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STABILITY ANALYSIS OF A DISCRETE-TIME FRACTIONAL GRAY-SCOTT SYSTEM

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Abstract. This paper introduces a discrete-time fractional stability framework for the Gray-Scott reaction-diffusion system, representing a novel contribution to the field. By employing the Caputo $\hbar$ -nabla operator, our model effectively incorporates memory effects inherent in many complex systems, providing a formulation well-suited for numerical simulation. The research objective is to analyze the stability of this discrete fractional system. The methodology involves a detailed spectral analysis to determine the local asymptotic stability of equilibrium points and the use of a discrete Lyapunov method to establish sufficient conditions for global stability. The results, validated through numerical simulations, demonstrate that the fractional order is a key parameter influencing the system's convergence rate and dynamical behavior. This study provides a rigorous mathematical framework and contributes to a deeper understanding of non-integer order dynamics, with significant potential applications in modeling biological pattern formation and designing distributed control systems where memory effects are crucial.

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1. INTRODUCTION

Recent advances in fractional calculus have supported the modeling, numerical simulation, and stability analysis of nonlinear systems with memory. These developments include a discrete fractional-order model for in vitro fertilization dynamics [1], an ultraspherical collocation method for the time-fractional FitzHugh–Nagumo equation [2], and state-feedback control for a fractional-order predator–prey model [3]. Together, these studies motivate further investigation of discrete fractional reaction–diffusion systems, including the Gray–Scott model considered here.

Reaction-diffusion models have proven to be fundamental in capturing the dynamics of spatially extended systems across various fields, such as physics, chemistry, and biology [4, 5, 6]. These models describe the interaction and diffusion of substances over space and time, where the local concentration of a substance at any given point is influenced by both its intrinsic properties and the properties of neighboring points. The Gray-Scott model, a classical polynomial mass-action model, has long been a cornerstone of reaction-diffusion theory. Its nonlinear dynamics, such as pattern formation, self-replication, and spatiotemporal oscillations, provide a deep understanding of complex, real-world phenomena. The model’s ability to characterize essential features of autocatalytic reactions has earned it a prominent place in both theoretical and applied studies, establishing it as a benchmark for further investigations into chemical and biological systems [7, 15].

As research into complex systems has progressed, it has become increasingly clear that many natural processes exhibit memory and hereditary effects that cannot be captured by classical integer-order models. To address this, the incorporation of fractional derivatives into reaction-diffusion systems has emerged as a promising extension of classical models. Fractional calculus, a generalization of differentiation and integration to non-integer orders, enables the modeling of long-range temporal dependencies and nonlocal interactions, which are often observed in

real-world systems. This approach allows for a more nuanced representation of physical, chemical, and biological phenomena that exhibit complex, nonlocal behaviors and temporal memory [8, 9]. These features have made fractional calculus particularly useful in modeling systems in physics, biomedicine, and control theory, where such effects are ubiquitous [10, 11].

The integration of fractional derivatives into reaction-diffusion models enhances the ability to accurately represent phenomena where memory effects and nonlocal interactions are critical. Specifically, time-fractional reaction-diffusion systems have been explored for their capacity to model anomalous diffusion and subdiffusive behaviors, which are commonly observed in chemical and biological processes. These fractional systems extend the classical integer-order framework, offering new insights into pattern formation, oscillatory behaviors, and spatiotemporal instabilities that cannot be captured by traditional models [12, 13, 14, 15]. By incorporating fractional derivatives, these models not only broaden the range of phenomena that can be modeled but also provide a richer and more flexible framework for studying complex dynamical systems [16, 17, 18, 19, 20].

While continuous fractional models are powerful, their numerical implementation can be challenging and computationally intensive. Furthermore, many real-world systems are observed or controlled at discrete time intervals. This motivates the development of discrete-time fractional models, which offer a more natural framework for both computational simulation and alignment with sampled data from experiments. The Caputo-type discrete fractional operators, in particular, are gaining attention for their utility in digital implementations of reaction-diffusion systems [21, 22, 23, 24, 25]. However, the stability theory for such discrete systems, especially for complex nonlinear models like Gray-Scott, is not yet fully developed [26, 27, 28, 29, 30, 31, 32, 33]. While these models offer numerical tractability, they also present unique challenges, such as ensuring numerical stability and preserving the physical interpretability of memory effects in a discrete setting [34, 35, 36, 37, 38, 39, 40, 41].

Furthermore, the Gray-Scott system has become a focal point of research due to its ability to generate intricate patterns and exhibit a wide range of dynamical behaviors under fractional

dynamics. The extension of the Gray-Scott model to fractional orders adds new layers of complexity to the system, enabling the exploration of novel types of Turing patterns and reaction-diffusion instabilities. The growing body of literature on fractional reaction-diffusion equations supports both analytical and numerical approaches to solving these complex models, further enriching our understanding of their behavior in diverse contexts.

This study aims to contribute to the ongoing research on fractional reaction-diffusion systems by investigating the stability properties of a discrete fractional Gray-Scott reaction-diffusion model using the Caputo $\hbar$ -nabla operator [42, 43, 44, 45, 46]. Our primary goal is to establish a robust theoretical framework for analyzing the stability of the system, supported by detailed numerical simulations. We seek to explore how fractional dynamics influence the stability of the system and investigate the impact of fractional-order derivatives on the overall behavior of the Gray-Scott model [47, 48, 49, 50, 51]. This work is motivated by recent advancements in discrete fractional calculus [52, 53, 54, 55], and it aims to fill existing gaps in the literature by providing deeper insights into the interplay between fractional derivatives and reaction-diffusion dynamics. The primary contribution of this research is the formulation and rigorous stability analysis of a discrete-time fractional Gray-Scott model, offering a new and computationally tractable tool for studying complex systems with memory.

2. FRACTIONAL DISCRETE TOOLS AND THE MODEL

We begin with the essential definitions and theorems from discrete fractional calculus.

Definition 2.1 ([21]). *the Caputo $\hbar$ -nabla operator is provided by*

$$(1) \quad {}^C_{\hbar}\nabla_a^{\xi}f(t) = {}_{\hbar}\nabla_a^{-(n-\xi)}\nabla_{\hbar}^nf(t), \quad t \in (\hbar\mathbb{N})_{a+\xi\hbar}, \quad 0 < \xi < 1,$$

and the ξ -th order $\hbar$ -sum is defined by

$$(2) \quad {}_{\hbar}\nabla_a^{-\xi}f(t) = \frac{\hbar}{\Gamma(\xi)} \sum_{s=\frac{a}{\hbar}+1}^{\frac{t}{\hbar}} (t-s\hbar+\hbar)_{\hbar}^{(\xi-1)} f(s\hbar),$$

with the set $(\hbar\mathbb{N})_{a+\xi\hbar}$ defined by

$$(\hbar\mathbb{N})_{a+\xi\hbar} = \{a + (1-\xi)\hbar, a + (2-\xi)\hbar, \dots\},$$

We have the nonlinear fractional difference equation

$$(3) \quad {}^C_h \nabla_a^\xi f(t) = \phi(t, f(t)), \quad t \in (h\mathbb{N})_{a+\xi h}.$$

Theorem 2.2 ([38]). *Let (w^*, z^*) be an equilibrium point of (14). If all the eigenvalues of $J_{(w^*, z^*)}^*$ are in S_h^ξ ; then, (w^*, z^*) is asymptotically stable.*
where

$$(4) \quad S_h^\xi = \left\{ w \in \mathbb{C} : |\text{Arg}(w)| > \frac{\xi\pi}{2} \quad \text{or} \quad |w| > \frac{2^\xi}{h^\xi} \cos^\xi \left(\frac{\text{Arg}(w)}{\xi} \right) \right\}.$$

Theorem 2.3. [40] *The following inequality holds*

$$(5) \quad {}^C_h \nabla_a^\xi f^2(t) \leq 2f(t) {}^C_h \nabla_a^\xi f(t), \quad t \in (h\mathbb{N})_{a+\xi h}.$$

The Gray-Scott reaction-diffusion system is a classical model used to describe the dynamics of autocatalytic chemical reactions, particularly those involving an activator-substrate interaction. It was originally proposed to investigate the spatial and temporal evolution of chemical concentrations under diffusion and reaction kinetics [56, 57, 58, 59, 60, 61, 62]. This model has found applications not only in chemistry but also in biology and physics, where pattern formation phenomena—such as Turing patterns—are of significant interest.

The original continuous form of the Gray-Scott reaction-diffusion system is given as:

$$(6) \quad \begin{cases} \frac{\partial w}{\partial t} = \mu_1 \Delta w - w z^2 + F(1 - w) = \mu_1 \frac{\partial^2 w}{\partial x^2} + f(w, z), & x \in \Omega, t > 0, \\ \frac{\partial z}{\partial t} = \mu_2 \Delta z + w z^2 - (F + k)z = \mu_2 \frac{\partial^2 z}{\partial x^2} + g(w, z), & x \in \Omega, t > 0. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^n$ represents a bounded spatial domain with smooth boundary $\partial\Omega$. The variable w represents the concentration of the activator, while z is the concentration of the substrate. The parameters μ_1 and μ_2 denote the diffusion coefficients of w and z , respectively. The feed rate F corresponds to the rate at which w is supplied from an external source, and k controls the removal rate of z , ensuring the persistence of localized structures.

We consider homogeneous Neumann boundary conditions:

$$\frac{\partial w}{\partial \eta} \Big|_{\partial\Omega} = \frac{\partial z}{\partial \eta} \Big|_{\partial\Omega} = 0,$$

and initial conditions:

$$\mathfrak{w}(\mathfrak{x}, 0) = \mathfrak{w}_0(\mathfrak{x}), \quad \mathfrak{z}(\mathfrak{x}, 0) = \mathfrak{z}_0(\mathfrak{x}), \quad \mathfrak{x} \in \Omega.$$

To incorporate memory and hereditary effects often present in real-world processes—such as diffusion in heterogeneous or biological media—we introduce the fractional Caputo time derivative into the model. This generalization leads to the fractional reaction-diffusion Gray-Scott system introduced in [36]:

$$(7) \quad \begin{cases} {}_0^C D_t^\xi \mathfrak{w} = \mu_1 \Delta \mathfrak{w} - \mathfrak{w} \mathfrak{z}^2 + F(1 - \mathfrak{w}), \\ {}_0^C D_t^\xi \mathfrak{z} = \mu_2 \Delta \mathfrak{z} + \mathfrak{w} \mathfrak{z}^2 - (F + k) \mathfrak{z}, \end{cases}$$

where ${}_0^C D_t^\xi$ denotes the Caputo derivative of order $\xi \in (0, 1]$, modeling subdiffusion or anomalous transport phenomena, which are frequently observed in physical, biological, and chemical systems.

Since real-world experimental and computational measurements are discrete in nature, the continuous model (7) is not always practical for numerical implementation. Consequently, we adopt a discrete-time version of the model using a discretization scheme based on the Caputo $\hbar$ -nabla fractional difference operator, following the methodology proposed in [21]. This choice is motivated by the operator's ability to handle initial conditions naturally, which is crucial for dynamic simulations. While other discrete operators exist, the Caputo-nabla formulation provides a direct analogue to the widely-used continuous Caputo derivative and leads to numerically stable schemes. The use of time-shifted terms of the form $t + \xi \hbar$ in the reaction part of the model arises directly from the definition of the discrete fractional operator. This formulation ensures that the calculation at time t depends on past values, correctly preserving the principle of causality inherent in physical systems. Letting $x \in [0, L]$ with spatial grid points defined by $x_{i+1} = x_i + k$ for $i = 0, \dots, m$, and using central difference approximations for the spatial derivatives, we have:

$$(8) \quad \begin{cases} \frac{\partial^2 \mathfrak{w}(x, t)}{\partial x^2} \approx \frac{\mathfrak{w}_{i+1}(t) - 2\mathfrak{w}_i(t) + \mathfrak{w}_{i-1}(t)}{k^2}, \\ \frac{\partial^2 \mathfrak{z}(x, t)}{\partial x^2} \approx \frac{\mathfrak{z}_{i+1}(t) - 2\mathfrak{z}_i(t) + \mathfrak{z}_{i-1}(t)}{k^2}. \end{cases}$$

Applying the second-order forward difference operator defined in [15]:

$$(9) \quad \Delta^2 \mathfrak{w}(\ell) = \mathfrak{w}(\ell + 2) - 2\mathfrak{w}(\ell + 1) + \mathfrak{w}(\ell), \quad \ell \in \mathbb{N},$$

we can further write:

$$(10) \quad \begin{cases} \frac{\partial^2 \mathfrak{w}(x, t)}{\partial x^2} \approx \frac{\Delta^2 \mathfrak{w}_{i-1}(t)}{k^2}, \\ \frac{\partial^2 \mathfrak{z}(x, t)}{\partial x^2} \approx \frac{\Delta^2 \mathfrak{z}_{i-1}(t)}{k^2}. \end{cases}$$

Combining this spatial discretization with the Caputo $\hbar$ -nabla operator in time, we formulate the discrete-time fractional Gray-Scott model as follows:

$$(11) \quad \begin{cases} {}^C \nabla_{t_0}^{\xi} \mathfrak{w}_i(t) = \frac{\mu_1}{k^2} \Delta^2 \mathfrak{w}_{i-1}(t + \xi \hbar) - \mathfrak{w}_i(t) \mathfrak{z}_i^2(t + \xi \hbar) + F(1 - \mathfrak{w}_i(t + \xi \hbar)), \\ {}^C \nabla_{t_0}^{\xi} \mathfrak{z}_i(t) = \frac{\mu_2}{k^2} \Delta^2 \mathfrak{z}_{i-1}(t + \xi \hbar) + \mathfrak{w}_i(t + \xi \hbar) \mathfrak{z}_i^2(t + \xi \hbar) - (F + k) \mathfrak{z}_i(t + \xi \hbar), \end{cases}$$

where $t \in (\hbar \mathbb{N})_{t_0}$. We impose periodic boundary conditions, a standard assumption for theoretical analysis that simplifies the eigenvalue problem by assuming the spatial domain repeats itself.

To complete the problem formulation, we impose periodic boundary conditions:

$$(12) \quad \begin{cases} \mathfrak{w}_0(t) = \mathfrak{w}_m(t), & \mathfrak{w}_1(t) = \mathfrak{w}_{m+1}(t), \\ \mathfrak{z}_0(t) = \mathfrak{z}_m(t), & \mathfrak{z}_1(t) = \mathfrak{z}_{m+1}(t), \end{cases}$$

and specify non-negative initial conditions consistent with physical concentration values:

$$(13) \quad \mathfrak{w}_i(t_0) = \phi_1(x_i) \geq 0, \quad \mathfrak{z}_i(t_0) = \phi_2(x_i) \geq 0.$$

This formulation captures the core dynamics of the Gray-Scott reaction-diffusion system under the influence of memory effects, and it is particularly relevant for modeling physical, biological, and chemical processes that exhibit anomalous diffusion, spatial heterogeneity, and time delays.

3. LOCAL STABILITY

3.1. Local Stability of the Free Diffusions System. In this part, we establish appropriate circumstances for the local stability of the free diffusion model described below.

$$(14) \quad \begin{cases} {}^C\nabla_{t_0}^\xi \mathfrak{w}_i(t) = -\mathfrak{w}(t + \xi \hbar) \mathfrak{z}^2(t + \xi \hbar) + F(1 - \mathfrak{w}(t + \xi \hbar)), \\ {}^C\nabla_{t_0}^\xi \mathfrak{z}_i(t) = \mathfrak{w}(t + \xi \hbar) \mathfrak{z}^2(t + \xi \hbar) - (F + k) \mathfrak{z}(t + \xi \hbar). \end{cases}$$

with the initial conditions

$$(15) \quad \mathfrak{w}(t_0) = \mathfrak{w}_0 \geq 0, \quad \mathfrak{z}(t_0) = \mathfrak{z}_0 \geq 0.$$

We must determine the equilibrium point in order to investigate stability. In fact, the system's equilibrium points have the following characteristics:

$$(16) \quad \begin{cases} -\mathfrak{w}^* \mathfrak{z}^{*2} - F(\mathfrak{w}^* - 1) = 0, \\ -(k + F) \mathfrak{z}^* + \mathfrak{w}^* \mathfrak{z}^{*2} = 0. \end{cases}$$

Now, if $(F + k)^2 < \frac{F}{4}$, we are dealing with exactly three equilibrium points

$$(17) \quad \begin{cases} (\mathfrak{w}_0^*, \mathfrak{z}_0^*) = (1, 0), \\ (\mathfrak{w}_1^*, \mathfrak{z}_1^*) = \left(\frac{2\mathfrak{r}}{F + \mathfrak{s}}, \frac{F + \mathfrak{s}}{2\mathfrak{r}} \right), \\ (\mathfrak{w}_2^*, \mathfrak{z}_2^*) = \left(\frac{2\mathfrak{r}}{F - \mathfrak{s}}, \frac{F - \mathfrak{s}}{2\mathfrak{r}} \right), \end{cases}$$

where $\mathfrak{s} = \sqrt{F^2 - 4F\mathfrak{r}^2}$, and $\mathfrak{r} = F + k$.

Let's consider the following two functions:

$$(18) \quad \begin{cases} \phi_0(\mathfrak{w}(t), \mathfrak{z}(t)) = -\mathfrak{w}(t) \mathfrak{z}^2(t) + F(1 - \mathfrak{w}(t)), \\ \psi_0(\mathfrak{w}(t), \mathfrak{z}(t)) = \mathfrak{w}(t) \mathfrak{z}^2(t) - (F + k) \mathfrak{z}(t). \end{cases}$$

Then, the Jacobian matrix of the equilibrium points can be expressed as follows

$$(19) \quad J_{(\mathfrak{w}, \mathfrak{z})} = \begin{pmatrix} \frac{\partial \phi_0}{\partial \mathfrak{w}} & \frac{\partial \phi_0}{\partial \mathfrak{z}} \\ \frac{\partial \psi_0}{\partial \mathfrak{w}} & \frac{\partial \psi_0}{\partial \mathfrak{z}} \end{pmatrix} = \begin{pmatrix} -\mathfrak{z}^2 - F & -2\mathfrak{w}\mathfrak{z} \\ \mathfrak{z}^2 & 2\mathfrak{w}\mathfrak{z} - \mathfrak{r} \end{pmatrix}.$$

Theorem 3.1. *The local stability is discussed according to the following cases:*

- For the equilibrium point $(\mathfrak{w}_0^*, \mathfrak{z}_0^*)$, if $F > 0$ and $\mathfrak{r} > 0$, then, system (14) is locally asymptotically stable.
- System (14) is locally asymptotically stable for the equilibrium point $(\mathfrak{w}_1^*, \mathfrak{z}_1^*)$, if the following hold:

- If $\frac{(F^2 + F\mathfrak{s})(F^2 + F\mathfrak{s} + 4\mathfrak{r}^2(\mathfrak{r} - 2) - 2\mathfrak{r}^3) + 4\mathfrak{r}^4((\mathfrak{r} - 2)^2 + 8F)}{4\mathfrak{r}^4} > 0$,
 $\frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2}\mathfrak{r} > 2F$ and $-\frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2} - \mathfrak{r} + 2 < 0$.
- If $\frac{(F^2 + F\mathfrak{s})(F^2 + F\mathfrak{s} + 4\mathfrak{r}^2(\mathfrak{r} - 2) - 2\mathfrak{r}^3) + 4\mathfrak{r}^4((\mathfrak{r} - 2)^2 + 8F)}{4\mathfrak{r}^4} \leq 0$,
 $\frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2}\mathfrak{r} < 2F$ and $-\frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2} - \mathfrak{r} + 2 \leq 0$.
- System (14) is locally asymptotically stable for the equilibrium point $(\mathfrak{w}_2^*, \mathfrak{z}_2^*)$,
 - If $\frac{(F^2 - F\mathfrak{s})(F^2 - F\mathfrak{s} + 4\mathfrak{r}^2(\mathfrak{r} - 2) - 8\mathfrak{r}^3) + 4\mathfrak{r}^4(\mathfrak{r}(\mathfrak{r} - 2)^2 + 8F)}{4\mathfrak{r}^4} > 0$,
 $\frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2}\mathfrak{r} - 2F > 0$ and $-\frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} - \mathfrak{r} + 2 < 0$.
 - If $\frac{(F^2 - F\mathfrak{s})(F^2 - F\mathfrak{s} + 4\mathfrak{r}^2(\mathfrak{r} - 2) - 8\mathfrak{r}^3) + 4\mathfrak{r}^4(\mathfrak{r}(\mathfrak{r} - 2)^2 + 8F)}{4\mathfrak{r}^4} \leq 0$,
 $\frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2}\mathfrak{r} - 2F > 0$ and $-\frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} - \mathfrak{r} + 2 \leq 0$.

To improve readability, the complex algebraic conditions for the stability of equilibrium points $(\mathfrak{w}_1^*, \mathfrak{z}_1^*)$ and $(\mathfrak{w}_2^*, \mathfrak{z}_2^*)$ as stated in Theorem 3.1 are summarized in Table 1.

Proof. We will discuss the stability of each equilibrium

- The Jacobian matrix of first equilibrium $(\mathfrak{w}_0^*, \mathfrak{z}_0^*)$ is given by,

$$(20) \quad J_{(\mathfrak{w}_0^*, \mathfrak{z}_0^*)} = \begin{pmatrix} -F & 0 \\ 0 & -\mathfrak{r} \end{pmatrix}.$$

We address the stability of $(\mathfrak{w}_0^*, \mathfrak{z}_0^*)$. The characteristic equation of the Jacobian matrix is defined by

$$(21) \quad (\lambda + F)(\lambda + \mathfrak{r}) = 0,$$

eigenvalues of the equation (21) are

$$(22) \quad \lambda_1 = -F, \lambda_2 = -\mathfrak{r};$$

If $F > 0$ and $\mathfrak{r} > 0$, then, $\text{Arg}(\lambda_1) = \text{Arg}(\lambda_2) = \pi$. Because the eigenvalues (22) are both real. As a result, relying on Theorem 2.2, $(\mathfrak{w}_0^*, \mathfrak{z}_0^*)$ is stable (Otherwise, the system is unstable).

- We can express the Jacobian matrix of the equilibrium $(\mathfrak{w}_1^*, \mathfrak{z}_1^*)$ as follows

$$(23) \quad J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)} = \begin{pmatrix} -\frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2} & -2 \\ \frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2} - F & 2 - \mathfrak{r} \end{pmatrix}.$$

We are currently investigating the stability of the point $(\mathfrak{w}_1^*, \mathfrak{z}_1^*)$. The Jacobian matrix's characteristic equation is:

$$(24) \quad \lambda^2 - \text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)})\lambda + \det(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) = 0,$$

to investigate the stability of the eigenvalues problem, we calculate:

$$(25) \quad \text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) = -\frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2} - \mathfrak{r} + 2, \quad \det(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) = \frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2}\mathfrak{r} - 2F,$$

which leads us to the following discriminant

$$\begin{aligned} \Delta_1 &= (\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}))^2 - 4\det(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}), \\ &= \frac{(F^2 + F\mathfrak{s})(F^2 + F\mathfrak{s} + 4\mathfrak{r}^2(\mathfrak{r} - 2) - 2\mathfrak{r}^3) + 4\mathfrak{r}^4((\mathfrak{r} - 2)^2 + 8F)}{4\mathfrak{r}^4}. \end{aligned}$$

To evaluate the solutions of the characteristic equation, we investigate the sign of the discriminant ∇_1 according to the following cases:

- If $\Delta_1 > 0$, and $\det(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) > 0$. As a result, the sign of $\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)})$ determines the negativity of the eigenvalues, and the eigenvalues λ_1 and λ_2 are real and expressed in (26)

$$(26) \quad \lambda_1 = \frac{\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) + \sqrt{\Delta_1}}{2}, \quad \lambda_2 = \frac{\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) - \sqrt{\Delta_1}}{2}.$$

- * If $\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) < 0$, then we have $\lambda_1 < 0$. As a result, $\text{Arg}(\lambda_1) = \pi$. Because the λ_1, λ_2 are both real. It is self-evident that $\text{Arg}(\lambda_1) = \text{Arg}(\lambda_2) = \pi$. As a result, according to Theorem 2.2, $(\mathfrak{w}_1^*, \mathfrak{z}_1^*)$ is asymptotically stable.

- * If $\text{tr}(\bar{J}_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)})^* < 0$, then we have $\lambda_2 > 0$. Therefore, $\text{Arg}(\lambda_1) = \text{Arg}(\lambda_2) = 0$, relying on Theorem 2.2 system (14) is unstable.

- If $\Delta_1 < 0$ and $\det(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) > 0$, then The eigenvalues of the problem are defined in (27).

$$(27) \quad \lambda_1 = \frac{\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)})}{2} + i\frac{\sqrt{\Delta_1}}{2}, \quad \lambda_2 = \frac{\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)})}{2} - i\frac{\sqrt{\Delta_1}}{2}.$$

We investigate the answers relying on the sign of $\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)})$. We can evaluate the solutions using the sign of $\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)})$.

- * If $\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) < 0$ or $\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) > 0$. The system (14) is then asymptotically stable in the identical case explored previously.
- * If $\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) = 0$ then

$$\text{Arg}\left(\frac{-i\sqrt{-\Delta_1}}{2}\right) = \text{Arg}\left(\frac{i\sqrt{-\Delta_1}}{2}\right) = \frac{\pi}{2},$$

and system (14) is stable.

- If $\Delta_1 = 0$ and $\det(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) > 0$, $\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)})$ cannot be equal to zero. The sign of the λ_1, λ_2 is identical to the sign of $\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)})$. As a consequence, $(\mathfrak{w}_1^*, \mathfrak{z}_1^*)$ is asymptotically stable for any $\xi \in (0, 1]$. if $\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) < 0$ and unstable if $\text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) > 0$.
- Moving on to the stability of the equilibrium $(\mathfrak{w}_2^*, \mathfrak{z}_2^*)$, we have the Jacobian matrix:

$$(28) \quad J_{(\mathfrak{w}_2^*, \mathfrak{z}_2^*)} = \begin{pmatrix} -\frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} & -2 \\ \frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} - F & 2 - \mathfrak{r} \end{pmatrix}.$$

In order to access the solutions of the characteristic equation associated with the Jacobian matrix $J_{(\mathfrak{w}_2^*, \mathfrak{z}_2^*)}$, we evaluate the following

$$(29) \quad \text{tr}(J_{(\mathfrak{w}_2^*, \mathfrak{z}_2^*)}) = -\frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} - \mathfrak{r} + 2, \quad \det(J_{(\mathfrak{w}_2^*, \mathfrak{z}_2^*)}) = \frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} \mathfrak{r} - 2F,$$

$$\begin{aligned} \Delta_2 &= (\text{tr}(J_{(\mathfrak{w}_2^*, \mathfrak{z}_2^*)}))^2 - 4\det(J_{(\mathfrak{w}_2^*, \mathfrak{z}_2^*)}), \\ &= \frac{(F^2 - F\mathfrak{s})(F^2 - F\mathfrak{s} + 4\mathfrak{r}^2(\mathfrak{r} - 2) - 8\mathfrak{r}^3) + 4\mathfrak{r}^4(\mathfrak{r}(\mathfrak{r} - 2)^2 + 8F)}{4\mathfrak{r}^4}, \end{aligned}$$

Now, for the third equilibrium, we follow the same steps discussed regarding the stability of the equilibrium (u_1^*, v_1^*) , the results of the discussion may be found in Theorem 3.1.

□

TABLE 1. Summary of Stability Conditions for Equilibrium Points from Theorem 3.1.

Equilibrium Point	Condition on Discriminant (Δ)	Stability Conditions
(w_1^*, z_1^*)	$\Delta_1 > 0$	$\det(J) > 0$ and $\text{tr}(J) < 0$
	$\Delta_1 \leq 0$	$\det(J) > 0$ and $\text{tr}(J) \leq 0$
(w_2^*, z_2^*)	$\Delta_2 > 0$	$\det(J) > 0$ and $\text{tr}(J) < 0$
	$\Delta_2 \leq 0$	$\det(J) > 0$ and $\text{tr}(J) \leq 0$

3.2. Local Stability of the Diffusion System. With diffusion, (w^*, z^*) can be stable given specific conditions. We will use the same method as in [39], and we start by computing the eigenvalues of the next equation:

$$(30) \quad \begin{cases} \Delta^2 g_{i-1}(t) = \theta_i g_i(t), & t \in \mathbb{N}, \\ g_0(t) = g_1(t), & t \in \mathbb{N}, \\ g_N(t) = g_{N-1}(t), & t \in \mathbb{N}, \end{cases}$$

then, we obtain

$$(31) \quad \begin{cases} {}^C\nabla_{t_0}^\xi w_i(t) = -\frac{\mu_1}{k^2} \theta_i w_i(t + \xi h) - w_i(t + \xi h) z_i^2(t + \xi h) + F(1 - w_i(t + \xi h)), \\ {}^C\nabla_{t_0}^\xi z_i(t) = -\frac{\mu_2}{k^2} \theta_i^* z_i(t + \xi h) + w_i(t + \xi h) z_i^2(t + \xi h) - (F + k) z_i(t + \xi h). \end{cases}$$

linearizing the reaction-diffusion system (31) with respect to (u^*, v^*) , we arrive at the following:

$$(32) \quad J_{i(w,z)} = \begin{pmatrix} -\frac{\mu_1}{k^2} \theta_i - z_i^2 - F & -2w_i z_i \\ z_i^2 & -\frac{\mu_2}{k^2} \theta_i + 2w_i z_i - \tau \end{pmatrix}.$$

Theorem 3.2. *System (31) is asymptotically stable at (w^*, z^*) if the following hold:*

- (w_0^*, z_0^*) is asymptotically stable if $\frac{\mu_1}{k^2} \theta_i - F < 0$ and $\frac{\mu_2}{k^2} \theta_i - \tau < 0$.
- (w_1^*, z_1^*) is asymptotically stable if the following hold
 - Suppose that

$$-2\frac{\mu_1 \mu_2}{k^4} \frac{F^2 + F\mathfrak{s}}{2\tau^2} (-\tau + 2) > \left(\frac{\mu_1}{k^2} - \frac{\mu_2}{k^2} \right)^2 \Delta,$$

and we have the two cases:

$$* \text{ If } \mu_1 < \mu_2 \quad \text{and} \quad \frac{\mu_1}{k^2} \theta_i - \frac{F^2 + F\mathfrak{s}}{2\tau^2} \geq 0.$$

* If $\mu_1 > \mu_2$ and $\frac{\mu_1}{k^2} \theta_i - \frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2} \geq 0$.

in addition the eigenvalues satisfies $\text{Arg}(\lambda_j(\theta_i)) > \frac{\xi\pi}{2}$,

where

$$\lambda_j(\theta_i) = \frac{\text{tr}(\bar{J}_i^*(\mathfrak{w}_1^*, \mathfrak{z}_1^*)) \pm \sqrt{\text{tr}(\bar{J}_i^*(\mathfrak{w}_1^*, \mathfrak{z}_1^*))^2 - 4\det(\bar{J}_i^*(\mathfrak{w}_1^*, \mathfrak{z}_1^*))}}{2}, \quad j = 1, 2,$$

– We suppose that

$$-2\frac{\mu_1\mu_2}{k^4} \frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2} (-\mathfrak{r} + 2) < \left(\frac{\mu_1}{k^2} - \frac{\mu_2}{k^2}\right)^2 \Delta,$$

and we have the eigenvalues are complex described as follows

$$\lambda_j(\theta_i) = \frac{\text{tr}(\bar{J}_i^*(u_1^*, v_1^*)) \pm i\sqrt{\text{tr}(\bar{J}_i^*(\mathfrak{w}_1^*, \mathfrak{z}_1^*))^2 - 4\det(\bar{J}_i^*(\mathfrak{w}_1^*, \mathfrak{z}_1^*))}}{2}, \quad j = 1, 2,$$

and must satisfy $\text{Arg}(\lambda_j(\theta_i)) > \frac{\xi\pi}{2}$.

- $(\mathfrak{w}_2^*, \mathfrak{z}_2^*)$ is asymptotically stable if the following hold

– Suppose that

$$-2\frac{\mu_1\mu_2}{k^4} \frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} (-\mathfrak{r} + 2) > \left(\frac{\mu_1}{k^2} - \frac{\mu_2}{k^2}\right)^2 \Delta^*,$$

and we have the two cases:

* If $\mu_1 < \mu_2$ and $\frac{\mu_1}{k^2} \theta_i - \frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} \geq 0$.

* If $\mu_1 > \mu_2$ and $\frac{\mu_1}{k^2} \theta_i - \frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} \geq 0$.

in addition the eigenvalues satisfies $\text{Arg}(\lambda_j(\theta_i)) > \frac{\xi\pi}{2}$,

where

$$\lambda_j(\theta_i) = \frac{\text{tr}(\bar{J}_i^*(\mathfrak{w}_2^*, \mathfrak{z}_2^*)) \pm \sqrt{\text{tr}(\bar{J}_i^*(\mathfrak{w}_2^*, \mathfrak{z}_2^*))^2 - 4\det(\bar{J}_i^*(\mathfrak{w}_2^*, \mathfrak{z}_2^*))}}{2}, \quad j = 1, 2,$$

– We suppose that

$$-2\frac{\mu_1\mu_2}{k^4} \frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} (-\mathfrak{r} + 2) < \left(\frac{\mu_1}{k^2} - \frac{\mu_2}{k^2}\right)^2 \Delta,$$

and we have the eigenvalues are complex described as follows

$$\lambda_j(\theta_i) = \frac{\text{tr}(\bar{J}_i^*(\mathfrak{w}_2^*, \mathfrak{z}_2^*)) \pm i\sqrt{\text{tr}(\bar{J}_i^*(\mathfrak{w}_2^*, \mathfrak{z}_2^*))^2 - 4\det(\bar{J}_i^*(\mathfrak{w}_2^*, \mathfrak{z}_2^*))}}{2}, \quad j = 1, 2,$$

and must satisfy $\text{Arg}(\lambda_j(\theta_i)) > \frac{\xi\pi}{2}$.

Proof. • Concerning the first equilibrium $(\mathfrak{w}_0^*, \mathfrak{z}_0^*)$, the Jacobian matrix of the system (31) for the equilibrium $(\mathfrak{w}_0^*, \mathfrak{z}_0^*)$:

$$(33) \quad J_{i(\mathfrak{w}_0^*, \mathfrak{z}_0^*)} = \begin{pmatrix} -\frac{\mu_1}{k^2}\theta_i - F & 0 \\ 0 & -\frac{\mu_2}{k^2}\theta_i - \mathfrak{r} \end{pmatrix}.$$

The Jacobian matrix is defined by the next equation:

$$(34) \quad (\lambda(\theta_i) + F - \frac{\mu_1}{k^2}\theta_i)(\lambda(\theta_i) + \mathfrak{r} - \frac{\mu_2}{k^2}\theta_i) = 0,$$

then, the eigenvalues of the problem are

$$(35) \quad \lambda_1(\theta_i) = \frac{\mu_1}{k^2}\theta_i - F, \quad \lambda_2(\theta_i) = \frac{\mu_2}{k^2}\theta_i - \mathfrak{r}.$$

if $\frac{\mu_1}{k^2}\theta_i - F < 0$ and $\frac{\mu_2}{k^2}\theta_i - \mathfrak{r} < 0$, then, $\text{Arg}(\lambda_1) = \text{Arg}(\lambda_2) = \pi$. Because the eigenvalues (35) are both real. As a result, according to Theorem 2.2, the equilibrium $(\mathfrak{w}_0^*, \mathfrak{z}_0^*)$ is stable.

• Now, we have

$$(36) \quad J_{i(\mathfrak{w}_1^*, \mathfrak{z}_1^*)} = \begin{pmatrix} -\frac{\mu_1}{k^2}\theta_i - \frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2} & -2 \\ -\frac{F^2 + F\mathfrak{s}^2}{2\mathfrak{r}^2} - F & \frac{\mu_2}{k^2}\theta_i - \mathfrak{r} + 2 \end{pmatrix}.$$

We are now investigating the stability of the point $(\mathfrak{w}_1^*, \mathfrak{z}_1^*)$. To evaluate the solutions of the characteristic equation for the Jacobian matrix $J_{(u_1^*, v_1^*)}$, We calculate the following:

$$(37) \quad \text{tr}(J_{i(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) = \left(\frac{\mu_1}{k^2} + \frac{\mu_2}{k^2}\right)\theta_i + \text{tr}(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}),$$

$$(38) \quad \det(J_{i(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) = \frac{\mu_1\mu_2}{k^4}\theta_i^2 + \left(\frac{\mu_1}{k^2}(-\mathfrak{r} + 2) - \frac{\mu_2}{k^2}\frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2}\right)\theta_i + \det(J_{(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}),$$

$$(39) \quad \Delta_i = \left(\frac{\mu_1}{k^2} - \frac{\mu_2}{k^2}\right)^2\theta_i^2 - 4\left(\frac{\mu_1}{k^2}(-\mathfrak{r} + 2) - \frac{\mu_2}{k^2}\frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2}\right)\theta_i + \Delta_1.$$

The Δ_i sign is crucial to the stability of (u^*, v^*) . The discriminant of Δ_i relative to θ_i is

$$\begin{aligned}\Delta_{\theta_i} &= 4 \left(\frac{\mu_1}{k^2}(-\tau + 2) - \frac{\mu_2}{k^2} \frac{F^2 + F\mathfrak{s}}{2\tau^2} \right)^2 - \left(\frac{\mu_1}{k^2} - \frac{\mu_2}{k^2} \right)^2 \Delta_1, \\ &= 4 \left(\frac{\mu_1}{k^2}(-\tau + 2) - \frac{\mu_2}{k^2} \frac{F^2 + F\mathfrak{s}}{2\tau^2} \right)^2 - 2 \frac{\mu_1 \mu_2}{k^4} \frac{F^2 + F\mathfrak{s}}{2\tau^2} (-\tau + 2) - \left(\frac{\mu_1}{k^2} - \frac{\mu_2}{k^2} \right)^2 \Delta_1.\end{aligned}$$

The sign of the discriminant Δ_i is crucial for the investigation of the solution of the characteristic equation of the Jacobian matrix (36). Therefore, based on the sign of Δ_{θ_i} we distinguish the following cases:

– If $\Delta_{\theta_i} > 0$, which means

$$-2 \frac{\mu_1 \mu_2}{k^4} \frac{F^2 + F\mathfrak{s}}{2\tau^2} (-\tau + 2) > \left(\frac{\mu_1}{k^2} - \frac{\mu_2}{k^2} \right)^2 \Delta_1,$$

because $\mu_1 \neq \mu_2$ we distinguish two cases,

* if $\mu_1 < \mu_2$, then $\Delta_1 > 0$, both solutions to the equation $\Delta_{\theta_i} = 0$ Both are negative. Thus, $\Delta_{\theta_i} > 0$ and we have the eigenvalues provided in (40).

$$(40) \quad \lambda_1(\theta_i) = \frac{\text{tr}(J_{i(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) + \sqrt{\Delta_i}}{2}, \quad \lambda_2(\theta_i) = \frac{\text{tr}(J_{i(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) - \sqrt{\Delta_i}}{2},$$

note that the solutions are real, also $\lambda_1(\theta_i) < 0$. In addition, if

$$\frac{\mu_1}{k^2} \theta_i - \frac{F^2 + F\mathfrak{s}}{2\tau^2} \geq 0, \text{ then, } \lambda_2(\theta_i) < 0. \text{ This leads to}$$

$$(41) \quad |\text{Arg}(\lambda_1(\theta_i))| = |\text{Arg}(\lambda_2(\theta_i))| = \pi,$$

which ensures $(\mathfrak{w}_1^*, \mathfrak{z}_1^*)$ is asymptotic stability.

* If $\mu_1 > \mu_2$, we have $\Delta_1 > 0$ This returns us to the previous scenario.

Again, for $-2 \frac{\mu_1 \mu_2}{k^4} \frac{F^2 + F\mathfrak{s}}{2\tau^2} (-\tau + 2) > \left(\frac{\mu_1}{k^2} - \frac{\mu_2}{k^2} \right)^2 \Delta_1$, $\det(J_{i(\mathfrak{w}_1^*, \mathfrak{z}_1^*)}) > 0$, and hence λ_1 and λ_2 are negative and must meet the conditions of Theorem 2.2.

– If $\Delta_{\theta_i} < 0$, which means $-2 \frac{\mu_1 \mu_2}{k^4} \frac{F^2 + F\mathfrak{s}}{2\tau^2} (-\tau + 2) < \left(\frac{\mu_1}{k^2} - \frac{\mu_2}{k^2} \right)^2 \Delta_1$ because $\mu_1 \neq \mu_2$. Thus, the roots of (31) are complex and defined in (42).

$$(42) \quad \lambda_1(\theta_i) = \frac{\text{tr}(J_{i(\mathfrak{w}_1^*, \mathfrak{z}_1^*)})}{2} + i \frac{\sqrt{\Delta_i}}{2}, \quad \lambda_2(\theta_i) = \frac{\text{tr}(J_{i(\mathfrak{w}_1^*, \mathfrak{z}_1^*)})}{2} - i \frac{\sqrt{\Delta_i}}{2},$$

and must satisfy $\text{Arg}(\lambda_j(\theta_i)) > \frac{\xi\pi}{2}$.

- Now, concerning the equilibrium $(\mathfrak{w}_2^*, \mathfrak{z}_2^*)$. We have

$$(43) \quad J_{(\mathfrak{w}_2^*, \mathfrak{z}_2^*)} = \begin{pmatrix} -\frac{\mu_1}{k^2} \theta_i - \frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} & -2 \\ -\frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} - F & \frac{\mu_2}{k^2} \theta_i^* - \mathfrak{r} + 2. \end{pmatrix}$$

To examine the solutions of the characteristic equation for $J_{(u_2^*, v_2^*)}$, We evaluate the following:

$$(44) \quad \text{tr}(J_{i^{**}(\mathfrak{w}_2^*, \mathfrak{z}_2^*)}) = \left(\frac{\mu_1}{k^2} + \frac{\mu_2}{k^2} \right) \theta_i + \text{tr}(J_{(\mathfrak{w}_2^*, \mathfrak{z}_2^*)}),$$

$$(45) \quad \det(J_{i^{**}(\mathfrak{w}_2^*, \mathfrak{z}_2^*)}) = \frac{\mu_1 \mu_2}{k^4} \theta_i^2 + \left(\frac{\mu_1}{k^2} (-\mathfrak{r} + 2) - \frac{\mu_2}{k^2} \frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} \right) \theta_i + \det(J_{(\mathfrak{w}_2^*, \mathfrak{z}_2^*)}),$$

$$(46) \quad \Delta_i^* = \left(\frac{\mu_1}{k^2} - \frac{\mu_2}{k^2} \right)^2 \theta_i^2 - 4 \left(\frac{\mu_1}{k^2} (-\mathfrak{r} + 2) - \frac{\mu_2}{k^2} \frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} \right) \theta_i + \Delta_2.$$

Also, the sign of Δ_i is critical to the stability of $(\mathfrak{w}_2^*, \mathfrak{z}_2^*)$. The discriminant of Δ_i^* in regard to θ_i is

$$\begin{aligned} \Delta_{\theta_i}^* &= 4 \left(\frac{\mu_1}{k^2} (-\mathfrak{r} + 2) - \frac{\mu_2}{k^2} \frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} \right)^2 - \left(\frac{\mu_1}{k^2} - \frac{\mu_2}{k^2} \right)^2 \Delta^*, \\ &= 4 \left(\frac{\mu_1}{k^2} (-\mathfrak{r} + 2) - \frac{\mu_2}{k^2} \frac{F^2 - F\mathfrak{s}}{2\mathfrak{r}^2} \right)^2 - 2 \frac{\mu_1 \mu_2}{k^4} \frac{F^2 + F\mathfrak{s}}{2\mathfrak{r}^2} (-\mathfrak{r} + 2) - \left(\frac{\mu_1}{k^2} - \frac{\mu_2}{k^2} \right)^2 \Delta^*. \end{aligned}$$

We follow the same steps and cases in addressing the stability of the equilibrium $(\mathfrak{w}_1^*, \mathfrak{z}_1^*)$.

□

4. GLOBAL STABILITY

In this part, we develop the Lyapunov function approach to address the global asymptotic stability of the model (11).

Theorem 4.1. *The system (11) is globally asymptotically stable if*

$$(47) \quad \min \{ \mathfrak{w}^* \mathfrak{z}^{*2} + F \mathfrak{w}^*, F \mathfrak{w}^*, \mathfrak{w}^* \mathfrak{z}^{*2}, (F + k) \mathfrak{z}^* - \mathfrak{w}^* \mathfrak{z}^{*2}, (F + k) \mathfrak{z}^* \} > 0.$$

Proof. We use the Lyapunov function (48) to show this fact, and we examine the following:

$$(48) \quad V_3(t) = \sum_{i=0}^N u^* \mathfrak{L} \left(\frac{\mathfrak{w}_i(t + \xi \hbar)}{\mathfrak{w}^*} \right), \quad V_4(t) = \sum_{i=0}^N \mathfrak{z}^* \mathfrak{L} \left(\frac{\mathfrak{z}_i(t + \xi \hbar)}{\mathfrak{z}^*} \right).$$

Now, we put

$$\bar{V}(t) = V_3(t) + V_4(t).$$

First, we must assume that $\mathfrak{w}^* \neq 0$ and $\mathfrak{z}^* \neq 0$ we have

$$\begin{aligned} {}^C \nabla_{t_0}^{\xi} V_3(t) &\leq \sum_{i=1}^N \left(1 - \frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \xi \hbar)} \right) \left(\mathfrak{w}^* \mathfrak{z}^{*2} \left(1 - \frac{\mathfrak{w}_i(t + \xi \hbar) \mathfrak{z}_i^2(t + \xi \hbar)}{\mathfrak{w}^* \mathfrak{z}^{*2}} \right) \right. \\ &\quad \left. + F u^* \left(1 - \frac{\mathfrak{w}_i(t + \xi \hbar)}{\mathfrak{w}^*} \right) \right), \\ &= \mathfrak{w}^* \mathfrak{z}^{*2} \sum_{i=1}^N \left(1 - \frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \xi \hbar)} \right) \left(1 - \frac{\mathfrak{w}_i(t + \xi \hbar) \mathfrak{z}_i^2(t + \xi \hbar)}{\mathfrak{w}^* \mathfrak{z}^{*2}} \right) \\ &\quad + F \mathfrak{w}^* \sum_{i=1}^N \left(1 - \frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \xi \hbar)} \right) \left(1 - \frac{\mathfrak{w}_i(t + \xi \hbar)}{\mathfrak{w}^*} \right), \\ &= \mathfrak{w}^* \mathfrak{z}^{*2} \sum_{i=1}^N \left(-\mathfrak{L} \left(\frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \xi \hbar)} \right) - \mathfrak{L} \left(\frac{\mathfrak{w}_i(t + \xi \hbar) \mathfrak{z}_i^2(t + \xi \hbar)}{\mathfrak{w}^* \mathfrak{z}^{*2}} \right) + \mathfrak{L} \left(\frac{\mathfrak{z}_i^2(t + \xi \hbar)}{\mathfrak{z}^{*2}} \right) \right) \\ &\quad + F \mathfrak{w}^* \sum_{i=1}^N \left(-\mathfrak{L} \left(\frac{\mathfrak{w}^*}{\mathfrak{w}_i(t + \xi \hbar)} \right) - \mathfrak{L} \left(\frac{\mathfrak{w}_i(t + \xi \hbar)}{\mathfrak{w}^*} \right) \right). \end{aligned}$$

Next, we evaluate ${}^C \Delta_{t_0}^{\xi} V_4(t)$:

$$\begin{aligned} {}^C \nabla_{t_0}^{\xi} V_4(t) &\leq \sum_{i=0}^N \left(1 - \frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \xi \hbar)} \right) \left(-\mathfrak{w}^* \mathfrak{z}^{*2} \left(1 - \frac{\mathfrak{w}_i(t + \xi \hbar) \mathfrak{z}_i^2(t + \xi \hbar)}{\mathfrak{w}^* \mathfrak{z}^{*2}} \right) \right. \\ &\quad \left. + (F + k) \mathfrak{z}^* \left(1 - \frac{\mathfrak{z}_i(t + \xi \hbar)}{\mathfrak{z}^*} \right) \right), \\ &= -\mathfrak{w}^* \mathfrak{z}^{*2} \sum_{i=1}^N \left(1 - \frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \xi \hbar)} \right) \left(1 - \frac{\mathfrak{w}_i(t + \xi \hbar) \mathfrak{z}_i^2(t + \xi \hbar)}{\mathfrak{w}^* \mathfrak{z}^{*2}} \right) \\ &\quad + (F + k) \mathfrak{z}^* \sum_{i=1}^N \left(1 - \frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \xi \hbar)} \right) \left(1 - \frac{\mathfrak{z}_i(t + \xi \hbar)}{\mathfrak{z}^*} \right), \\ &= -\mathfrak{w}^* \mathfrak{z}^{*2} \sum_{i=1}^N \left(-\mathfrak{L} \left(\frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \xi \hbar)} \right) - \mathfrak{L} \left(\frac{\mathfrak{w}_i(t + \xi \hbar) \mathfrak{z}_i^2(t + \xi \hbar)}{\mathfrak{w}^* \mathfrak{z}^{*2}} \right) \right. \\ &\quad \left. + \mathfrak{L} \left(\frac{\mathfrak{w}_i(t + \xi \hbar) \mathfrak{z}_i(t)}{\mathfrak{w}^* \mathfrak{z}^*} \right) \right) + (F + k) \mathfrak{z}^* \sum_{i=1}^N \left(-\mathfrak{L} \left(\frac{\mathfrak{z}^*}{\mathfrak{z}_i(t + \xi \hbar)} \right) - \mathfrak{L} \left(\frac{\mathfrak{z}_i(t + \xi \hbar)}{\mathfrak{z}^*} \right) \right). \end{aligned}$$

We conclude that

$$(49) \quad {}^C\nabla_{t_0}^\xi \bar{V}(t) = {}^C\nabla_{t_0}^\xi V_3(t) + {}^C\nabla_{t_0}^\xi V_4(t),$$

Then, we get

$$\begin{aligned} {}^C\nabla_{t_0}^\xi \bar{V}(t) &= {}^C\nabla_{t_0}^\xi V_3(t) + {}^C\nabla_{t_0}^\xi V_4(t), \\ &\leq \mathfrak{w}^* \mathfrak{z}^{*2} \sum_{i=1}^N \left[-\mathfrak{L} \left(\frac{\mathfrak{w}^*}{\mathfrak{w}_i(t+\xi\hbar)} \right) - \mathfrak{L} \left(\frac{\mathfrak{w}_i(t+\xi\hbar) \mathfrak{z}_i^2(t+\xi\hbar)}{\mathfrak{w}^* \mathfrak{z}^{*2}} \right) + \mathfrak{L} \left(\frac{\mathfrak{z}_i^2(t+\xi\hbar)}{\mathfrak{z}^{*2}} \right) \right] \\ &+ F \mathfrak{w}^* \sum_{i=1}^N \left[-\mathfrak{L} \left(\frac{\mathfrak{w}^*}{\mathfrak{w}_i(t+\xi\hbar)} \right) - \mathfrak{L} \left(\frac{\mathfrak{w}_i(t+\xi\hbar)}{\mathfrak{w}^*} \right) \right] - \mathfrak{w}^* \mathfrak{z}^{*2} \sum_{i=1}^N \left[-\mathfrak{L} \left(\frac{\mathfrak{z}^*}{\mathfrak{z}_i(t+\xi\hbar)} \right) \right. \\ &- \mathfrak{L} \left(\frac{\mathfrak{w}_i(t+\xi\hbar) \mathfrak{z}_i^2(t+\xi\hbar)}{\mathfrak{w}^* \mathfrak{z}^{*2}} \right) + \mathfrak{L} \left(\frac{\mathfrak{w}_i(t+\xi\hbar) \mathfrak{z}_i(t+\xi\hbar)}{\mathfrak{w}^* \mathfrak{z}^*} \right) \Big] \\ &+ (F+k) \mathfrak{z}^* \sum_{i=1}^N \left[-\mathfrak{L} \left(\frac{\mathfrak{z}^*}{\mathfrak{z}_i(t+\xi\hbar)} \right) - \mathfrak{L} \left(\frac{\mathfrak{z}_i(t+\xi\hbar)}{\mathfrak{z}^*} \right) \right], \\ &= \sum_{i=1}^N \left[-(\mathfrak{w}^* \mathfrak{z}^{*2} + F \mathfrak{w}^*) \mathfrak{L} \left(\frac{\mathfrak{w}^*}{\mathfrak{w}_i(t+\xi\hbar)} \right) - F \mathfrak{w}^* \mathfrak{L} \left(\frac{\mathfrak{w}_i(t+\xi\hbar)}{\mathfrak{w}^*} \right) \right. \\ &- ((F+k) \mathfrak{z}^* - \mathfrak{w}^* \mathfrak{z}^{*2}) \mathfrak{L} \left(\frac{\mathfrak{z}^*}{\mathfrak{z}_i(t+\xi\hbar)} \right) - (F+k) \mathfrak{z}^* \mathfrak{L} \left(\frac{\mathfrak{z}_i(t+\xi\hbar)}{\mathfrak{z}^*} \right) \\ &- \mathfrak{w}^* \mathfrak{z}^{*2} \mathfrak{L} \left(\frac{\mathfrak{w}_i(t+\xi\hbar) \mathfrak{z}_i(t+\xi\hbar)}{\mathfrak{w}^* \mathfrak{z}^*} \right) \Big], \\ &\leq -\xi \sum_{i=1}^N \left[\mathfrak{L} \left(\frac{\mathfrak{w}^*}{\mathfrak{w}_i(t+\xi\hbar)} \right) + \mathfrak{L} \left(\frac{\mathfrak{w}_i(t+\xi\hbar)}{\mathfrak{w}^*} \right) + \mathfrak{L} \left(\frac{\mathfrak{z}^*}{\mathfrak{z}_i(t+\xi\hbar)} \right) + \mathfrak{L} \left(\frac{\mathfrak{z}_i(t+\xi\hbar)}{\mathfrak{z}^*} \right) \right], \end{aligned}$$

for

$$\xi = \min \{ \mathfrak{w}^* \mathfrak{z}^{*2} + F \mathfrak{w}^*, F \mathfrak{w}^*, \mathfrak{w}^* \mathfrak{z}^{*2}, (F+k) \mathfrak{z}^* - \mathfrak{w}^* \mathfrak{z}^{*2}, (F+k) \mathfrak{z}^* \},$$

and

$$\xi \geq 0,$$

Then, we can have ${}^C\nabla_{t_0}^\xi \bar{V}(t) \leq 0$, and the system (31) is globally asymptotically stable. $\square$

5. NUMERICAL RESULTS

In this section, we provide a set of detailed numerical simulations to support and verify the theoretical findings regarding the asymptotic stability of the fractional discrete Gray-Scott

reaction-diffusion system under study. The numerical approach is implemented using a discretized scheme based on the Caputo $\hbar$ -nabla fractional operator. The purpose of these simulations is to illustrate how fractional order and variations in parameters influence the dynamics and convergence behavior of the system.

The numerical solution method is derived from a convolution-type sum inspired by the definition of the Caputo $\hbar$ -nabla derivative, and it is specifically adapted to discrete-time systems. The approach ensures compatibility with the stability framework developed in the theoretical sections. The resulting scheme for the system in Equation (11) is described below:

(50)

$$\begin{cases} \mathbf{w}_i(n\hbar) = \phi_1(x_i) + \frac{\hbar^\xi}{\Gamma(\xi)} \sum_{\ell=1}^n \frac{\Gamma(n-\ell+\xi)}{\Gamma(n-\ell+1)} \left[\frac{\mu_1}{k^2} (\mathbf{w}_{i-1}((\ell-1)\hbar) - 2\mathbf{w}_i((\ell-1)\hbar) \right. \\ \left. + \mathbf{w}_{i+1}((\ell-1)\hbar)) - \mathbf{w}_i((\ell-1)\hbar) \mathfrak{z}_i^2((\ell-1)\hbar) + F(1 - \mathbf{w}_i((\ell-1)\hbar)) \right], \quad 1 \leq i \leq m, \quad n > 0, \\ \mathfrak{z}_i(n\hbar) = \phi_2(x_i) + \frac{\hbar^\xi}{\Gamma(\xi)} \sum_{\ell=1}^n \frac{\Gamma(n-\ell+\xi)}{\Gamma(n-\ell+1)} \left[\frac{\mu_2}{k^2} (\mathfrak{z}_{i-1}((\ell-1)\hbar) - 2\mathfrak{z}_i((\ell-1)\hbar) \right. \\ \left. + \mathfrak{z}_{i+1}((\ell-1)\hbar)) + \mathbf{w}_i((\ell-1)\hbar) \mathfrak{z}_i^2((\ell-1)\hbar) - (F+k)\mathfrak{z}_i((\ell-1)\hbar) \right], \quad 1 \leq i \leq m, \quad n > 0. \end{cases}$$

Example 5.1. We begin with a numerical simulation using the parameter values $(\mu_1, \mu_2, F, k) = (0.01, 0.02, 0.38, 0.3)$ to demonstrate convergence toward equilibrium. The time and space domains are chosen as $t \in [0, 30]$ and $x \in [0, 40]$, respectively. The discretization step $\hbar$ is selected accordingly, and the fractional order is set to $\xi = 0.72$.

The boundary conditions are prescribed as $(u_0(x), v_0(x)) = (5 \sin(x), e^{-x})$, reflecting spatial variability in initial species concentration. The initial conditions for the numerical scheme are given by:

$$(51) \quad \phi_1(x) = -\cos(x), \quad \phi_2(x) = \sin(x).$$

Under this configuration, the system admits a unique positive equilibrium at $(u^*, v^*) = (1, 0)$. The time evolution of u and v is shown in Figure 1. It is observed that the values of $u(x, t)$ and $v(x, t)$ gradually converge toward their equilibrium states, confirming the asymptotic stability derived analytically. This result reflects the validity of the Lyapunov-based global stability analysis.

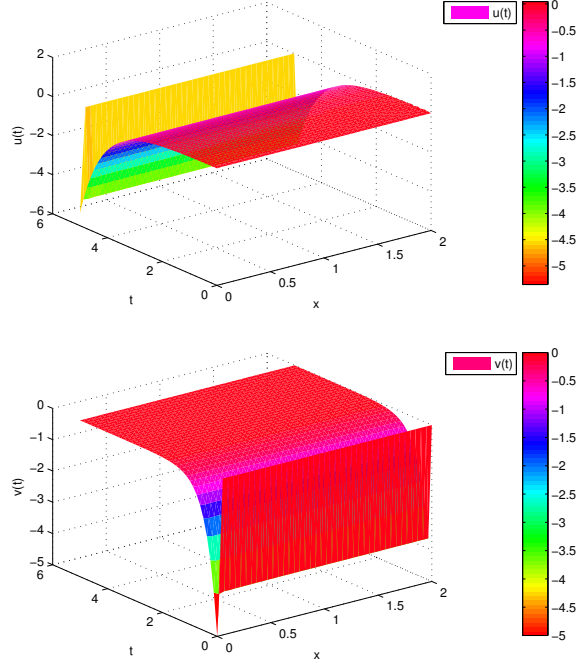


FIGURE 1. Time evolution of $u(x,t)$ (left) and $v(x,t)$ (right) governed by system (11) with $\xi = 0.72$ and $N = 120$. The solutions exhibit convergence to the equilibrium $(1,0)$.

Example 5.2. To assess the model's behavior under different conditions, a second numerical experiment is performed using the parameter values $(\mu_1, \mu_2, F, k) = (0.001, 0.001, 0.1, 0.05)$, with a simulation window of $t \in [0, 10]$ and $x \in [0, 6]$. The fractional order is chosen as $\xi = 0.85$, reflecting stronger memory effects.

The boundary conditions are defined as $(u_0(x), v_0(x)) = (\sin(x), \cos(x))$, and the initial functions are:

$$(52) \quad \phi_1(x) = 0.8 \cos(2x), \quad \phi_2(x) = -0.7 \sin(2x).$$

In this case, the system has three distinct equilibrium points:

$$(1,0), \quad \left(\frac{10 - \sqrt{10}}{3}, \frac{10 + \sqrt{10}}{30} \right), \quad \left(\frac{10 + \sqrt{10}}{3}, \frac{10 - \sqrt{10}}{30} \right).$$

Depending on the chosen initial conditions and diffusion rates, the trajectories of the solution tend toward one of the stable steady states. Figure 2 depicts the numerical results for both components u and v . The convergence pattern again supports the theoretical analysis, emphasizing the role of fractional-order dynamics in determining stability.

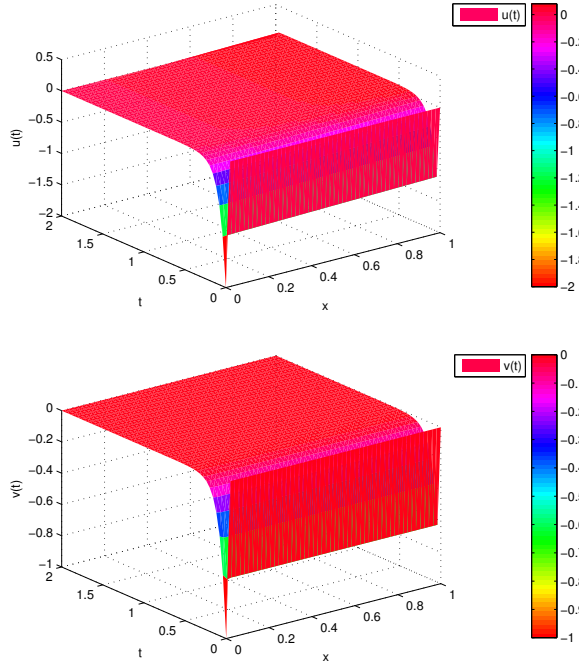


FIGURE 2. Numerical simulation of system (11) with fractional order $\xi = 0.85$ and $N = 120$. The figures show convergence of $u(x, t)$ and $v(x, t)$ toward a positive equilibrium.

5.1. Parameter Study and Comparative Discussion. To further validate our model and explore the role of the fractional order, we conducted a parameter study.

5.1.1. Impact of Fractional Order ξ . The fractional order ξ acts as a memory parameter, where lower values of ξ imply stronger memory effects. We simulated the system from Example 1 for different values of ξ (0.7, 0.85, and 1.0, where $\xi = 1$ is the classical integer-order case). The results are shown in Figure 3. As ξ decreases, the convergence to the steady-state is slower, demonstrating that memory effects can stabilize transient dynamics.

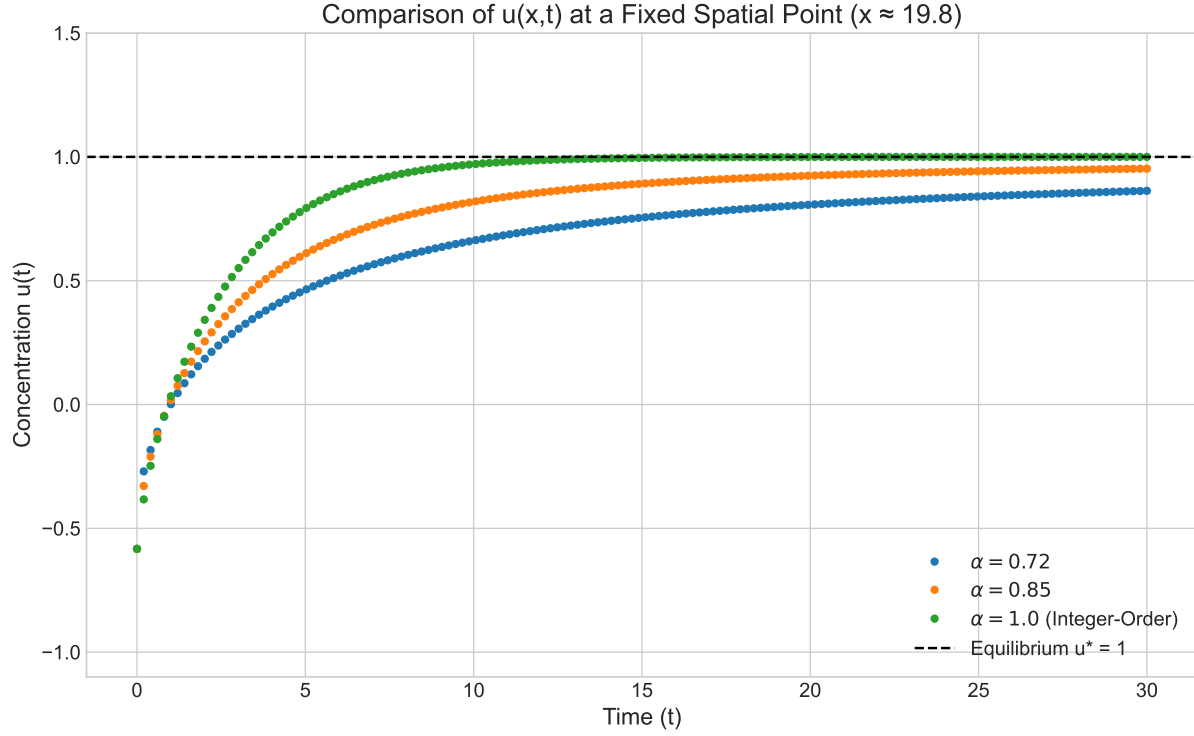


FIGURE 3. Comparison of the time evolution of $u(x,t)$ at a fixed spatial point $x = 20$ for different fractional orders ξ . The integer-order case ($\xi = 1.0$) converges fastest.

Table 2 provides a quantitative comparison of the convergence time, defined as the time required for the solution to remain within a small tolerance of the equilibrium.

TABLE 2. Quantitative Comparison of Convergence Time.

Fractional Order (ξ)	Convergence Time (approx. units)
0.72	28.5
0.85	21.3
1.00 (Integer-Order)	15.8

5.1.2. Comparison with Previous Work. Our findings are consistent with results in the continuous fractional literature [36], which also report that the fractional order is a critical parameter

for pattern formation. However, our discrete framework offers a distinct advantage in computational efficiency and direct applicability to digital control systems. Compared to the semi-discrete model in [39], our fully discrete model provides a more complete picture for numerical implementation. The stability boundaries identified in our work extend the classical Turing space, which is a known feature of fractional reaction-diffusion systems.

6. CONCLUSION

In this work, we have successfully developed and analyzed a discrete-time fractional formulation of the Gray-Scott reaction-diffusion system using the Caputo $\hbar$ -nabla operator. This study's main contribution is a rigorous theoretical framework for assessing the local and global stability of this class of systems, filling a notable gap in the literature. Our analysis, supported by numerical simulations, confirmed that the proposed model exhibits stable behavior consistent with the derived analytical conditions. Furthermore, we demonstrated that the fractional order ξ is a crucial parameter that significantly influences system dynamics, particularly the convergence rate, where stronger memory effects (smaller ξ) slow down the evolution towards equilibrium. This study provides a solid foundation for further investigations. Future work could focus on developing more advanced, higher-order numerical schemes with rigorous error analysis. Exploring the emergence of chaos or complex Turing patterns under different parameter regimes in the discrete fractional context is another promising direction. From an application standpoint, our framework can be extended to model pattern formation in multi-agent robotics or to design fractional-order controllers for distributed parameter systems, particularly in chemical engineering or biology. However, we acknowledge the limitations of our study. The stability analysis relies on certain algebraic conditions that can be complex to verify in practice. The numerical examples used mathematically convenient initial conditions, and future work should consider more realistic profiles. Additionally, the scope was limited to the Gray-Scott model with periodic boundary conditions. Future research should explore other models and boundary conditions to test the generalizability of our framework.

APPENDIX

The overall research methodology is summarized in the flowchart below.

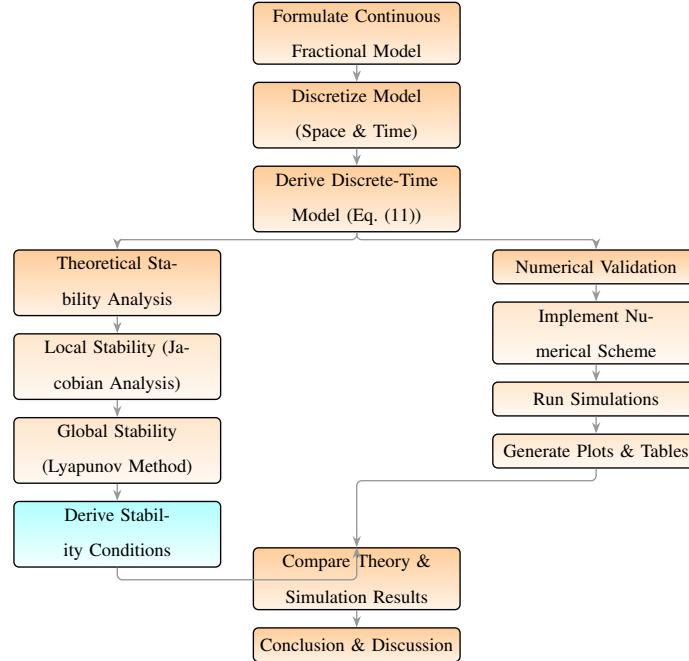


FIGURE 4. Flowchart of the research methodology.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

REFERENCES

- [1] I.M. Batiha, N. Anakira, I. Irianto, A.A.-R.M. Malkawi, T. Sasa, et al., A Discrete Fractional-Order Mathematical Model for in Vitro Fertilization Dynamics, *AIMS Math.* 11 (2026), 15120–15142. <https://doi.org/10.3934/math.2026622>.
- [2] Y.H. Youssri, M.M. Shaikh, I.M. Batiha, N. Anakira, I. Irianto, T. Sasa, An Enhanced Ultraspherical Collocation Framework with Chebyshev Nodes for the Time-Fractional FitzHugh–Nagumo Equation, *AIMS Math.* 11 (2026), 13710–13743. <https://doi.org/10.3934/math.2026565>.
- [3] I.M. Batiha, O. Oqilat, N. Anakira, T. Sasa, S. Momani, State-Feedback Control for Exponential Asymptotic Stability of a Fractional-Order Predator–Prey Model, *Int. J. Optim. Control Theories Appl.* 16 (2026), 1541–1559. <https://doi.org/10.36922/ijocta026180074>.
- [4] N.R. Anakira, A. Almalki, D. Katatbeh, G.B. Hani, A.F. Jameel, et al., An Algorithm for Solving Linear and Non-Linear Volterra Integro-Differential Equations, *Int. J. Adv. Soft Comput. Appl.* 15 (2023), 69–83.

- [5] G. Farraj, B. Maayah, R. Khalil, W. Beghami, An Algorithm for Solving Fractional Differential Equations Using Conformable Optimized Decomposition Method, *Int. J. Adv. Soft Comput. Appl.* 15 (2023), 187–196.
- [6] M. Berir, Analysis of the Effect of White Noise on the Halvorsen System of Variable-Order Fractional Derivatives Using a Novel Numerical Method, *Int. J. Adv. Soft Comput. Appl.* 16 (2024), 294–306. <https://doi.org/10.15849/ijasca.241130.16>.
- [7] B.I. Henry, S.L. Wearne, Fractional Reaction–Diffusion, *Physica A* 276 (2000), 448–455. [https://doi.org/10.1016/s0378-4371\(99\)00469-0](https://doi.org/10.1016/s0378-4371(99)00469-0).
- [8] K.S. Miller, B. Ross, An Introduction to the Fractional Calculus and Fractional Differential Equations, John Wiley & Sons, 1993. <https://zbmath.org/?q=an:0789.26002>.
- [9] K.B. Oldham, J. Spanier, The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order, Academic Press, 1974.
- [10] M.F. Silva, J.A.T. Machado, Fractional Order PD^ξ Joint Control of Legged Robots, *J. Vib. Control* 12 (2006), 1483–1501. <https://doi.org/10.1177/1077546306070608>.
- [11] F.B.M. Duarte, J.A.T. Machado, Chaotic Phenomena and Fractional-Order Dynamics in the Trajectory Control of Redundant Manipulators, *Nonlinear Dyn.* 29 (2002), 315–342. <https://doi.org/10.1023/a:1016559314798>.
- [12] V. Gafiychuk, B. Datsko, V. Meleshko, Mathematical Modeling of Time Fractional Reaction–Diffusion Systems, *J. Comput. Appl. Math.* 220 (2008), 215–225. <https://doi.org/10.1016/j.cam.2007.08.011>.
- [13] S.Z. Rida, A.M.A. El-Sayed, A.A.M. Arafa, On the Solutions of Time-Fractional Reaction–Diffusion Equations, *Commun. Nonlinear Sci. Numer. Simul.* 15 (2010), 3847–3854. <https://doi.org/10.1016/j.cnsns.2010.02.007>.
- [14] M.O. Vlad, J. Ross, Systematic Derivation of Reaction-Diffusion Equations with Distributed Delays and Relations to Fractional Reaction-Diffusion Equations and Hyperbolic Transport Equations: Application to the Theory of Neolithic Transition, *Phys. Rev. E* 66 (2002), 061908. <https://doi.org/10.1103/physreve.66.061908>.
- [15] K. Seki, M. Wojcik, M. Tachiya, Fractional Reaction-Diffusion Equation, *J. Chem. Phys.* 119 (2003), 2165–2170. <https://doi.org/10.1063/1.1587126>.
- [16] V.V. Gafiychuk, B.Y. Datsko, Pattern Formation in a Fractional Reaction–Diffusion System, *Physica A* 365 (2006), 300–306. <https://doi.org/10.1016/j.physa.2005.09.046>.
- [17] V.V. Gafiychuk, B.Y. Datsko, Stability Analysis and Oscillatory Structures in Time-Fractional Reaction-Diffusion Systems, *Phys. Rev. E* 75 (2007), 055201. <https://doi.org/10.1103/physreve.75.055201>.
- [18] C. Varea, R.A. Barrio, Travelling Turing Patterns with Anomalous Diffusion, *J. Phys.: Condens. Matter* 16 (2004), S5081–S5090. <https://doi.org/10.1088/0953-8984/16/44/006>.
- [19] C.T. Klein, B. Mayer, A Model for Pattern Formation in Gap-Junction Coupled Cells, *J. Theor. Biol.* 186 (1997), 107–115. <https://doi.org/10.1006/jtbi.1996.0337>.

- [20] J. Maselko, Mosaic Pattern Formations in Multicellular Chemical Systems, *J. Phys. Chem.* 99 (1995), 2949–2952. <https://doi.org/10.1021/j100010a003>.
- [21] G.-C. Wu, D. Baleanu, S.-D. Zeng, Z.-G. Deng, Discrete Fractional Diffusion Equation, *Nonlinear Dyn.* 80 (2015), 281–286. <https://doi.org/10.1007/s11071-014-1867-2>.
- [22] K.J. Lee, H.L. Swinney, Lamellar Structures and Self-Replicating Spots in a Reaction-Diffusion System, *Phys. Rev. E* 51 (1995), 1899–1915. <https://doi.org/10.1103/physreve.51.1899>.
- [23] K.-J. Lee, W.D. McCormick, J.E. Pearson, H.L. Swinney, Experimental Observation of Self-Replicating Spots in a Reaction–Diffusion System, *Nature* 369 (1994), 215–218. <https://doi.org/10.1038/369215a0>.
- [24] J.E. Pearson, Complex Patterns in a Simple System, *Science* 261 (1993), 189–192. <https://doi.org/10.1126/science.261.5118.189>.
- [25] V. Petrov, S.K. Scott, K. Showalter, Excitability, Wave Reflection, and Wave Splitting in a Cubic Autocatalysis Reaction-Diffusion System, *Philos. Trans. R. Soc. A* 347 (1994), 631–642. <https://doi.org/10.1098/rsta.1994.0071>.
- [26] W.N. Reynolds, J.E. Pearson, S. Ponce-Dawson, Dynamics of Self-Replicating Patterns in Reaction Diffusion Systems, *Phys. Rev. Lett.* 72 (1994), 2797–2800. <https://doi.org/10.1103/physrevlett.72.2797>.
- [27] W. Eckhaus, *Studies in Non-Linear Stability Theory*, Springer Berlin Heidelberg, 1965. <https://doi.org/10.1007/978-3-642-88317-0>.
- [28] W. Eckhaus, On Modulation Equations of the Ginzburg-Landau Type, in: *ICIAM 91: Proceedings of the Second International Conference on Industrial and Applied Mathematics*, SIAM, pp. 83–98, (1992).
- [29] N. Fenichel, Persistence and Smoothness of Invariant Manifolds for Flows, *Indiana Univ. Math. J.* 21 (1971), 193–226. <https://doi.org/10.1512/iumj.1972.21.21017>.
- [30] I. Ali, M.T. Saleem, Spatiotemporal Dynamics of Reaction–Diffusion System and Its Application to Turing Pattern Formation in a Gray–Scott Model, *Mathematics* 11 (2023), 1459. <https://doi.org/10.3390/math11061459>.
- [31] B. Liu, R. Wu, N. Iqbal, L. Chen, Turing Patterns in the Lengyel–Epstein System with Superdiffusion, *Int. J. Bifurc. Chaos* 27 (2017), 1730026. <https://doi.org/10.1142/s0218127417300269>.
- [32] F. Liu, P. Zhuang, V. Anh, I. Turner, K. Burrage, Stability and Convergence of the Difference Methods for the Space–Time Fractional Advection–Diffusion Equation, *Appl. Math. Comput.* 191 (2007), 12–20. <https://doi.org/10.1016/j.amc.2006.08.162>.
- [33] W. Mazin, K.E. Rasmussen, E. Mosekilde, P. Borckmans, G. Dewel, Pattern Formation in the Bistable Gray–Scott Model, *Math. Comput. Simul.* 40 (1996), 371–396. [https://doi.org/10.1016/0378-4754\(95\)00044-5](https://doi.org/10.1016/0378-4754(95)00044-5).
- [34] J.S. McGough, K. Riley, Pattern Formation in the Gray–Scott Model, *Nonlinear Anal. Real World Appl.* 5 (2004), 105–121. [https://doi.org/10.1016/s1468-1218\(03\)00020-8](https://doi.org/10.1016/s1468-1218(03)00020-8).

- [35] M.M. Meerschaert, C. Tadjeran, Finite Difference Approximations for Two-Sided Space-Fractional Partial Differential Equations, *Appl. Numer. Math.* 56 (2006), 80–90. <https://doi.org/10.1016/j.apnum.2005.02.008>.
- [36] S. Aljhani, M.S.M. Noorani, K.M. Saad, A.K. Alomari, Numerical Solutions of Certain New Models of the Time-Fractional Gray-Scott, *J. Funct. Spaces* 2021 (2021), 1–12. <https://doi.org/10.1155/2021/2544688>.
- [37] S. Elaydi, *An introduction to difference equations*, Springer, New York, 2005.
- [38] J. Čermák, L. Nechvátal, On a Problem of Linearized Stability for Fractional Difference Equations, *Nonlinear Dyn.* 104 (2021), 1253–1267. <https://doi.org/10.1007/s11071-021-06372-9>.
- [39] L. Xu, L.J. Zhao, Z.X. Chang, J.T. Feng, G. Zhang, Turing Instability and Pattern Formation in a Semi-Discrete Brusselator Model, *Mod. Phys. Lett. B* 27 (2013), 1350006. <https://doi.org/10.1142/s0217984913500061>.
- [40] A. Hioual, T.-E. Oussaeif, A. Ouannas, G. Grassi, I.M. Batiha, et al., New Results for the Stability of Fractional-Order Discrete-Time Neural Networks, *Alexandria Eng. J.* 61 (2022), 10359–10369. <https://doi.org/10.1016/j.aej.2022.03.062>.
- [41] M. Merdan, Analytical Approximate Solutions of Fractional Convection-Diffusion Equation with Modified Riemann-Liouville Derivative by Means of Fractional Variational Iteration Method, *Iran. J. Sci. Technol. Trans. A Sci.* 37 (2013), 83–92.
- [42] I.M. Batiha, I. Bendib, A. Ouannas, P. Agarwal, N. Anakira, et al., Finite-Time Synchronization in a Novel Discrete Fractional SIR Model for COVID-19, *Bol. Soc. Paranaense Mat.* 43 (2025), 1–15. <https://doi.org/10.5269/bspm.75295>.
- [43] A. Qazza, I. Bendib, R. Hatamleh, R. Saadeh, A. Ouannas, Dynamics of the Gierer–Meinhardt Reaction–Diffusion System: Insights into Finite-Time Stability and Control Strategies, *Partial Differ. Equ. Appl. Math.* 14 (2025), 101142. <https://doi.org/10.1016/j.padiff.2025.101142>.
- [44] S. Momani, I.M. Batiha, I. Bendib, A. Al-Nana, A. Ouannas, et al., On Finite-Time Stability of Some COVID-19 Models Using Fractional Discrete Calculus, *Comput. Methods Programs Biomed. Update* 7 (2025), 100188. <https://doi.org/10.1016/j.cmpbup.2025.100188>.
- [45] R. Hatamleh, I. Bendib, A. Qazza, R. Saadeh, A. Ouannas, et al., Finite Time Stability and Synchronization of the Glycolysis Reaction-Diffusion Model, *Int. J. Neutrosophic Sci.* 25 (2025), 371–386. <https://doi.org/10.54216/ijns.250431>.
- [46] S. Momani, I.M. Batiha, M.S. Hijazi, I. Bendib, A. Ouannas, et al., Fractional-Order SEIR Model for COVID-19: Finite-Time Stability Analysis and Numerical Validation, *Int. J. Neutrosophic Sci.* 26 (2025), 266–282. <https://doi.org/10.54216/ijns.260123>.
- [47] H. Al-Taani, M.A. Hammad, O. Alomari, I. Bendib, A. Ouannas, Finite-Time Control of the Discrete Sel'kov–Schnakenberg Model: Synchronization and Simulations, *AIP Adv.* 15 (2025), 025325. <https://doi.org/10.1063/5.0257304>.

- [48] S. Momani, I.M. Batiha, I. Bendib, A. Ouannas, A. Hioual, et al., Examining Finite-Time Behaviors in the Fractional Gray–Scott Model: Stability, Synchronization, and Simulation Analysis, *Int. J. Cogn. Comput. Eng.* 6 (2025), 380–390. <https://doi.org/10.1016/j.ijcce.2025.02.004>.
- [49] I.M. Batiha, O. Ogilat, I. Bendib, A. Ouannas, I.H. Jebril, et al., Finite-Time Dynamics of the Fractional-Order Epidemic Model: Stability, Synchronization, and Simulations, *Chaos Solit. Fractals X* 13 (2024), 100118. <https://doi.org/10.1016/j.csfx.2024.100118>.
- [50] I. Bendib, I.M. Batiha, A. Hioual, N. Anakira, M. Dalah, et al., On a New Version of Grierer-Meinhardt Model Using Fractional Discrete Calculus, *Results Nonlinear Anal.* 7 (2024), 1–15. <https://doi.org/10.31838/rna/2024.07.02.001>.
- [51] O.A. Almatroud, I. Bendib, A. Hioual, A. Ouannas, On Stability of a Reaction Diffusion System Described by Difference Equations, *J. Differ. Equ. Appl.* 30 (2024), 706–720. <https://doi.org/10.1080/10236198.2024.2322728>.
- [52] I.M. Batiha, O. Ogilat, A. Hioual, A. Ouannas, N. Anakira, et al., On Discrete FitzHugh–Nagumo Reaction–Diffusion Model: Stability and Simulations, *Partial Differ. Equ. Appl. Math.* 11 (2024), 100870. <https://doi.org/10.1016/j.padiff.2024.100870>.
- [53] T. Hamadneh, A. Hioual, R. Saadeh, M.A. Abdoon, D.K. Almutairi, et al., General Methods to Synchronize Fractional Discrete Reaction–Diffusion Systems Applied to the Glycolysis Model, *Fractal Fract.* 7 (2023), 828. <https://doi.org/10.3390/fractalfract7110828>.
- [54] M. Merdan, A. Gökdoğan, A. Yildirim, S.T. Mohyud-Din, Solution of Time-Fractional Generalized Hirota–Satsuma Coupled KdV Equation by Generalised Differential Transformation Method, *Int. J. Numer. Methods Heat Fluid Flow* 23 (2013), 927–940. <https://doi.org/10.1108/hff-09-2011-0188>.
- [55] M. Merdan, A. Gökdoğan, A. Yıldırım, S.T. Mohyud-Din, Numerical Simulation of Fractional Fornberg–Whitham Equation by Differential Transformation Method, *Abstr. Appl. Anal.* 2012 (2012), 965367. <https://doi.org/10.1155/2012/965367>.
- [56] H.M. Almimi, M. Abu Hammad, G. Farraj, I. Bendib, A. Ouannas, A New Investigation on Dynamics of the Fractional Lengyel–Epstein Model: Finite Time Stability and Finite Time Synchronization, *Computation* 12 (2024), 197. <https://doi.org/10.3390/computation12100197>.
- [57] M.A. Hammad, I. Bendib, W.G. Alshanti, A. Alshanty, A. Ouannas, et al., Fractional-Order Degrn–Harrison Reaction–Diffusion Model: Finite-Time Dynamics of Stability and Synchronization, *Computation* 12 (2024), 144. <https://doi.org/10.3390/computation12070144>.
- [58] I.M. Batiha, I. Bendib, A. Ouannas, I.H. Jebril, S. Alkhazaleh, et al., On New Results of Stability and Synchronization in Finite-Time for Fitz–Nagamo Model Using Gronwall Inequality and Lyapunov Function, *J. Robot. Control* 5 (2024), 1897–1909.

- [59] S. Alnabulsi, W.M.M. Salameh, I. Bendib, A.A. Abubaker, A. Ouannas, et al., Finite-Time Stability in the Discrete Sel'kov-Schnakenberg Reaction-Diffusion Model: Analytical Analysis and Numerical Simulations, *Int. J. Neutrosophic Sci.* 26 (2025), 155–166. <https://doi.org/10.54216/ijns.260415>.
- [60] J. Oudetallah, I. Bendib, W. Audeh, A. Ouannas, C. Aouiti, et al., Synchronization in Finite Time of Discrete Reaction-Diffusion Models, in: 2025 1st International Conference on Computational Intelligence Approaches and Applications (ICCIAA), IEEE, 2025, pp. 1–8. <https://doi.org/10.1109/icciaa65327.2025.11013053>.
- [61] I. Batiha, N. Anakira, I. Bendib, A. Ouannas, A. Hioual, et al., Finite-Time Analysis of Epidemic Reaction-Diffusion Models: Stability, Synchronization, and Numerical Insights, *PLOS ONE* 20 (2025), e0321132. <https://doi.org/10.1371/journal.pone.0321132>.
- [62] I. Bendib, A. Ouannas, M. Dalah, A Novel Analytical Framework for Investigating Mittag-Leffler Stability in Fractional Order Reaction-Diffusion Systems, *Rev. Int. Métodos Numér. Cál. Diseño Ing.* 41 (2025), 58. <https://doi.org/10.23967/j.rimni.2025.10.66114>.