2+2\alpha_n\langle T_i(PT_i)^{k-1}\tau_{n+1}-q, j(x_{n+1} \\
&\quad -q)\rangle+2\alpha_n\langle\gamma_n u_n-q, jx_{n+1}-q\rangle
\end{aligned}
$$

$$\begin{aligned}
&= (1-\alpha_n-\gamma_n)^2\|x_n-q\|^2+2\alpha_n\langle T_i(PT_i)^{n-1}\tau_{n+1}-T_i(PT_i)^{k-1}x_{n+1}\\
&\quad +(T_i(PT_i)^{k-1}x_{n+1}-q),j(x_{n+1}-q)\rangle+2\alpha_n\gamma_n\|u_n-q\|\|(x_{n+1}-q)\|\\
&= (1-\alpha_n-\gamma_n)^2\|x_n-q\|^2+2\alpha_n\langle T_i(PT_i)^{k-1}\tau_{n+1}\\
&\quad -T_i(PT_i)^{k-1}x_{n+1},j(x_{n+1}-q)\rangle+2\alpha_n\langle T_i(PT_i)^{k-1}x_{n+1}-q,j(x_{n+1}-q)\rangle\\
&\quad +2\alpha_n\gamma_n\|u_n-q\|\|(x_{n+1}-q)\|\\
&\leq (1-\alpha_n-\gamma_n)^2\|x_n-q\|^2+2\alpha_n\langle T_i(PT_i)^{k-1}\tau_{n+1}\\
&\quad -T_i(PT_i)^{k-1}x_{n+1},j(x_{n+1}-q)\rangle+2\alpha_n[\langle T_i(PT_i)^{k-1}x_{n+1}-q,j(x_{n+1}-q)\rangle\\
&\quad -\|x_{n+1}-q\|^2-\frac{\mu_n}{2}\phi(\|x_{n+1}-q\|)-\frac{\xi_n}{2}]+2\alpha_n\|x_{n+1}-q\|^2\\
&\quad +\alpha_n\mu_n\phi(\|x_{n+1}-q\|)+\xi_n+2\alpha_n\gamma_nM\|x_{n+1}-q\|\\
&\leq (1-\alpha_n-\gamma_n)^2\|x_n-q\|^2+2\alpha_nL\|\tau_{n+1}-x_{n+1}\|\|x_{n+1}-q\|\\
&\quad +2\alpha_n[\langle T_i(PT_i)^{k-1}x_{n+1}-q,j(x_{n+1}-q)\rangle-\|x_{n+1}-q\|^2-\frac{\mu_{in}}{2}\phi(\|x_{n+1}-q\|)\\
&\quad -\frac{\xi_n}{2}+\phi(\|x_{n+1}-q\|)]-2\alpha_n\phi(\|x_{n+1}-q\|)+2\alpha_n\|x_{n+1}-q\|^2\\
&\quad +\alpha_n\mu_{in}\phi(\|x_{n+1}-q\|)+\xi_n+2\alpha_n\gamma_nM\|x_{n+1}-q\| \qquad (3.10)
\end{aligned}$$

(3.9) and (3.10) imply that

$$\begin{aligned}
\|x_{n+1}-q\|^2 &\leq (1-\alpha_n-\gamma_n)^2\|x_n-q\|^2\\
&\quad +2\alpha_nL\{[(1+L+2\beta_nL)(1+\alpha'_nL^2)+(\alpha'_n+2\alpha_n(1+\alpha'_n)L^2)L^2]\|x_n-q\|]\\
&\quad +[(2+L+2\beta_nL+2\alpha_nL^2)\gamma'_n+\gamma_n]M\}\|x_{n+1}-q\|\\
&\quad +2\alpha_n[\langle T_i(PT_i)^{k-1}x_{n+1}-q,j(x_{n+1}-q)\rangle-\|x_{n+1}-q\|^2-\frac{\mu_n}{2}\phi(\|x_{n+1}-q\|)\\
&\quad -\frac{\xi_n}{2}+\phi(\|x_{n+1}-q\|)]-2\alpha_n\phi(\|x_{n+1}-q\|)+2\alpha_n\|x_{n+1}-q\|^2\\
&\quad +\alpha_n\mu_n\phi(\|x_{n+1}-q\|)+\xi_n+2\alpha_n\gamma_nM\|x_{n+1}-q\|
\end{aligned}$$

$$
\begin{aligned}
&= (1-(\alpha_n+\gamma_n))^2\|x_n-q\|^2\\
&\quad +2\{\{\alpha_n L[(1+L+2\beta_n L)(1+\alpha_n' L^2)+(\alpha_n'+2\alpha_n(1+\alpha_n')L^2))L^2]\|x_n-q\|]\\
&\quad +\alpha_n L[(2+L+2\beta_n L+2\alpha_n L^2)\gamma_n'+\gamma_n]M\}+2\alpha_n\gamma_n M\}\|x_{n+1}-q\|\\
&\quad +2\alpha_n[\langle T_i(PT_i)^{k-1}x_{n+1}-q, j(x_{n+1}-q)\rangle-\|x_{n+1}-q\|^2-\frac{\mu_n}{2}\phi(\|x_{n+1}-q\|)\\
&\quad -\frac{\xi_n}{2}+\phi(\|x_{n+1}-q\|)]-2\alpha_n\phi(\|x_{n+1}-q\|)+2\alpha_n\|x_{n+1}-q\|^2\\
&\quad +\alpha_n\mu_n\phi(\|x_{n+1}-q\|)+\xi_n\\
&= [1+(\alpha_n+\gamma_n)^2]\|x_n-q\|^2-2(\alpha_n+\gamma_n)\|x_n-x_{n+1}+x_{n+1}-q\|^2\\
&\quad +2\{\{\alpha_n L[(1+L+2\beta_n L)(1+\alpha_n' L^2)+(\alpha_n'+2\alpha_n(1+\alpha_n')L^2))L^2]\|x_n-q\|]\\
&\quad +\alpha_n L[(2+L+2\beta_n L+2\alpha_n L^2)\gamma_n'+\gamma_n]M\}+2\alpha_n\gamma_n M\}\|x_{n+1}-q\|\\
&\quad +2\alpha_n[\langle T_i(PT_i)^{k-1}x_{n+1}-q, j(x_{n+1}-q)\rangle-\|x_{n+1}-q\|^2-\frac{\mu_n}{2}\phi(\|x_{n+1}-q\|)\\
&\quad -\frac{\xi_n}{2}+\phi(\|x_{n+1}-q\|)]-2\alpha_n\phi(\|x_{n+1}-q\|)+2\alpha_n\|x_{n+1}-q\|^2\\
&\quad +\alpha_n\mu_n\phi(\|x_{n+1}-q\|)+\xi_n\\
&\leq [1+(\alpha_n+\gamma_n)^2]\|x_n-q\|^2-2(\alpha_n-\gamma_n'+\gamma_n'+\gamma_n)\|x_{n+1}-q\|^2\\
&\quad +2\{\{\alpha_n L[(1+L+2\beta_n L)(1+\alpha_n' L^2)+(\alpha_n'+2\alpha_n(1+\alpha_n')L^2))L^2]\|x_n-q\|]\\
&\quad +\alpha_n L[(2+L+2\beta_n L+2\alpha_n L^2)\gamma_n'+\gamma_n]M\}+2\alpha_n\gamma_n M\}\|x_{n+1}-q\|\\
&\quad +2\alpha_n[\langle T_i(PT_i)^{k-1}x_{n+1}-q, j(x_{n+1}-q)\rangle-\|x_{n+1}-q\|^2-\frac{\mu_n}{2}\phi(\|x_{n+1}-q\|)\\
&\quad -\frac{\xi_n}{2}+\phi(\|x_{n+1}-q\|)]-2\alpha_n\phi(\|x_{n+1}-q\|)+2\alpha_n\|x_{n+1}-q\|^2\\
&\quad +\alpha_n\mu_n\phi(\|x_{n+1}-q\|)+\xi_n
\end{aligned}
$$

$$
\begin{aligned}
&= [1+(\alpha_n+\gamma_n)^2]\|x_n-q\|^2-2(\alpha_n-\gamma_n')\|x_{n+1}-q\|^2-2(\gamma_n'+\gamma_n)\|x_{n+1}-q\|^2 \\
&\quad +2\{\{\alpha_n L[(1+L+2\beta_n L)(1+\alpha_n' L^2)+(\alpha_n'+2\alpha_n(1+\alpha_n')L^2))L^2]\|x_n-q\|] \\
&\quad +\alpha_n L[(2+L+2\beta_n L+2\alpha_n L^2)\gamma_n'+\gamma_n]M\}+2\alpha_n\gamma_n M\}\|x_{n+1}-q\| \\
&\quad +2\alpha_n[\langle T_i(PT_i)^{k-1}x_{n+1}-q,j(x_{n+1}-q)\rangle-\|x_{n+1}-q\|^2-\frac{\mu_n}{2}\phi(\|x_{n+1}-q\|) \\
&\quad -\frac{\xi_n}{2}+\phi(\|x_{n+1}-q\|)]-2\alpha_n\phi(\|x_{n+1}-q\|)+2\alpha_n\|x_{n+1}-q\|^2 \\
&\quad +\alpha_n\mu_n\phi(\|x_{n+1}-q\|)+\xi_n \\
&\leq [1+(\alpha_n+\gamma_n)^2]\|x_n-q\|^2-2\alpha_n\|x_{n+1}-q\|^2-2\|x_{n+1}-q\|^2 \\
&\quad +2\{\{\alpha_n L[(1+L+2\beta_n L)(1+\alpha_n' L^2)+(\alpha_n'+2\alpha_n(1+\alpha_n')L^2))L^2]\|x_n-q\|] \\
&\quad +\alpha_n L[(2+L+2\beta_n L+2\alpha_n L^2)\gamma_n'+\gamma_n]M\}+2\alpha_n\gamma_n M\}\|x_{n+1}-q\| \\
&\quad +2\alpha_n[\langle T_i(PT_i)^{k-1}x_{n+1}-q,j(x_{n+1}-q)\rangle-\|x_{n+1}-q\|^2-\frac{\mu_n}{2}\phi(\|x_{n+1}-q\|)+2\alpha_n\|x_{n+1}-q\|^2 \\
&\quad -\frac{\xi_n}{2}+\phi(\|x_{n+1}-q\|)]-2\alpha_n\phi(\|x_{n+1}-q\|)+2\alpha_n\|x_{n+1}-q\|+\alpha_n\mu_n\phi(\|x_{n+1}-q\|)+\xi_n
\end{aligned} \tag{3.11}
$$

This implies that

$$
\begin{aligned}
\|x_{n+1}-q|^2 &\leq [1+(\alpha_n+\gamma_n)^2]\|x_n-q\|^2-\|x_{n+1}-q\|^2 \\
&\quad +2\{\{\alpha_n L[(1+L+2\beta_n L)(1+\alpha_n' L^2)+(\alpha_n'+2\alpha_n(1+\alpha_n')L^2)L^2]\|x_n-q\|] \\
&\quad +\alpha_n L[(2+L+2\beta_n L+2\alpha_n L^2)\gamma_n'+\gamma_n]M\}+2\alpha_n\gamma_n M\}\|x_{n+1}-q\| \\
&\quad +2\alpha_n[\langle T_i(PT_i)^{k-1}x_{n+1}-q,j(x_{n+1}-q)\rangle-\|x_{n+1}-q\|^2-\frac{\mu_n}{2}\phi(\|x_{n+1}-q\|) \\
&\quad -\frac{\xi_n}{2}+\phi(\|x_{n+1}-q\|)]-2\alpha_n\phi(\|x_{n+1}-q\|) \\
&\quad +\alpha_n\mu_n\phi(\|x_{n+1}-q\|)+\xi_n
\end{aligned} \tag{3.12}
$$

Let $a_n=\|x_n-q\|^2, \phi(t)=\phi(\sqrt{t}), \nu_n=\alpha_n L[(1+L+2\beta_n L)(1+\alpha_n' L^2)+\alpha_n L(\alpha_n'+2\alpha_n L(1+\alpha_n' L^2))L^2$ and $\omega_n=\alpha_n L[(2+L+2\beta_n L+2\alpha_n L^2)\gamma_n'+\gamma_n]M+2\alpha_n\gamma_n M$.

Using the above information, (3.12) becomes

$$\begin{aligned} a_{n+1} &\leq (1+\alpha_n^2+2\alpha_n\gamma_n+\gamma_n^2)a_n - a_{n+1} + 2(\nu_n\|x_n-q\|+\omega_n)\|x_{n+1}-q\| \\ &\quad +2\alpha_n\sigma_n+\alpha_n\mu_n\phi(a_{n+1})+\xi_n-2\alpha_n\phi(a_{n+1}) \end{aligned} \tag{3.13}$$

By using $2ab \leq a^2+b^2$, (3.13) becomes

$$\begin{aligned} a_{n+1} &\leq (1+\alpha_n^2+2\alpha_n\gamma_n+\gamma_n^2)a_n - a_{n+1} + (\nu_n\|x_n-q\|+\omega_n)^2 + \|x_{n+1}-q\|^2 \\ &\quad +2\alpha_n\sigma_n+\alpha_n\mu_n\phi(a_{n+1})+\xi_n-2\alpha_n\phi(a_{n+1}) \\ &= (1+\alpha_n^2+2\alpha_n\gamma_n+\gamma_n^2)a_n - a_{n+1} + \nu_n^2\|x_n-q\|^2 + 2\nu_n\omega_n\|x_n-q\| + \omega_n^2 \\ &\quad +\|x_{n+1}-q\|^2+2\alpha_n\sigma_n+\alpha_n\mu_n\phi(a_{n+1})+\xi_n-2\alpha_n\phi(a_{n+1}) \\ &\leq (1+\alpha_n^2+2\alpha_n\gamma_n+\gamma_n^2)a_n - a_{n+1} + \nu_n^2\|x_n-q\|^2 + \nu_n^2+\omega_n^2\|x_n-q\|^2 + \omega_n^2 \\ &\quad +\|x_{n+1}-q\|^2+2\alpha_n\sigma_n+\alpha_n\mu_n\phi(a_{n+1})+\xi_n-2\alpha_n\phi(a_{n+1}) \\ &= (1+\alpha_n^2+2\alpha_n\gamma_n+\gamma_n^2)a_n - a_{n+1} + \nu_n^2 a_n + \nu_n^2 + \omega_n^2 a_n + \omega_n^2 + a_{n+1} \\ &\quad +2\alpha_n\sigma_n-\alpha_n(2-\mu_n)\phi(a_{n+1})+\xi_n \\ &= (1+\alpha_n^2+2\alpha_n\gamma_n+\gamma_n^2)a_n + \nu_n^2 a_n + \nu_n^2 + \omega_n^2 a_n + \omega_n^2 \\ &\quad +2\alpha_n\sigma_n-\alpha_n(2-\mu_n)\phi(a_{n+1})+\xi_n \end{aligned}$$

$$\Rightarrow a_{n+1} \leq (1+b_n)a_n - \lambda_n\phi(a_{n+1}) + c_n, \tag{3.14}$$

where $b_n = \alpha_n^2+2\alpha_n\gamma_n+\gamma_n^2+\nu_n^2+\omega_n^2, \lambda_n = \alpha_n(2-\mu_n)$ and $c_n = \nu_n^2+\omega_n^2+2\alpha_n\sigma_n+\xi_n$

From conditions [(i),(iii)], we have

$$\sum_{n=1}^{\infty} b_n < \infty, \sum_{n=1}^{\infty} c_n < \infty, \sum_{n=1}^{\infty} \lambda_n = \infty. \tag{3.15}$$

Again, from (3.14), (3.15) and Lemma 2.2, we obtain

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \|x_n-q\| = 0.$$

Thus, $\lim_{n\to\infty} x_n = q \in F = \cap_{i1}^{N} F((T_i \cap F(S_i))$.

(Necessity) Suppose that $\lim_{n\to\infty} x_n = q \in F$. Then, we can choose an arbitrary continous strictly increasing function $\phi : [0,\infty) \longrightarrow [0,\infty)$, with $\phi(0) = 0$, such that $\phi((t)) = t$

and $\lim_{n\to\infty}\phi(\|x_{n+1}-q\|)=0$. Since each $T_i(i=1,2,\cdots,N)$ is total asymptotically quasi-pseudocontractive-type mapping, for any $q\in F(T_i)\supseteq F$, we obtain

$$
\begin{aligned}
0 \;\geq\; & \limsup_{n\to\infty}\inf_{j(x_{n+1}-q)\in J(x_{n+1}-q)}(\langle T_i(PT_i)^{k-1}x_{n+1}-q, j(x-q)\rangle-\|x_{n+1}-q\| \\
& -\frac{\mu_{in}}{2}\phi(\|x_{n+1}-q\|)-\frac{\xi_n}{2})+\phi(\|x_{n+1}-q\|)).
\end{aligned} \tag{3.16}
$$

Hence,

$$
\begin{aligned}
0 &= 0+0 \\
&\geq \limsup_{n\to\infty}\inf_{j(x_{n+1}-q)\in J(x_{n+1}-q)}(\langle T_i(PT_i)^{k-1}x_{n+1}-q, j(x-q)\rangle-\|x_{n+1}-q\| \\
&\quad -\frac{\mu_{in}}{2}\phi(\|x_{n+1}-q\|)-\frac{\xi_n}{2})]+\lim_{n\to\infty}\phi(\|x_{n+1}-q\|) \\
&= .\limsup_{n\to\infty}\inf_{j(x_{n+1}-q)\in J(x_{n+1}-q)}[\langle T_i(PT_i)^{k-1}x_{n+1}-q, j(x-q)\rangle-\|x_{n+1}-q\| \\
&\quad -\frac{\mu_{in}}{2}\phi(\|x_{n+1}-q\|)-\frac{\xi_n}{2}+\phi(\|x_{n+1}-q\|)]
\end{aligned}
$$

This completes the proof. □

Corollary 3.2. Let E be a real Banach space and K a nonexpansive retract of E. Let $T_i:K\longrightarrow E$ $(i=1,2,\cdots,N)$ be N-uniformly L_i''-Lipschitizian nonself mappings and each T_i be total asymptotically quasi pseudocontractive-type nonself mapping with the sequence $\mu_{in}\in[1,\infty):\mu_{in}\to 1$ $(i=1,2,\cdots,N)$ as $n\to\infty$. Let $\{x_n\}$ be a sequence defined by

$$
\begin{cases}
x_1\in K \\
\qquad x_{n+1}=P((1-\alpha_n-\gamma_n)x_n+\alpha_nT_i(PT_i)^{k-1}\tau_{n+1}+\gamma_nu_n) \\
\qquad y_{n+1}=P((1-\alpha_n'-\gamma_n')x_n+\alpha_n'T_i(PT_i)^{k-1}\rho_{n+1}+\gamma_n'v_n)
\end{cases}, \tag{3.17}
$$

where $\tau_{n+1}=(1-\beta_n)y_n+\beta_nT_i(PT_i)^{k-1}y_{n+1}, \rho_{n+1}=(1-\beta_n')x_n+\beta_n'T_i(PT_i)^{k-1}x_n$ $\{\alpha_n\},\{\gamma_n\},\{\beta_n\},\{\alpha_n'\},\{\gamma_n'\},\{\beta_n'\}\in[0,1]$ and $\{u_n\},\{v_n\}\subset K$ are two bounded sequences. Suppose $F=\cap_{i=1}^{N}(F(S_i)\cap F(T_i))\neq\emptyset$. If the following conditions hold:

i. $\sum_{n=1}^{\infty}(\mu_n^2-1)<\infty, 0<\beta<\alpha_n<1, \sum_{n=1}^{\infty}\alpha_n=\infty, \sum_{n=1}^{\infty}\alpha_n^2<\infty, \sum_{n=1}^{\infty}\gamma_n<\infty, \sum_{n=1}^{\infty}\gamma_n'<\infty$;

ii. $\gamma_n+\gamma_n'\geq 1, \alpha_n+\gamma_n\geq 1, \alpha_n'+\gamma_n'\geq 1$;

iii. $\sum_{n=1}^{\infty}\alpha_n\beta_n<\infty, \sum_{n=1}^{\infty}\alpha_n\alpha_n'<\infty, \sum_{n=1}^{\infty}\alpha_n\gamma_n'<\infty, \alpha_n\geq\gamma_n'$.

Then, the sequence defined by (3.17) converges strongly to the fixed point $q\in F$ if and only if there exists a strictly increasing function $\phi:[0,\infty)\longrightarrow[0,\infty)$ with $\phi(0)=0$ such that

$$\begin{aligned} 0 \geq & \limsup_{n\to\infty} \inf_{j(x_{n+1}-q)\in J(x_{n+1}-q)} (\langle T_i(PT_i)^{k-1}x_{n+1}-q, j(x-q)\rangle - \|x_{n+1}-q\| \\ & -\frac{\mu_{in}}{2}\phi(\|x_{n+1}-q\|)-\frac{\xi_n}{2})+\phi(\|x_{n+1}-q\|)). \end{aligned}$$

Proof. Let $S_i=I$, where I is an identity mapping, in (3.2). Then, the results follows as in the proof of Theorem 3.1 by putting $S_i=I$. □

Remark 3.1. If $\mu_n=\xi_n=0$ and $S_i=I$, where I is an identity mapping, then (3.2) reduces to (12) in [26]. Thus (13), (14), (15) and (16) in [26] are special cases of (3.2).

Corollary 3.3. Let E be a real Banach space and K a nonexpansive retract of E. Let $T_i:K\longrightarrow E$ $(i=1,2,\cdots,N)$ be N-uniformly L_i''-Lipschitizian nonself mappings and each T_i be total asymptotically quasi pseudocontractive-type nonself mapping with the sequence $\mu_{in}\in[1,\infty):\mu_{in}\to 1$ $(i=1,2,\cdots,N)$ as $n\to\infty$. Let $\{x_n\}$ be a sequence defined by

$$\left.\begin{aligned} & x_1\in K \\ & x_{n+1}=P((1-\alpha_n-\gamma_n)x_n+\alpha_nT_i(PT_i)^{k-1}y_n+\gamma_nu_n) \\ & y_n=P((1-\alpha_n'-\gamma_n')x_n+\alpha_n'T_i(PT_i)^{k-1}x_n+\gamma_n'v_n) \end{aligned}\right\}, \tag{3.18}$$

where $\{\alpha_n\},\{\gamma_n\},\{\beta_n\}\in[0,1]$ and $\{u_n\},\{v_n\}\subset K$ are two bounded sequences. Suppose $F=F(T_1)\cap F(T_2)\neq\emptyset$. If the following conditions hold:

i. $\sum_{n=1}^{\infty}\mu_{in}<\infty, \sum_{n=1}^{\infty}\alpha_n=\infty, \sum_{n=1}^{\infty}\alpha_n^2<\infty, \sum_{n=1}^{\infty}\xi_n<\infty, \sum_{n=1}^{\infty}\gamma_n<\infty, \sum_{n=1}^{\infty}\gamma_n'<\infty$;

ii. $\gamma_n+\gamma_n'\geq 1, \alpha_n+\gamma_n\geq 1, \alpha_n'+\gamma_n'\geq 1$;

iii. $\sum_{n=1}^{\infty}\alpha_n\alpha_n'<\infty, \sum_{n=1}^{\infty}\alpha_n\gamma_n'<\infty, \alpha_n\geq\gamma_n'$.

Then, the sequence defined by (3.18) converges strongly to the fixed point $q \in F$ if and only if there exists a strictly increasing function $\phi : [0,\infty) \longrightarrow [0,\infty)$ with $\phi(0) = 0$ such that

$$\begin{aligned} 0 \geq & \limsup_{n\to\infty} \inf_{j(x_{n+1}-q)\in J(x_{n+1}-q)} (\langle T_i(PT_i)^{k-1}x_{n+1} - q, j(x-q)\rangle - \|x_{n+1} - q\| \\ & -\frac{\mu_{in}}{2}\phi(\|x_{n+1} - q\|) - \frac{\xi_n}{2}) + \phi(\|x_{n+1} - q\|)). \end{aligned}$$

Proof. Let $S_i = I$, where I is an identity mapping, and $\beta_n = \beta_n' = 0$ in (3.2). Then, the results follows as in the proof of Theorem 3.1 by putting $S_i = I$, where I is an identity mapping, and $\beta_n = \beta_n' = 0$. □

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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