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THE BIFURCATION ANALYSIS OF FOOD CHAIN PREY - PREDATOR MODEL INVOLVING HUNTING COOPERATION WITH TOXIN AND DISEASE IN TOP PREDATOR

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Abstract: This paper establishes the local bifurcation (LB) conditions for an eco-epidemiological food chain model involving prey, predator, and a top predator (susceptible and infected), incorporating fear, hunting cooperation, toxicity and anti-predator via Holling type IV and Sokol-Howell functional responses. The analysis identifies twelve equilibrium points (EPs), characterizing their asymptotic stability and identifying transcritical bifurcation (TB) occurs at M_0 , M_6 and M_7 , whereas both (TB) and pitchfork bifurcation (PB) are observed near M_1, M_2, M_3 , M_4, M_5 and M_8 . Furthermore saddle-node bifurcation (SNB) manifests at M_9 , M_{10} and M_{11} , In addition to the analytical determination of the thresholds that lead to the occurrence of Hopf branching (HB) near the positive steady state. Finally, a thorough numerical simulation verifies the analytical results. These findings shed light on how biological variables and disease affect biological community survival and pattern alterations.

Keywords: prey-predator; Sokol-Howell; hunting cooperation; anti-predator; toxin; disease; bifurcation.

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1. INTRODUCTION

In recent decades, the analysis of the dynamic behavior of food chain models has emerged as one of the most compelling subjects in mathematical ecology and the dynamics of biological communities. Natural ecosystems are not static entities but rather exist in a state of continuous transformation and adaptation driven by complex and intertwined biological and environmental factors. These dynamics exert profound influences on the nature of interspecific interactions and their stability over time [1]. Consequently, there is an urgent need to construct robust mathematical frameworks capable of capturing this complexity and uncovering the underlying mechanisms governing fundamental dynamic shifts [2, 3].

Food chain models serve as highly suitable mathematical frameworks for investigating predator prey interactions across multiple trophic levels. Building upon the classical Lotka Volterra framework, researchers have developed more realistic models that integrate complex functional responses and behavioral adaptations [3, 4]. Numerous studies have enriched this field; for instance, [4] examined a predator prey model featuring a Holling type III functional response under optimal harvesting. Recently, hunting cooperation among predators has garnered significant attention, as field observations confirm that many predatory species such as wolves, lions, and certain cetaceans rely on coordinated collective strategies. [5] explored this phenomenon within a generalist predator model, revealing that hunting cooperation fundamentally alters persistence and extinction thresholds and may lead to multi stable dynamic patterns. Similarly, [6] investigated the synergy between cooperation, the fear effect, and the Allee effect, finding that their interaction generates rich and diverse bifurcating behaviors. Furthermore, [1] provided a rigorous analytical dimension by examining multi parameter bifurcations in models incorporating both Allee and fear effects.

Beyond cooperation, anti predator behaviors represent a critical adaptive dimension. Prey species do not respond passively to predation pressure but instead employ defensive strategies such as herding, altering movement patterns, and collective warning signals. These behaviors reduce the effective predation rate and redistribute population pressure. [7] demonstrated that anti predator behavior, when coupled with the fear effect in models with Holling type IV and Crowley Martin

functional responses, induces qualitative shifts in stability regions. [8] further confirmed that the interaction between fear and anti predator behavior creates new bifurcation pathways in stage structured models. Complementarily, [9] conducted local bifurcation analyses on stage structured models, uncovering various bifurcation types dependent on parameter values.

The complexity of these systems intensifies with the introduction of disease within the top predator population, shifting the framework toward an eco epidemiological perspective as established by [10]. The spread of infection not only impairs predatory efficiency but also reshapes the entire system structure; it reduces the infected predators ability to cooperate and alleviates pressure on the middle trophic prey. [11, 12] explored this dimension, concluding that such cooperation fundamentally redraws bifurcation maps and persistence conditions. Moreover, [13] indicated that eco epidemiological models involving harvesting exhibit local stabilities distinct from disease free models. In a similar vein, [14] analyzed stability in migration models featuring Holling type III responses amidst competition and toxicity.

Environmental toxins further jeopardize the equilibrium of natural food chains. Toxins, whether biological or anthropogenic, directly impact predator growth rates or reduce their functional efficiency, potentially driving the system toward extinction or oscillation [15, 16] Kadhim and Majeed (2020) proved that exceeding specific toxicant thresholds triggers Hopf bifurcations, leading to sustained oscillatory cycles. Additionally, [17] identified novel stability regions emerging from the interaction between toxicity and anti-predator behavior.

Methodologically, bifurcation theory remains indispensable for understanding these complex systems, as it identifies critical parameter thresholds where qualitative transitions such as moving from stable equilibrium to periodic oscillations or chaos occur [18]. [19] documented bifurcation conditions in the Sokol Howell model involving fear and toxins, delineating parameter regions where stability is lost. Furthermore, [20] showed that one prey, two predator models under toxicity and harvesting exhibit saddle node and Hopf bifurcations. The role of competing predators in reshaping ecological dynamics and bifurcation structures was also highlighted by [21-23], while [15] analyzed discrete time models, revealing dynamic patterns absent in continuous time

frameworks.

Building upon this rich research heritage, the present study proposes an integrated three level food chain model. This framework combines hunting cooperation, anti-predator behavior, environmental toxicity, and disease prevalence in the top predator within a single mathematical structure. The model utilizes the Sokol Howell functional response due to its realistic capacity to account for predator saturation and adaptation [19, 24].

In the preceding chapter, we introduced and examined an ecological-epidemiological model that integrates fear, toxicity, cooperative hunting, established anti-predator and strategies featuring a diseased apex predator, while predators utilize two distinct types of functional responses to consume prey. This work aims to validate the analytical findings using numerical simulations and to ascertain the requirements for Hopf bifurcation at a positive equilibrium point, as well as local bifurcation around the equilibria of the mathematical model presented in (5.1).

2. MATHEMATICAL MODEL

Model of a food chain comprising prey, predator, and apex predator (susceptible and infected), has been shown and presented in [24]. At time T , which of the following best describes the total population density $x_1(T), x_2(T), x_s(T)$ and $x_l(T)$ in that order.

$$\left. \begin{aligned} \frac{dx_1}{dT} &= a_1 x_1 \left(1 - \frac{x_1}{k_1}\right) \left(\frac{1}{1+f_1 x_2}\right) - \frac{b_1 x_1 x_2}{c_1 + x_1^2} - \rho_1 x_1 x_2^2 - \delta_1 x_1^3 - d_1 x_1, \\ \frac{dx_2}{dT} &= a_2 x_2 \left(1 - \frac{x_2}{k_2}\right) \left(\frac{1}{1+f_2 x_s}\right) + \frac{g_1 x_1 x_2}{c_1 + x_1^2} - \eta x_1 x_2 + e_1 \rho_1 x_1 x_2^2 - \rho_2 x_2 x_s^2 - \delta_2 x_2^2 - d_2 x_2 - \frac{b_2 x_2 x_s}{c_2 + \gamma_1 x_2^2}, \\ \frac{dx_s}{dT} &= \frac{g_2 x_2 x_s}{c_2 + \gamma_1 x_2^2} + e_2 \rho_2 x_2 x_s^2 - \delta_3 x_s - \beta x_s x_l - d_3 x_s, \\ \frac{dx_l}{dT} &= \beta x_s x_l - d_4 x_l. \end{aligned} \right\} \quad (1)$$

With the following initial condition $x_1(T) \geq 0$, $x_2(T) \geq 0$, $x_s(T) \geq 0$ and $x_l(T) \geq 0$. Table (1) demonstrates the biological significance of the characteristics of system (1):

Table (1) System parameters and their biological meaning.

Parameters	Biological meaning
$a_n > 0, n = 1,2$	The rates of growth of both the prey and the predator in order
$k_n > 0, n = 1,2$	The carrying capacity of the prey and the predator accordingly
$f_n > 0, n = 1,2$	The fear range of prey species from predator and predator species from susceptible top predator in order
$b_v > 0, v = 1,2$	The prey and predator attack rates in a specific order
$c_1 > 0$	The saturation coefficient of Holling type <i>IV</i> function response of the prey and the predator accordingly
$\rho_v > 0, v = 1,2$	The hunting cooperation in prey and predator in order
$g_v > 0, g_v \leq b_v$	The rates at which food is converted into energy differ for both the predator and the apex predator
η	The anti-predator rate of the prey
$\gamma_1 > 0$	The interference constant
$c_2 > 0$	The saturation coefficient between the susceptible top predator and the predator's Sokol-Howell function response
$0 < e_v < 1, v = 1,2$	The conversion rate by attack prey and predator in order.
$\delta_v > 0, v = 1,2,3$	The toxin rate of the prey, predator and Susceptible top predator in order
$\beta > 0$	The disease transmission rate
$d_v > 0, v = 1,2$	The death rates of prey and predator by external sources in order
$d_3 > 0$	The natural death rate of susceptible top predator
$d_4 > 0$	The death rate of infected top predator by disease

3. LOCAL BIFURCATION ANALYSIS (LBA)

This part examined (LBA) of model (1), focusing on equilibrium point changes as dynamic behavior parameters change. We want to develop higher order conditions ensuring the formation of the dominant local bifurcations, as Sotomayor's theorem shows. The Jacobean matrix (JM) $J = [n_{ij}]_{4 \times 4}$ of system (1) is shown in Eq. (13.1) in [24].

It is obvious that for any vector that is not zero, $\Omega = (\Omega_1, \Omega_2, \Omega_3, \Omega_4)^T$:

$$D^2F(M, \mu)(\Omega, \Omega) = [K_{i1}]_{4 \times 1}, \quad (2)$$

$$K_{11} = 2 \left[\left(\frac{-a_1}{k_1(1+f_1x_2)} - \frac{b_1x_1x_2(3c_1-x_1^2)}{(c_1+x_1^2)^3} - 3\delta_1x_1 \right) \Omega_1^2 + \left(\left(1 - \frac{2x_1}{k_1} \right) \frac{-a_1f_1}{(1+f_1x_2)^2} + \frac{b_1(c_1-x_1^2)}{(c_1+x_1^2)^2} - 2\rho_1x_2 \right) \Omega_1\Omega_2 + \right. \\ \left. \left(a_1x_1 \left(1 - \frac{x_1}{k_1} \right) \frac{f_1^2}{(1+f_1x_2)^3} - \rho_1x_1 \right) \Omega_2^2 \right],$$

$$K_{21} = 2 \left[\frac{g_1 x_1 x_2 (x_1^2 - 3c_1)}{(c_1 + x_1)^3} \Omega_1^2 + \left(\frac{g_1 (c_1 - x_1^2)}{(c_1 + x_1^2)^2} - \eta + 2e_1 \rho_1 x_2 \right) \Omega_1 \Omega_2 + \left(\frac{-a_2}{k_2 (1 + f_2 x_s)} + e_1 \rho_1 x_1 - \delta_2 + \right. \right. \\ \left. \left. \frac{b_2 \gamma_1 x_2 x_s [3c_2 - \gamma_1 x_2^2]}{(c_2 + \gamma_1 x_2^2)^3} \right) \Omega_2^2 + \left(\left(1 - \frac{2x_2}{k_2} \right) \frac{-a_2 f_2}{(1 + f_2 x_s)^2} - 2\rho_2 x_s - \frac{b_2 (c_2 - \gamma_1 x_2^2)}{(c_2 + \gamma_1 x_2^2)^2} \right) \Omega_2 \Omega_3 + \left(a_2 x_2 \left(1 - \right. \right. \right. \\ \left. \left. \left. \frac{x_2}{k_2} \right) \frac{f_2^2}{(1 + f_1 x_s)^3} - \rho_2 x_2 \right) \Omega_3^2 \right],$$

$$K_{31} = 2 \left[\frac{g_2 \gamma_1 x_2 x_s (\gamma_1 x_2^2 - 3c_2)}{(c_2 + \gamma_1 x_2^2)^3} \Omega_2^2 + \left(\frac{g_2 (c_2 - \gamma_1 x_2^2)}{(c_2 + \gamma_1 x_2^2)^2} + 2e_2 \rho_2 x_s \right) \Omega_2 \Omega_3 + 2e_2 \rho_2 x_2 \Omega_3^2 - 2\beta \Omega_3 \Omega_4 \right],$$

$$K_{41} = 2\beta \Omega_3 \Omega_4,$$

and

$$D^3 F(M, \mu)(\Omega, \Omega, \Omega) = [R_{i1}]_{4 \times 1}, \quad (3)$$

$$R_{11} = 6 \left[\left(\frac{b_1 x_2 [c_1 (6x_1^2 - c_1) - x_1^4]}{(c_1 + x_1^2)^4} - \delta_1 \right) \Omega_1^3 + \left(\frac{a_1 f_1}{k_1 (1 + f_1 x_2)^2} - \frac{b_1 x_1 (3c_1 - x_1^2)}{(c_1 + x_1^2)^3} \right) \Omega_1^2 \Omega_2 + \left(\left(1 - \frac{2x_1}{k_1} \right) \frac{a_1 f_1^2}{(1 + f_1 x_2)^3} - \right. \right. \\ \left. \left. \rho_1 \right) \Omega_1 \Omega_2^2 - a_1 x_1 \left(1 - \frac{x_1}{k_1} \right) \frac{f_1^3}{(1 + f_1 x_2)^4} \Omega_2^3 \right],$$

$$R_{21} = 6 \left[g_1 x_2 \frac{c_2 (6x_1^2 - c_1) - x_1^4}{(c_1 + x_1^2)^4} \Omega_1^3 - g_1 x_1 \frac{(3c_1 - x_1^2)}{(c_1 + x_1^2)^3} \Omega_1^2 \Omega_2 + e_1 \rho_1 \Omega_1 \Omega_2^2 + \right. \\ \left. b_2 \gamma_1 x_s \frac{c_2 (c_2 - 6\gamma_1 x_2^2) + \gamma_1^2 x_2^4}{(c_2 + \gamma_1 x_2^2)^4} \Omega_2^3 + \left(\frac{2a_2 f_2}{k_2 (1 + f_2 x_s)^2} + b_2 \gamma_1 x_2 \frac{(3c_1 - \gamma_1 x_1^2)}{(c_2 + \gamma_1 x_2^2)^3} \right) \Omega_2^2 \Omega_3 + \left(\left(1 - \frac{2x_2}{k_2} \right) \frac{a_2 f_2^2}{(1 + f_2 x_s)^3} - \right. \right. \\ \left. \left. \rho_2 \right) \Omega_2 \Omega_3^2 - 6 \left(1 - \frac{x_2}{k_2} \right) \frac{a_2 f_2^3 x_2}{(1 + f_2 x_s)^4} \Omega_3^3 \right],$$

$$R_{31} = 6 \left[g_2 \gamma_1 x_s \frac{c_2 (c_2 - 6\gamma_1 x_2^2) + \gamma_1^2 x_2^4}{(c_2 + \gamma_1 x_2^2)^4} \Omega_2^3 - g_2 \gamma_1 x_2 \frac{3c_2 - \gamma_1 x_2^2}{(c_2 + \gamma_1 x_2^2)^3} \Omega_2^2 \Omega_3 + e_2 \rho_2 \Omega_2 \Omega_3^2 \right],$$

$$R_{41} = 0,$$

where $M = (x_1, x_2, x_s, x_l)^T$ and μ is a parameter can be.

Theorem (1): Suppose that condition (4) in [24] holds, then system (1) at the (EP) $M_0 = (0, 0, 0, 0)$ with $a_1^0 = a_1 = d_1$ has (TB), but (SNB) or (PB) cannot be done.

Proof: By employing (JM) J_0 , as specified in Eq. (14.1) in [24] the eigenvalue ($\lambda_{0x_1} = 0$) of system

(1) at $a_1^0 = a_1 = d_1$, then J_0 with $a_1^0 = a_1$ becomes

$$J_0^0 = J_0(M_0, a_1^0) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a_2 - d_2 & 0 & 0 \\ 0 & 0 & -(\delta_3 + d_3) & 0 \\ 0 & 0 & 0 & -d_4 \end{bmatrix}.$$

When the eigenvalue $\lambda_{0x_1} = 0$ then assume the eigenvector be written in $\Omega^{o[0]} = \left(\Omega_1^{o[0]}, \Omega_2^{o[0]}, \Omega_3^{o[0]}, \Omega_4^{o[0]} \right)^T$.

Hence $(J_0^o - \lambda_{0x_1} I) \Omega^{o[0]} = 0$, by condition (4) in [24] that gives $\Omega^{o[0]} = \left(\Omega_1^{o[0]}, 0, 0, 0 \right)^T$ consider $\Omega_1^{o[0]} \neq 0$ real number.

For $\lambda_{0x_1} = 0$, $\mathcal{U}^{o[0]} = \left(\mathcal{U}_1^{o[0]}, \mathcal{U}_2^{o[0]}, \mathcal{U}_3^{o[0]}, \mathcal{U}_4^{o[0]} \right)^T$ is the eigenvector of J_0^{oT} .

We find $(J_0^{oT} - \lambda_{0x_1} I) \mathcal{U}^{o[0]} = 0$, by condition (4) in [24] we get $\mathcal{U}^{o[0]} = \left(\mathcal{U}_1^{o[0]}, 0, 0, 0 \right)^T$, consider $\mathcal{U}_1^{o[0]} \neq 0$ real number.

$$\begin{aligned} \text{Currently, suppose } \frac{\partial f}{\partial a_1} &= f_{a_1}(M, a_1) = \left(\frac{\partial f_1}{\partial a_1}, \frac{\partial f_2}{\partial a_1}, \frac{\partial f_3}{\partial a_1}, \frac{\partial f_4}{\partial a_1} \right)^T \\ &= \left(x_1 \left(1 - \frac{x_1}{k_1} \right) \frac{1}{1+f_1 x_2}, 0, 0, 0 \right)^T. \end{aligned}$$

So, $f_{a_1}(M_0, a_1^o) = (0, 0, 0, 0)^T$ then $\left(\mathcal{U}^{o[0]} \right)^T f_{a_1}(M_0, a_1^o) = 0$.

As a result, using "Sotomayor's theorem," we infer that the prerequisites for a (SNB) are not met.

The first criterion for (TB) is met.

Right now,

$$Df_{a_1}(M, a_1) = \begin{bmatrix} \left(1 - \frac{2x_1}{k_1} \right) \frac{1}{(1+f_1 x_2)} & x_1 \left(1 - \frac{x_1}{k_1} \right) \frac{-f_1}{(1+f_1 x_2)^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $Df_{a_1}(M, a_1)$ is the derivative of $f_{a_1}(M, a_1)$ with respect to $M = (x_1, x_2, x_s, x_l)^T$.

Further, it is observed that

$$Df_{a_1}(M_0, a_1^o) \Omega^{o[0]} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \Omega_1^{o[0]} \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \Omega_1^{o[0]} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ so}$$

$$\left(\mathcal{U}^{o[0]} \right)^T \left[Df_{a_1}(M_0, a_1^o) \Omega^{o[0]} \right] = \left(\mathcal{U}_1^{o[0]}, 0, 0, 0 \right) \left(\Omega_1^{o[0]}, 0, 0, 0 \right)^T = \Omega_1^{o[0]} \mathcal{U}_1^{o[0]} \neq 0.$$

Moreover, by substituting $\Omega^{o[0]}$ in (2) we get:

$$D^2 f(M_0, a_1^o) \left(\Omega^{o[0]}, \Omega^{o[0]} \right) = \begin{bmatrix} \frac{-a_1}{k_1} (\Omega_1^{o[0]})^2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, it obtains that

$$\left(\mathcal{U}^{o[0]} \right)^T \left[D^2 f(M_0, a_1^o) \left(\Omega^{o[0]}, \Omega^{o[0]} \right) \right] = \frac{-a_1}{k_1} \mathcal{U}_1^{o[0]} (\Omega_1^{o[0]})^2 \neq 0.$$

According to "Sotomayor's theorem," system (1) at M_0 exhibits (TB) at $a_1^o = a_1$.

Theorem (2): Assume that conditions (15.2) in [24] when another condition are fulfilled:

$$\frac{g_1 \bar{x}_1}{c_1 + \bar{x}_1^2} < \eta \bar{x}_1 + d_2, \quad (4)$$

$$\frac{k_1}{2} < \bar{x}_1 < k_1, \quad (5)$$

$$c_1 < \bar{x}_1^2, \quad (6)$$

$$\left(\frac{g_1(\bar{x}_1^2 - c_1)}{(c_1 + \bar{x}_1^2)^2} + \eta \right) \left(\frac{J_{12}}{J_{11}} \right) \neq \left(\frac{a_2}{k_2} + \delta_2 \right), \quad (7)$$

$$\bar{x}_1^2 < 3c_1, \quad (8)$$

where

$$J_{12} = a_1 f_1 \bar{x}_1 \left(\frac{\bar{x}_1}{k_1} - 1 \right) - \frac{b_1 \bar{x}_1}{c_1 + \bar{x}_1^2}, \quad J_{11} = a_1 \left(1 - \frac{2\bar{x}_1}{k_1} \right) - 3\delta_1 \bar{x}_1^2 - d_1,$$

then system (1) at the (EP) $M_1 = (\bar{x}_1, 0, 0, 0)$ with $\bar{a}_2 = a_2 = -\frac{g_1 \bar{x}_1}{c_1 + \bar{x}_1^2} + \eta \bar{x}_1 + d_2$ has (TB) or

(PB) could happen, but (SNB) cannot be done.

Proof: By employing (JM) J_1 , as specified in Eq. (15.1) in [24], the eigenvalue ($\lambda_{1x_2} = 0$) of

system (1) at $\bar{a}_2 = a_2 = -\frac{g_1 \bar{x}_1}{c_1 + \bar{x}_1^2} + \eta \bar{x}_1 + d_2$, with condition (4) then J_1 with $\bar{a}_2 = a_2$ becomes

$$\bar{J}_1 = J_1(M_1, \bar{a}_2) = \begin{bmatrix} J_{11} & J_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -(\delta_3 + d_3) & 0 \\ 0 & 0 & 0 & -d_4 \end{bmatrix}.$$

When the eigenvalue $\lambda_{1x_2} = 0$ for $\bar{J}_1$ then assume the eigenvector be written in $\bar{\Omega}^{[1]} = (\bar{\Omega}_1^{[1]}, \bar{\Omega}_2^{[1]}, \bar{\Omega}_3^{[1]}, \bar{\Omega}_4^{[1]})^T$.

Hence $(\bar{J}_1 - \lambda_{1x_2} I)\bar{\Omega}^{[1]} = 0$, by condition (15.2) in [24] and condition (4) that gives $\bar{\Omega}^{[1]} = \left(\frac{-J_{12}}{J_{11}} \bar{\Omega}_2^{[1]}, \bar{\Omega}_2^{[1]}, 0, 0\right)^T$ consider $\bar{\Omega}_2^{[1]} \neq 0$ real number.

For $\lambda_{1x_2} = 0$, $\bar{U}^{[1]} = (\bar{U}_1^{[1]}, \bar{U}_2^{[1]}, \bar{U}_3^{[1]}, \bar{U}_4^{[1]})^T$ the eigenvector of is $\bar{J}_1^T$.

We find $(\bar{J}_1^T - \lambda_{1x_2} I)\bar{U}^{[1]} = 0$, we solving this equation for $\bar{U}^{[1]}$ and by condition (15.2) in [24] we obtain $\bar{U}^{[1]} = (0, \bar{U}_2^{[1]}, 0, 0)^T$, consider $\bar{U}_2^{[1]} \neq 0$ real number.

$$\begin{aligned} \text{Currently, suppose } \frac{\partial f}{\partial a_2} = f_{a_2}(M, a_2) &= \left(\frac{\partial f_1}{\partial a_2}, \frac{\partial f_2}{\partial a_2}, \frac{\partial f_3}{\partial a_2}, \frac{\partial f_4}{\partial a_2}\right)^T \\ &= \left(0, x_2 \left(1 - \frac{x_2}{k_2}\right) \left(\frac{1}{1+f_2 x_s}\right), 0, 0\right)^T. \end{aligned}$$

So, $f_{a_2}(M_1, \bar{a}_2) = (0, 0, 0, 0)^T$ then $(\bar{U}^{[1]})^T f_{a_2}(M_1, \bar{a}_2) = 0$.

As a result, using "Sotomayor's theorem," we infer that the prerequisites for a (SNB) are not met.

The first criterion for (TB) is met. Right now,

$$Df_{a_2}(M, a_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \left(1 - \frac{2x_2}{k_2}\right) \frac{1}{(1+f_2 x_s)} & x_2 \left(1 - \frac{x_2}{k_2}\right) \frac{-f_2}{(1+f_2 x_s)^2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $Df_{a_2}(M, a_2)$ is the derivative of $f_{a_2}(M, a_2)$ with respect to $M = (x_1, x_2, x_s, x_I)^T$.

Further, it is observed that

$$Df_{a_2}(M_0, \bar{a}_2)\bar{\Omega}^{[1]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{-J_{12}}{J_{11}} \bar{\Omega}_2^{[1]} \\ \bar{\Omega}_2^{[1]} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \bar{\Omega}_2^{[1]} \\ 0 \\ 0 \end{bmatrix}, \text{ so}$$

$$\begin{aligned} (\bar{U}^{[1]})^T [Df_{a_2}(M_1, \bar{a}_2)\bar{\Omega}^{[1]}] &= (0, \bar{U}_2^{[1]}, 0, 0)(0, \bar{\Omega}_2^{[1]}, 0, 0)^T \\ &= \bar{\Omega}_2^{[1]} \bar{U}_2^{[1]} \neq 0. \end{aligned}$$

Moreover, by substituting $\bar{\Omega}^{[1]}$ in (2) we get:

$$D^2 f(M_1, \bar{a}_2)(\bar{\Omega}^{[1]}, \bar{\Omega}^{[1]}) = \begin{bmatrix} \bar{K}'_1 \\ \bar{K}'_2 \\ 0 \\ 0 \end{bmatrix}, \text{ where}$$

$$\bar{K}'_1 = 2 \left[\left(\frac{-a_1}{k_1} - 3\delta_1 \bar{x}_1 \right) \left(\frac{-J_{12}}{J_{11}} \right)^2 + \left(\left(1 - \frac{2\bar{x}_1}{k_1} \right) (-a_1 f_1) + \frac{b_1(c_1 - \bar{x}_1^2)}{(c_1 + \bar{x}_1^2)^2} \right) \left(\frac{-J_{12}}{J_{11}} \right) + \left(a_1 f_1^2 \bar{x}_1 \left(1 - \frac{\bar{x}_1}{k_1} \right) - \rho_1 \bar{x}_1 \right) \right] (\bar{\Omega}_2^{[1]})^2,$$

$$\bar{K}'_2 = 2 \left[\left(\frac{g_1(c_1 - \bar{x}_1^2)}{(c_1 + \bar{x}_1^2)^2} - \eta \right) \left(\frac{-J_{12}}{J_{11}} \right) - \left(\frac{a_2}{k_2} + \delta_2 \right) \right] (\bar{\Omega}_2^{[1]})^2.$$

Hence, it obtains that by conditions (6) and (7)

$$(\bar{U}^{[1]})^T [D^2 f(M_1, \bar{a}_2)(\bar{\Omega}^{[1]}, \bar{\Omega}^{[1]})] = 2 \left[\left(\frac{g_1(\bar{x}_1^2 - c_1)}{(c_1 + \bar{x}_1^2)^2} + \eta \right) \frac{J_{12}}{J_{11}} - \left(\frac{a_2}{k_2} + \delta_2 \right) \right] \bar{U}_2^{[1]} (\bar{\Omega}_2^{[1]})^2 \neq 0.$$

As stated in "Sotomayor's theorem," system (1) at M_1 displays a (TB) at $\bar{a}_2 = a_2$.

Nonetheless, (TB) does not occur if conditions (6) and (7) are not satisfied, and by employing $\bar{\Omega}^{[1]}$ in Eq. (3), we obtain:

$$D^3 F(M_1, \bar{a}_2)(\bar{\Omega}^{[1]}, \bar{\Omega}^{[1]}, \bar{\Omega}^{[1]}) = \begin{bmatrix} \bar{R}'_1 \\ \bar{R}'_2 \\ 0 \\ 0 \end{bmatrix}, \text{ where}$$

$$\bar{R}'_1 = 6 \left[\delta_1 \left(\frac{J_{12}}{J_{11}} \right)^3 + \left(\frac{a_1 f_1}{k_1} + \frac{b_1 \bar{x}_1 (x_1 - 3c_1)}{(c_1 + \bar{x}_1^2)^3} \right) \left(\frac{J_{12}}{J_{11}} \right)^2 + \left(a_1 f_1^2 \left(\frac{2\bar{x}_1}{k_1} - 1 \right) + \rho_1 \right) \left(\frac{J_{12}}{J_{11}} \right) - a_1 f_1^3 \bar{x}_1 \left(1 - \frac{\bar{x}_1}{k_1} \right) \right] (\bar{\Omega}_2^{[1]})^3,$$

$$\bar{R}'_2 = 6 \left[g_1 \bar{x}_1 \frac{(\bar{x}_1^2 - 3c_1)}{(c_1 + \bar{x}_1^2)^3} \left(\frac{J_{12}}{J_{11}} \right)^2 - e_1 \rho_1 \left(\frac{J_{12}}{J_{11}} \right) \right] (\bar{\Omega}_2^{[1]})^3,$$

so according to condition (8)

$$(\bar{U}^{[2]})^T [D^3 F(M_1, \bar{a}_2)(\bar{\Omega}^{[1]}, \bar{\Omega}^{[1]}, \bar{\Omega}^{[1]})] = 6 \left[g_1 \bar{x}_1 \frac{(\bar{x}_1^2 - 3c_1)}{(c_1 + \bar{x}_1^2)^3} \left(\frac{J_{12}}{J_{11}} \right)^2 - e_1 \rho_1 \left(\frac{J_{12}}{J_{11}} \right) \right] \bar{U}_2^{[1]} (\bar{\Omega}_2^{[1]})^3 \neq 0.$$

Therefore, as stated in "Sotomayor's theorem," system (1) at M_1 displays a (PB) at $\bar{a}_2 = a_2$ but (SNB) cannot be displayed.

Theorem (3): suppose that conditions (16.3 – 16.4) in [24] when another condition are fulfilled:

$$\frac{g_1 \bar{x}_2}{c_1} + e_1 \rho_1 \bar{x}_2^2 > \eta \bar{x}_2, \quad (9)$$

$$\frac{b_1 \bar{r}}{c_1} \neq \frac{a_1}{k_1(1+f_1 \bar{x}_2)} + \left(\frac{a_1 f_1}{(1+f_1 \bar{x}_2)^2} + 2\rho_1 \bar{x}_2 \right) \bar{r}, \quad (10)$$

$$\frac{b_1 \bar{x}_2}{c_1^2} + \frac{a_1 f_1^2 \bar{r}^2}{(1+f_1 \bar{x}_2)^3} + \frac{a_1 f_1 \bar{r}}{k_1(1+f_1 \bar{x}_2)^2} \neq (\delta_1 + \rho_1 \bar{r}^2), \quad (11)$$

then system (1) at the (EP) $M_2 = (0, \bar{x}_2, 0, 0)$ with $\bar{a}_1 = a_1 = (1 + f_1 \bar{x}_2) \left(\frac{b_1 \bar{x}_2}{c_1} + \rho_1 \bar{x}_2^2 + d_1 \right)$ has (TB) or (PB) could happen, but (SNB) cannot be done.

Proof: By employing (JM) J_2 , as specified in Eq. (16.1) in [24] the eigenvalue ($\lambda_{2x_1} = 0$) of system (5.1) at $\bar{a}_1 = a_1 = (1 + f_1 \bar{x}_2) \left(\frac{b_1 \bar{x}_2}{c_1} + \rho_1 \bar{x}_2^2 + d_1 \right)$, then J_1 with $\bar{a}_1 = a_1$ becomes

$$\bar{J}_2 = J_2(M_2, \bar{a}_1) = [\bar{m}_{ij}]_{4 \times 4}.$$

Consider $\bar{m}_{ij} = m_{ij}$, $i, j = 1, 2, 3, 4$ as specified in Eq. (16.1) in [24] accept $\bar{m}_{11} = 0$.

When the eigenvalue $\lambda_{2x_1} = 0$ for $\bar{J}_2$ then assume the eigenvector be written in $\bar{\Omega}^{[2]} = (\bar{\Omega}_1^{[2]}, \bar{\Omega}_2^{[2]}, \bar{\Omega}_3^{[2]}, \bar{\Omega}_4^{[2]})^T$.

Hence $(\bar{J}_2 - \lambda_{2x_1} I) \bar{\Omega}^{[2]} = 0$, by conditions (16.3 – 16.4) in [24] and condition (9) that gives

$$\bar{\Omega}^{[2]} = \left(\bar{\Omega}_1^{[2]}, \bar{r} \bar{\Omega}_1^{[2]}, 0, 0 \right)^T \text{ where } \bar{r} = \frac{-\bar{m}_{21}}{\bar{m}_{22}} \text{ and the real constant } \bar{\Omega}_1^{[2]} \neq 0.$$

For $\lambda_{2x_1} = 0$, the eigenvector of $\bar{J}_2^T$ is $\bar{U}^{[2]} = (\bar{U}_1^{[2]}, \bar{U}_2^{[2]}, \bar{U}_3^{[2]}, \bar{U}_4^{[2]})^T$.

We find $(\bar{J}_2^T - \lambda_{2x_1} I) \bar{U}^{[2]} = 0$, we solving this equation for $\bar{U}^{[2]}$ and by condition (16.3 – 16.4) in [24] and condition (9) we obtain $\bar{U}^{[2]} = (\bar{U}_1^{[2]}, 0, 0, 0)^T$, consider $\bar{U}_1^{[2]} \neq 0$ real number.

$$\begin{aligned} \text{Currently, suppose } \frac{\partial f}{\partial a_1} &= f_{a_1}(M, a_1) = \left(\frac{\partial f_1}{\partial a_1}, \frac{\partial f_2}{\partial a_1}, \frac{\partial f_3}{\partial a_1}, \frac{\partial f_4}{\partial a_1} \right)^T \\ &= \left(x_1 \left(1 - \frac{x_1}{k_1} \right) \left(\frac{1}{1+f_1 x_2} \right), 0, 0, 0 \right)^T. \end{aligned}$$

So, $f_{a_1}(M_2, \bar{a}_1) = (0, 0, 0, 0)^T$ then $(\bar{U}^{[2]})^T f_{a_1}(M_2, \bar{a}_1) = 0$.

As a result, using "Sotomayor's theorem," we infer that the prerequisites for a (SNB) are not met.

The first criterion for (TB) is met.

Right now,

$$Df_{a_1}(M, a_1) = \begin{bmatrix} \left(1 - \frac{2x_1}{k_1}\right) \frac{1}{1+f_1x_2} & x_1 \left(1 - \frac{x_1}{k_1}\right) \left(\frac{-f_1}{(1+f_1x_2)^2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $Df_{a_1}(M, a_1)$ is the derivative of $f_{a_1}(M, a_1)$ with respect to $M = (x_1, x_2, x_s, x_l)^T$.

Further, it is observed that

$$Df_{a_1}(M_2, \bar{a}_1) \bar{\bar{\Omega}}^{[2]} = \begin{bmatrix} \frac{1}{1+f_1\bar{x}_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \bar{\bar{\Omega}}_1^{[2]} \\ \bar{\bar{\tau}} \bar{\bar{\Omega}}_1^{[2]} \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\bar{\bar{\Omega}}_1^{[2]}}{1+f_1\bar{x}_2} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ so}$$

$$(\bar{\bar{U}}^{[2]})^T [Df_{a_1}(M_1, \bar{a}_1) \bar{\bar{\Omega}}^{[2]}] = (\bar{\bar{U}}_1^{[2]}, 0, 0, 0) \left(\frac{\bar{\bar{\Omega}}_1^{[2]}}{1+f_1\bar{x}_2}, 0, 0, 0 \right)^T = \frac{\bar{\bar{\Omega}}_1^{[2]} \bar{\bar{U}}_1^{[2]}}{1+f_1\bar{x}_2} \neq 0.$$

Moreover, by substituting $\bar{\bar{\Omega}}^{[2]}$ in (2) we get:

$$D^2f(M_1, \bar{a}_1)(\bar{\bar{\Omega}}^{[2]}, \bar{\bar{\Omega}}^{[2]}) = \begin{bmatrix} 2 \left[\frac{-a_1}{k_1(1+f_1\bar{x}_2)} + \left(\frac{-a_1f_1}{(1+f_1\bar{x}_2)^2} + \frac{b_1}{c_1} - 2\rho_1\bar{x}_2 \right) \bar{\bar{\tau}} \right] (\bar{\bar{\Omega}}_1^{[2]})^2 \\ 2 \left[\left(\frac{g_1}{c_1} - \eta + 2e_1\rho_1\bar{x}_2 \right) + \left(\frac{-a_2}{k_2} - \delta_2 \right) \bar{\bar{\tau}} \right] \bar{\bar{\tau}} (\bar{\bar{\Omega}}_1^{[2]})^2 \\ 0 \\ 0 \end{bmatrix}$$

Hence, it obtains that by condition (10)

$$(\bar{\bar{U}}^{[2]})^T [D^2f(M_2, \bar{a}_1)(\bar{\bar{\Omega}}^{[2]}, \bar{\bar{\Omega}}^{[2]})] = 2 \left[\frac{-a_1}{k_1(1+f_1\bar{x}_2)} + \left(\frac{-a_1f_1}{(1+f_1\bar{x}_2)^2} + \frac{b_1}{c_1} - 2\rho_1\bar{x}_2 \right) \bar{\bar{\tau}} \right] (\bar{\bar{\Omega}}_1^{[2]})^2 \bar{\bar{U}}_1^{[2]} \neq 0.$$

As stated in "Sotomayor's theorem," system (1) at M_2 displays a (TB) at $\bar{a}_1 = a_1$.

Nonetheless, (TB) does not occur if conditions (10) are not satisfied, and by employing $\bar{\bar{\Omega}}^{[2]}$ in

Eq. (3), we obtain:

$$D^3F(M_2, \bar{a}_1)(\bar{\Omega}^{[2]}, \bar{\Omega}^{[2]}, \bar{\Omega}^{[2]}) = \begin{bmatrix} \bar{R}'_1 \\ 6 \left[\frac{g_1 \bar{x}_2}{c_2^2} + e_1 \rho_1 \bar{t}^2 \right] (\bar{\Omega}_1^{[2]})^3 \\ 0 \\ 0 \end{bmatrix},$$

Where

$$\bar{R}'_1 = 6 \left[\left(\frac{b_1 \bar{x}_2}{c_1^2} - \delta_1 \right) + \frac{a_1 f_1 \bar{t}}{k_1 (1 + f_1 \bar{x}_2)^2} + \left(\frac{a_1 f_1^2}{(1 + f_1 \bar{x}_2)^3} - \rho_1 \right) \bar{t}^2 \right] (\bar{\Omega}_1^{[2]})^3,$$

so according to condition (11)

$$(\bar{U}^{[2]})^T [D^3F(M_2, \bar{a}_1)(\bar{\Omega}^{[2]}, \bar{\Omega}^{[2]}, \bar{\Omega}^{[2]})] = 6 \left[\left(\frac{b_1 \bar{x}_2}{c_1^2} - \delta_1 \right) + \frac{a_1 f_1 \bar{t}}{k_1 (1 + f_1 \bar{x}_2)^2} + \left(\frac{a_1 f_1^2}{(1 + f_1 \bar{x}_2)^3} - \rho_1 \right) \bar{t}^2 \right] (\bar{\Omega}_1^{[2]})^3 \bar{U}_1^{[1]} \neq 0.$$

Therefore, as stated in ‘‘Sotomayor’s theorem’’ system (1) at M_2 displays (PB) at $\bar{a}_1 = a_1$, but (SNB) cannot be displayed.

Theorem (4): suppose that conditions (17.3 – 17.8) in [24] when another condition are fulfilled:

$$\frac{c_2 g_2 \check{x}_2}{(c_2 + \gamma_1 \check{x}_2^2)^2} + 2e_2 \rho_2 \check{x}_2 \neq \frac{g_2 \gamma_1 \check{x}_2^2}{(c_2 + \gamma_1 \check{x}_2^2)^2}, \quad (12)$$

$$\frac{3c_2 \check{x}_2 g_2 \gamma_1 \check{x}_2}{(c_2 + \gamma_1 \check{x}_2^2)^3} \neq e_2 \rho_2 + \frac{g_2 \check{x}_2 \gamma_1^2 \check{x}_2^3}{(c_2 + \gamma_1 \check{x}_2^2)^3}, \quad (13)$$

then system (1) at the (EP) $M_3 = (\check{x}_1, \check{x}_2, 0, 0)$, with $\check{g}_2 = g_2 = \frac{(c_2 + \gamma_1 \check{x}_2^2)(\delta_3 + d_3)}{\check{x}_2}$ has a (TB) or (PB) could happen, but (SNB) cannot be done.

Proof: By employing (JM) J_3 , as specified in Eq. (17.1) in [24] the eigenvalue ($\lambda_{3x_s} = 0$) of system

(1) at $\check{g}_2 = g_2 = \frac{(c_2 + \gamma_1 \check{x}_2^2)(\delta_3 + d_3)}{\check{x}_2}$, then J_3 with $\check{g}_2 = g_2$ becomes

$$\check{J}_3 = J_3(M_3, \check{g}_2) = [\check{p}_{ij}]_{4 \times 4}.$$

consider $\check{p}_{ij} = p_{ij}$, $i, j = 1, 2, 3, 4$ as specified in Eq. (17.1) in [24] accept $\check{p}_{33} = 0$.

When the eigenvalue $\lambda_{3x_s} = 0$ for $\check{J}_3$ then assume the eigenvector be written in $\check{\Omega}^{[3]} =$

$$(\check{\Omega}_1^{[3]}, \check{\Omega}_2^{[3]}, \check{\Omega}_3^{[3]}, \check{\Omega}_4^{[3]})^T.$$

Hence $(\check{J}_3 - \lambda_{3x_s} I) \check{\Omega}^{[3]} = 0$, by condition (17.3 – 17.8) in [24] that gives $\check{\Omega}^{[3]} =$

$$(\check{t}_1 \check{\Omega}_3^{[3]}, -\check{t}_2 \check{\Omega}_3^{[3]}, \check{\Omega}_3^{[3]}, 0)^T \text{ where } \check{t}_1 = \frac{\check{p}_{12} \check{p}_{23}}{\check{p}_{11} \check{p}_{22} - \check{p}_{12} \check{p}_{12}}, \quad \check{t}_2 = \frac{\check{p}_{11} \check{p}_{23}}{\check{p}_{11} \check{p}_{22} - \check{p}_{12} \check{p}_{12}} \text{ and the real constant}$$

$$\check{\Omega}_3^{[3]} \neq 0.$$

For $\lambda_{3x_s} = 0$, the eigenvector of $\tilde{J}_3^T$ is $\tilde{U}^{[3]} = (\tilde{U}_1^{[3]}, \tilde{U}_2^{[3]}, \tilde{U}_3^{[3]}, \tilde{U}_4^{[3]})^T$.

We find $(\tilde{J}_3^T - \lambda_{3x_s} I)\tilde{U}^{[3]} = 0$, we solving this equation for $\tilde{U}^{[3]}$ and by conditions

(17.4) and (17.6) in [24] we obtain $\tilde{U}^{[3]} = (0, 0, \tilde{U}_3^{[3]}, 0)^T$, consider $\tilde{U}_3^{[3]} \neq 0$ real number.

$$\begin{aligned} \text{Currently, assume } \frac{\partial f}{\partial g_2} = f_{g_2}(M, g_2) &= \left(\frac{\partial f_1}{\partial g_2}, \frac{\partial f_2}{\partial g_2}, \frac{\partial f_3}{\partial g_2}, \frac{\partial f_4}{\partial g_2} \right)^T \\ &= \left(0, 0, \frac{x_2 x_s}{c_2 + \gamma_1 x_2^2}, 0 \right)^T. \end{aligned}$$

So, $f_{g_2}(M_3, \check{g}_2) = (0, 0, 0, 0)^T$ then $(\tilde{U}^{[3]})^T f_{g_2}(M_3, \check{g}_2) = 0$.

As a result, using "Sotomayor's theorem," we infer that the prerequisites for a (SNB) are not met.

The first criterion for (TB) is met.

Right now,

$$Df_{g_2}(M, g_2) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \frac{(c_2 - \gamma_1 x_2^2)x_s}{(c_2 + \gamma_1 x_2^2)^2} & \frac{x_2}{(c_2 + \gamma_1 x_2^2)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $Df_{g_2}(M, g_2)$ is the derivative of $f_{g_2}(M, g_2)$ with respect to $M = (x_1, x_2, x_s, x_l)^T$.

Further, it is observed that

$$Df_{g_2}(M_2, \check{g}_2)\tilde{\Omega}^{[3]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\check{x}_2}{(c_2 + \gamma_1 \check{x}_2^2)} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\check{x}_1 \tilde{\Omega}_3^{[3]} \\ \check{x}_2 \tilde{\Omega}_3^{[3]} \\ \tilde{\Omega}_3^{[3]} \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \frac{\check{x}_2 \tilde{\Omega}_3^{[3]}}{(c_2 + \gamma_1 \check{x}_2^2)} \\ 0 \end{bmatrix}, \text{ so}$$

$$(\tilde{U}^{[3]})^T [Df_{g_2}(M_3, \check{g}_2)\tilde{\Omega}^{[3]}] = (0, 0, \tilde{U}_3^{[3]}, 0) \left(0, 0, \frac{\check{x}_2 \tilde{\Omega}_3^{[3]}}{(c_2 + \gamma_1 \check{x}_2^2)}, 0 \right)^T = \frac{\check{x}_2 \tilde{\Omega}_3^{[3]} \tilde{U}_3^{[3]}}{(c_2 + \gamma_1 \check{x}_2^2)} \neq 0.$$

Moreover, by substituting $\tilde{\Omega}^{[3]}$ in (2) we get:

$$D^2 f(M_3, \check{g}_2)(\tilde{\Omega}^{[3]}, \tilde{\Omega}^{[3]}) = \begin{bmatrix} \tilde{K}'_1 \\ \tilde{K}'_2 \\ 2 \left[\frac{g_2(\gamma_1 \check{x}_2^2 - c_2)}{(c_2 + \gamma_1 \check{x}_2^2)^2} \check{x}_2 + 2e_2 \rho_2 \check{x}_2 \right] (\tilde{\Omega}_3^{[3]})^2 \\ 0 \end{bmatrix}, \text{ where}$$

$$\begin{aligned}\tilde{K}'_1 &= 2 \left[\left(\frac{-a_1}{k_1(1+f_1\tilde{x}_2)} - \frac{b_1\tilde{x}_1\tilde{x}_2(3c_1-\tilde{x}_1^2)}{(c_1+\tilde{x}_1^2)^3} - 3\delta_1\tilde{x}_1 \right) (\tilde{t}_1)^2 + \left(\left(1 - \frac{2\tilde{x}_1}{k_1} \right) \frac{-a_1f_1}{(1+f_1\tilde{x}_2)^2} + \frac{b_1(c_1-\tilde{x}_1^2)}{(c_1+\tilde{x}_1^2)^2} - \right. \right. \\ &\quad \left. \left. 2\rho_1\tilde{x}_2 \right) (-\tilde{t}_1\tilde{t}_2) + \left(a_1\tilde{x}_1 \left(1 - \frac{\tilde{x}_1}{k_1} \right) \frac{f_1^2}{(1+f_1\tilde{x}_2)^3} - \rho_1\tilde{x}_1 \right) (\tilde{t}_2)^2 \right] (\tilde{\Omega}_3^{[3]})^2, \\ \tilde{K}'_2 &= 2 \left[\frac{g_1\tilde{x}_1\tilde{x}_2(\tilde{x}_1^2-3c_1)}{(c_1+\tilde{x}_1)^3} (\tilde{t}_1)^2 + \left(\frac{g_1(c_1-\tilde{x}_1^2)}{(c_1+\tilde{x}_1^2)^2} - \eta + 2e_1\rho_1\tilde{x}_2 \right) (-\tilde{t}_1\tilde{t}_2) + \left(\frac{-a_2}{k_2} + e_1\rho_1\tilde{x}_1 - \delta_2 \right) (\tilde{t}_2)^2 + \right. \\ &\quad \left. \left(a_2f_2 \left(1 - \frac{2\tilde{x}_2}{k_2} \right) + \frac{b_2(c_2-\gamma_1\tilde{x}_2^2)}{(c_2+\gamma_1\tilde{x}_2^2)^2} \right) \tilde{t}_2 + \left(a_2\tilde{x}_2 \left(1 - \frac{\tilde{x}_2}{k_2} \right) \frac{f_1^2}{(1+f_1\tilde{x}_2)^3} - \rho_2\tilde{x}_2 \right) \right] (\tilde{\Omega}_3^{[3]})^2.\end{aligned}$$

Hence, it obtains that by condition (12)

$$(\tilde{\mathcal{U}}^{[3]})^T [D^2 f(M_3, \tilde{g}_2)(\tilde{\Omega}^{[3]}, \tilde{\Omega}^{[3]})] = 2 \left[\frac{g_2\gamma_1\tilde{t}_2\tilde{x}_2^2}{(c_2+\gamma_1\tilde{x}_2^2)^2} + 2e_2\rho_2\tilde{x}_2 - \frac{c_2g_2\tilde{t}_2}{(c_2+\gamma_1\tilde{x}_2^2)^2} \right] = (\tilde{\Omega}_3^{[3]})^2 \tilde{\mathcal{U}}_3^{[3]} \neq 0.$$

As stated in "Sotomayor's theorem," system (1) at M_3 displays a (TB) at $\tilde{g}_2 = g_2$.

Nonetheless, (TB) does not occur if condition (12) is not satisfied, and by employing $\tilde{\Omega}^{[3]}$ in Eq.(3), we obtain:

$$\begin{aligned}D^3 F(M_3, \tilde{g}_2)(\tilde{\Omega}^{[3]}, \tilde{\Omega}^{[3]}, \tilde{\Omega}^{[3]}) &= \begin{bmatrix} \tilde{R}'_1 \\ \tilde{R}'_2 \\ 6 \left[g_2\gamma_1\tilde{x}_2 \frac{3c_2-\gamma_1\tilde{x}_2^2}{(c_2+\gamma_1\tilde{x}_2^2)^3} \tilde{t}_2 - e_2\rho_2 \right] \tilde{t}_2 (\tilde{\Omega}_3^{[3]})^3 \\ 0 \end{bmatrix}, \\ \tilde{R}'_1 &= 6 \left[\left(\frac{b_1\tilde{x}_2[c_1(6\tilde{x}_1^2-c_1)-\tilde{x}_1^4]}{(c_1+\tilde{x}_1^2)^4} - \delta_1 \right) (\tilde{t}_1)^3 + \left(\frac{a_1f_1}{k_1(1+f_1\tilde{x}_2)^2} - \frac{b_1\tilde{x}_1(3c_1-\tilde{x}_1^2)}{(c_1+\tilde{x}_1^2)^3} \right) (\tilde{t}_1)^2 (-\tilde{t}_2) + \left(\left(1 - \right. \right. \right. \\ &\quad \left. \left. \frac{2\tilde{x}_1}{k_1} \right) \frac{a_1f_1^2}{(1+f_1\tilde{x}_2)^3} - \rho_1 \right) (\tilde{t}_1)(\tilde{t}_2)^2 + a_1\tilde{x}_1 \left(1 - \frac{\tilde{x}_1}{k_1} \right) \frac{f_1^3}{(1+f_1\tilde{x}_2)^4} (\tilde{t}_2)^3 \right] (\tilde{\Omega}_3^{[3]})^3, \\ \tilde{R}'_2 &= 6 \left[g_1\tilde{x}_2 \frac{c_2(6\tilde{x}_1^2-c_1)-\tilde{x}_1^4}{(c_1+\tilde{x}_1^2)^4} (\tilde{t}_1)^3 - g_1\tilde{x}_1 \frac{(3c_1-\tilde{x}_1^2)}{(c_1+\tilde{x}_1^2)^3} (\tilde{t}_1)^2 (-\tilde{t}_2) + e_1\rho_1(\tilde{t}_1)(\tilde{t}_2)^2 + \left(\frac{2a_2f_2}{k_2} + \right. \right. \\ &\quad \left. \left. b_2\gamma_1\tilde{x}_2 \frac{(3c_1-\gamma_1\tilde{x}_1^2)}{(c_2+\gamma_1\tilde{x}_2^2)^3} \right) (\tilde{t}_2)^2 + \left(a_2f_2^2 \left(1 - \frac{2\tilde{x}_2}{k_2} \right) - \rho_2 \right) (-\tilde{t}_2) - 6a_2f_2^3\tilde{x}_2 \left(1 - \frac{\tilde{x}_2}{k_2} \right) \right] (\tilde{\Omega}_3^{[3]})^3, \\ (\tilde{\mathcal{U}}^{[3]})^T [D^3 F(M_3, \tilde{g}_2)(\tilde{\Omega}^{[3]}, \tilde{\Omega}^{[3]}, \tilde{\Omega}^{[3]})] &= 6 \left[\frac{3c_2\tilde{t}_2g_2\gamma_1\tilde{x}_2}{(c_2+\gamma_1\tilde{x}_2^2)^3} - e_2\rho_2 - \frac{g_2\tilde{t}_2\gamma_1^2\tilde{x}_2^3}{(c_2+\gamma_1\tilde{x}_2^2)^3} \right] \tilde{t}_2 (\tilde{\Omega}_3^{[3]})^3 \tilde{\mathcal{U}}_3^{[3]} \neq 0.\end{aligned}$$

Therefore, as stated in "Sotomayor's theorem" and condition (13) then system (1) at M_3 displays a (PB) at $\tilde{g}_2 = g_2$, but (SNB) cannot be displayed.

Theorem (5): suppose that conditions (18.3 – 18.7) in [24] when another condition are fulfilled:

$$\frac{g_1\hat{x}_2}{c_1} + e_1\rho_1\hat{x}_2^2 > \eta\hat{x}_2, \quad (14)$$

$$\frac{b_1 \hat{t}_1}{c_1} \neq \frac{a_1}{k_1(1+f_1 \hat{x}_2)} + \left(\frac{a_1 f_1}{(1+f_1 \hat{x}_2)^2} + 2\rho_1 \hat{x}_2 \right) \hat{t}_1, \quad (15)$$

$$\frac{a_1 f_1 \hat{t}_1}{k_1(1+f_1 \hat{x}_2)^2} + \frac{a_1 f_1^2 (\hat{t}_1)^2}{(1+f_1 \hat{x}_2)^3} \neq \frac{b_1 \hat{x}_2}{c_1^2} + \delta_1 + \rho_1 (\hat{t}_1)^2, \quad (16)$$

then system (1) at the (EP) $M_4 = (0, \hat{x}_2, \hat{x}_s, 0)$ with $\hat{a}_1 = a_1 = (1 + f_1 \hat{x}_2) \left(\frac{b_1 \hat{x}_2}{c_1} + \rho_1 \hat{x}_2^2 + d_1 \right)$, has (TB) or (PB) could happen, but (SNB) cannot be done.

Proof: By employing (JM) J_4 , as specified in Eq. (18.1) in [24] the eigenvalue ($\lambda_{4x_1} = 0$) of system

(1) at $\hat{a}_1 = a_1 = (1 + f_1 \hat{x}_2) \left(\frac{b_1 \hat{x}_2}{c_1} + \rho_1 \hat{x}_2^2 + d_1 \right)$, then J_4 with $\hat{a}_1 = a_1$ becomes

$$\hat{J}_4 = J_4(M_4, \hat{a}_1) = [\hat{q}_{ij}]_{4 \times 4}.$$

consider $\hat{q}_{ij} = q_{ij}$, $i, j = 1, 2, 3, 4$ as specified in Eq. (18.1) in [24] accept $\hat{q}_{11} = 0$.

When the eigenvalue $\lambda_{4x_1} = 0$ for $\hat{J}_4$ then assume the eigenvector be written in $\hat{\Omega}^{[4]} = (\hat{\Omega}_1^{[4]}, \hat{\Omega}_2^{[4]}, \hat{\Omega}_3^{[4]}, \hat{\Omega}_4^{[4]})^T$.

Hence $(\hat{J}_4 - \lambda_{4x_1} I) \hat{\Omega}^{[4]} = 0$, by condition (18.3 – 18.7) in [24] and condition (14) that gives

$\hat{\Omega}^{[4]} = (\hat{\Omega}_1^{[4]}, \hat{t}_1 \hat{\Omega}_1^{[4]}, \hat{t}_2 \hat{\Omega}_1^{[4]}, 0)^T$ where $\hat{t}_1 = \frac{\hat{q}_{21} \hat{q}_{33}}{\hat{q}_{23} \hat{q}_{32} - \hat{q}_{22} \hat{q}_{33}}$, $\hat{t}_2 = (\frac{-\hat{q}_{32}}{\hat{q}_{33}}) \hat{t}_1$ and the real constant

$$\hat{\Omega}_1^{[4]} \neq 0.$$

For $\lambda_{4x_1} = 0$, the eigenvector of $\hat{J}_4^T$ is $\hat{\mathcal{U}}^{[4]} = (\hat{\mathcal{U}}_1^{[4]}, \hat{\mathcal{U}}_2^{[4]}, \hat{\mathcal{U}}_3^{[4]}, \hat{\mathcal{U}}_4^{[4]})^T$.

We find $(\hat{J}_4^T - \lambda_{4x_1} I) \hat{\mathcal{U}}^{[4]} = 0$, we solving this equation for $\hat{\mathcal{U}}^{[4]}$ and by condition (18.3 – 18.7) in

[24] and condition (14) we obtain $\hat{\mathcal{U}}^{[4]} = (\hat{\mathcal{U}}_1^{[4]}, 0, 0, 0)^T$, consider $\hat{\mathcal{U}}_1^{[4]} \neq 0$ real number.

$$\begin{aligned} \text{Currently, suppose } \frac{\partial f}{\partial a_1} &= f_{a_1}(M, a_1) = \left(\frac{\partial f_1}{\partial a_1}, \frac{\partial f_2}{\partial a_1}, \frac{\partial f_3}{\partial a_1}, \frac{\partial f_4}{\partial a_1} \right)^T \\ &= \left(x_1 \left(1 - \frac{x_1}{k_1} \right) \left(\frac{1}{1+f_1 x_2} \right), 0, 0, 0 \right)^T. \end{aligned}$$

So, $f_{a_1}(M_4, \hat{a}_1) = (0, 0, 0, 0)^T$ then $(\hat{\mathcal{U}}^{[4]})^T f_{a_1}(M_4, \hat{a}_1) = 0$.

As a result, using "Sotomayor's theorem," we infer that the prerequisites for a (SNB) are not met.

The first criterion for (TB) is met.

Right now,

$$Df_{a_1}(M, a_1) = \begin{bmatrix} \left(1 - \frac{2x_1}{k_1}\right)\left(\frac{1}{1+f_1x_2}\right) & \left(1 - \frac{x_1}{k_1}\right)\left(\frac{-f_1x_1}{(1+f_1x_2)^2}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $Df_{a_1}(M, a_1)$ is the derivative of $f_{a_1}(M, a_1)$ with respect to $M = (x_1, x_2, x_s, x_l)^T$.

Further, it is observed that

$$Df_{a_1}(M_4, \hat{a}_1)\hat{\Omega}^{[4]} = \begin{bmatrix} \frac{1}{1+f_1\hat{x}_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \hat{\Omega}_1^{[4]} \\ \hat{t}_1 \hat{\Omega}_1^{[4]} \\ \hat{t}_2 \hat{\Omega}_1^{[4]} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\hat{\Omega}_1^{[4]}}{1+f_1\hat{x}_2} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ so}$$

$$(\hat{\mathbf{U}}^{[4]})^T [Df_{a_1}(M_4, \hat{a}_1)\hat{\Omega}^{[4]}] = (\hat{\mathbf{U}}_1^{[4]}, 0, 0, 0) \left(\frac{\hat{\Omega}_1^{[4]}}{1+f_1\hat{x}_2}, 0, 0, 0 \right)^T = \frac{\hat{\Omega}_1^{[4]}\hat{\mathbf{U}}_1^{[4]}}{1+f_1\hat{x}_2} \neq 0.$$

Moreover, by substituting $\hat{\Omega}^{[4]}$ in (2) we get:

$$D^2f(M_4, \hat{a}_1)(\hat{\Omega}^{[4]}, \hat{\Omega}^{[4]}) = \begin{bmatrix} \hat{K}'_1 \\ \hat{K}'_2 \\ \hat{K}'_3 \\ 0 \end{bmatrix}, \text{ where}$$

$$\hat{K}'_1 = 2 \left[\frac{-a_1}{k_1(1+f_1\hat{x}_2)} + \left(\frac{-a_1f_1}{(1+f_1\hat{x}_2)^2} + \frac{b_1}{c_1} - 2\rho_1\hat{x}_2 \right) \hat{t}_1 \right] (\hat{\Omega}_1^{[4]})^2,$$

$$\hat{K}'_2 = 2 \left[\left(\frac{g_1}{c_1} - \eta + 2e_1\rho_1\hat{x}_2 \right) \hat{t}_1 + \left(\frac{-a_2}{k_2} - \delta_2 + \frac{b_2\gamma_1\hat{x}_2\hat{x}_s[3c_2-\gamma_1\hat{x}_2^2]}{(c_2+\gamma_1\hat{x}_2^2)^3} \right) (\hat{t}_1)^2 + \left(\left(1 - \frac{2\hat{x}_2}{k_2} \right) \frac{-a_2f_2}{(1+f_2\hat{x}_s)^2} - \right.$$

$$\left. 2\rho_2\hat{x}_s - \frac{b_2(c_2-\gamma_1\hat{x}_2^2)}{(c_2+\gamma_1\hat{x}_2^2)^2} \right) \hat{t}_1\hat{t}_2 + \left(a_2\hat{x}_2 \left(1 - \frac{\hat{x}_2}{k_2} \right) \frac{f_2^2}{(1+f_1\hat{x}_s)^3} - \rho_2\hat{x}_2 \right) (\hat{t}_2)^2 \right] (\hat{\Omega}_1^{[4]})^2,$$

$$\hat{K}'_3 = 2 \left[\frac{g_2\gamma_1\hat{x}_2\hat{x}_s(\gamma_1\hat{x}_2^2-3c_2)}{(c_2+\gamma_1\hat{x}_2^2)^3} (\hat{t}_1)^2 + \left(\frac{g_2(c_2-\gamma_1\hat{x}_2^2)}{(c_2+\gamma_1\hat{x}_2^2)^2} + 2e_2\rho_2\hat{x}_2 \right) \hat{t}_1\hat{t}_2 + 2e_2\rho_2\hat{x}_2(\hat{t}_2)^2 \right] (\hat{\Omega}_1^{[4]})^2.$$

Hence, it obtains that by condition (15)

$$(\hat{\mathbf{U}}^{[4]})^T [D^2f(M_4, \hat{a}_1)(\hat{\Omega}^{[4]}, \hat{\Omega}^{[4]})] = 2 \left[\frac{-a_1}{k_1(1+f_1\hat{x}_2)} + \left(\frac{-a_1f_1}{(1+f_1\hat{x}_2)^2} + \frac{b_1}{c_1} - 2\rho_1\hat{x}_2 \right) \hat{t}_1 \right] (\hat{\Omega}_1^{[4]})^2 \hat{\mathbf{U}}_1^{[4]} \neq 0.$$

As stated in "Sotomayor's theorem," system (1) at M_4 displays a (TB) at $\hat{a}_1 = a_1$.

Nonetheless, (TB) does not occur if condition (15) is not satisfied, and by employing $\widehat{\Omega}^{[4]}$ in Eq. (3), we obtain:

$$D^3F(M_3, \check{g}_2)(\widehat{\Omega}^{[4]}, \widehat{\Omega}^{[4]}, \widehat{\Omega}^{[4]}) = \begin{bmatrix} 6 \left[-\left(\frac{b_1 \hat{x}_2}{c_1^2} + \delta_1\right) + \frac{a_1 f_1 \hat{t}_1}{k_1(1+f_1 \hat{x}_2)^2} + \left(\frac{a_1 f_1^2}{(1+f_1 \hat{x}_2)^3} - \rho_1\right) (\hat{t}_1)^2 \right] (\widehat{\Omega}_1^{[4]})^3 \\ \hat{R}'_1 \\ \hat{R}'_2 \\ 0 \end{bmatrix},$$

where

$$\begin{aligned} \hat{R}'_1 &= 6 \left[\frac{-g_1 \hat{x}_2}{c_2^2} + e_1 \rho_1 (\hat{t}_1)^2 + b_2 \gamma_1 \hat{x}_s \frac{c_2(c_2 - 6\gamma_1 \hat{x}_2^2) + \gamma_1^2 \hat{x}_2^4}{(c_2 + \hat{x}_2^2)^4} (\hat{t}_1)^3 + \left(\frac{2a_2 f_2}{k_2(1+f_2 \hat{x}_s)^2} + \right. \right. \\ &\quad \left. b_2 \gamma_1 \hat{x}_2 \frac{(3c_1 - \gamma_1 \hat{x}_2^2)}{(c_2 + \gamma_1 \hat{x}_2^2)^3} \right) (\hat{t}_1)^2 \hat{t}_2 + \left(\left(1 - \frac{2\hat{x}_2}{k_2}\right) \frac{a_2 f_2^2}{(1+f_2 \hat{x}_s)^3} - \rho_2 \right) \hat{t}_1 (\hat{t}_2)^2 - 6 \left(1 - \frac{\hat{x}_2}{k_2}\right) \frac{a_2 f_2^3 \hat{x}_2}{(1+f_2 \hat{x}_s)^4} (\hat{t}_2)^3 \right] (\widehat{\Omega}_1^{[4]})^3, \\ \hat{R}'_2 &= 6 \left[g_2 \gamma_1 \hat{x}_s \frac{c_2(c_2 - 6\gamma_1 \hat{x}_2^2) + \gamma_1^2 \hat{x}_2^4}{(c_2 + \gamma_1 \hat{x}_2^2)^4} (\hat{t}_1)^3 - g_2 \gamma_1 \hat{x}_2 \frac{3c_2 - \gamma_1 \hat{x}_2^2}{(c_2 + \gamma_1 \hat{x}_2^2)^3} (\hat{t}_1)^2 \hat{t}_2 + e_2 \rho_2 \hat{t}_1 (\hat{t}_2)^2 \right] (\widehat{\Omega}_1^{[4]})^3, \end{aligned}$$

so according to condition (16)

$$\begin{aligned} (\widehat{\mathcal{O}}^{[4]})^T [D^3F(M_4, \hat{a}_1)(\widehat{\Omega}^{[4]}, \widehat{\Omega}^{[4]}, \widehat{\Omega}^{[4]})] &= 6 \left[-\left(\frac{b_1 \hat{x}_2}{c_1^2} + \delta_1\right) + \frac{a_1 f_1 \hat{t}_1}{k_1(1+f_1 \hat{x}_2)^2} + \right. \\ &\quad \left. \left(\frac{a_1 f_1^2}{(1+f_1 \hat{x}_2)^3} - \rho_1 \right) (\hat{t}_1)^2 \right] (\widehat{\Omega}_1^{[4]})^3 \widehat{\mathcal{O}}_1^{[4]} \neq 0. \end{aligned}$$

Therefore, as stated in ‘‘Sotomayor’s theorem’’ system (1) at M_4 displays a (PB) at $\hat{a}_1 = a_1$, but (SNB) cannot be displayed, similarly for the (EP) M_5 .

Theorem (6): suppose that conditions (19.3 – 19.10) in [24] holds, then system (1) at the (EP) $M_6 = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_s, 0)$ with $\tilde{d}_4 = d_4 = \beta \tilde{x}_s$, has (TB) could happen, but (SNB) or (PB) cannot be done

Proof: By employing (JM) J_6 , as specified in Eq. (19.1) in [24] the eigenvalue ($\lambda_{6x_l} = 0$) of system (1) at $\tilde{d}_4 = d_4 = \beta \tilde{x}_s$, then J_6 with $\tilde{d}_4 = d_4$ becomes

$$\tilde{J}_6 = J_6(M_6, \tilde{d}_4) = [\tilde{r}_{ij}]_{4 \times 4}.$$

where $\tilde{r}_{ij} = r_{ij}$, $i, j = 1, 2, 3, 4$ as specified in Eq. (19.1) in [24] accept $\tilde{r}_{44} = 0$.

When the eigenvalue $\lambda_{6x_l} = 0$ then assume the eigenvector be written in $\tilde{\Omega}^{[6]} =$

$$\left(\tilde{\Omega}_1^{[6]}, \tilde{\Omega}_2^{[6]}, \tilde{\Omega}_3^{[6]}, \tilde{\Omega}_4^{[6]} \right)^T.$$

Hence $(\tilde{J}_6 - \lambda_{6x_I} I) \tilde{\Omega}^{[6]} = 0$, by condition (19.3 – 19.10) in [24] that gives

$$\tilde{\Omega}^{[6]} = \left(\tilde{\tau}_1 \tilde{\Omega}_4^{[6]}, -\tilde{\tau}_2 \tilde{\Omega}_4^{[6]}, -\tilde{\tau}_3 \tilde{\Omega}_4^{[6]}, \tilde{\Omega}_4^{[6]} \right)^T \quad \text{where} \quad \tilde{\tau}_1 = \frac{\tilde{r}_{12} \tilde{r}_{23} \tilde{\tau}_3}{\tilde{r}_{11} \tilde{r}_{22} - \tilde{r}_{12} \tilde{r}_{21}}, \quad \tilde{\tau}_2 = \frac{\tilde{r}_{11} \tilde{r}_{23} \tilde{\tau}_3}{\tilde{r}_{11} \tilde{r}_{22} - \tilde{r}_{12} \tilde{r}_{21}}, \quad \tilde{\tau}_3 = \frac{\beta \tilde{x}_s (\tilde{r}_{11} \tilde{r}_{22} - \tilde{r}_{12} \tilde{r}_{21})}{\tilde{r}_{11} \tilde{r}_{23} \tilde{r}_{32} + \tilde{r}_{33} (\tilde{r}_{12} \tilde{r}_{21} - \tilde{r}_{11} \tilde{r}_{22})} \quad \text{and the real constant } \tilde{\Omega}_4^{[6]} \neq 0.$$

For $\lambda_{6x_I} = 0$, the eigenvector of $\tilde{J}_6^T$ is $\tilde{U}^{[6]} = \left(\tilde{U}_1^{[6]}, \tilde{U}_2^{[6]}, \tilde{U}_3^{[6]}, \tilde{U}_4^{[6]} \right)^T$.

We find $(\tilde{J}_6^T - \lambda_{6x_I} I) \tilde{U}^{[6]} = 0$, we solving this equation for $\tilde{U}^{[6]}$ and by condition (19.4) and (19.7)

in [24] we obtain $\tilde{U}^{[6]} = \left(0, 0, 0, \tilde{U}_4^{[6]} \right)^T$, consider $\tilde{U}^{[6]} \neq 0$ real number.

$$\begin{aligned} \text{Currently, suppose } \frac{\partial f}{\partial d_4} &= f_{d_4}(M, d_4) = \left(\frac{\partial f_1}{\partial d_4}, \frac{\partial f_2}{\partial d_4}, \frac{\partial f_3}{\partial d_4}, \frac{\partial f_4}{\partial d_4} \right)^T \\ &= (0, 0, 0, -x_I)^T. \end{aligned}$$

So, $f_{d_4}(M_6, \tilde{d}_4) = (0, 0, 0, 0)^T$ then $(\tilde{U}^{[6]})^T f_{d_4}(M_6, \tilde{d}_4) = 0$.

As a result, using "Sotomayor's theorem," we infer that the prerequisites for a (SNB) are not met.

The first criterion for (TB) is met.

Right now,

$$Df_{d_4}(M, d_4) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix},$$

where $Df_{d_4}(M, d_4)$ is the derivative of $f_{d_4}(M, d_4)$ with respect to $M = (x_1, x_2, x_s, x_I)^T$.

Further, it is observed that

$$Df_{d_4}(M_6, \tilde{d}_4) \tilde{\Omega}^{[6]} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \tilde{\tau}_1 \tilde{\Omega}_4^{[6]} \\ -\tilde{\tau}_2 \tilde{\Omega}_4^{[6]} \\ -\tilde{\tau}_3 \tilde{\Omega}_4^{[6]} \\ \tilde{\Omega}_4^{[6]} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\tilde{\Omega}_4^{[6]} \end{bmatrix}, \text{ so}$$

$$(\tilde{U}^{[6]})^T [Df_{d_4}(M_6, \tilde{d}_4) \tilde{\Omega}^{[6]}] = \left(0, 0, 0, \tilde{U}_4^{[6]} \right) \left(0, 0, 0, -\tilde{\Omega}_4^{[6]} \right)^T = -\tilde{\Omega}_4^{[6]} \tilde{U}_4^{[6]} \neq 0.$$

Moreover, by substituting $\tilde{\Omega}_4^{[6]}$ in (2) we get:

$$D^2 f(M_6, \tilde{d}_4)(\tilde{\Omega}^{[6]}, \tilde{\Omega}^{[6]}) = \begin{bmatrix} \tilde{K}'_1 \\ \tilde{K}'_2 \\ \tilde{K}'_3 \\ \tilde{K}'_4 \end{bmatrix}, \text{ where}$$

$$\begin{aligned} \tilde{K}'_1 &= 2 \left[\left(\frac{-a_1}{k_1(1+f_1\tilde{x}_2)} - \frac{b_1\tilde{x}_1\tilde{x}_2(3c_1-\tilde{x}_1^2)}{(c_1+\tilde{x}_1^2)^3} - 3\delta_1\tilde{x}_1 \right) (\tilde{t}_1)^2 + \left(\left(1 - \frac{2\tilde{x}_1}{k_1} \right) \frac{-a_1f_1}{(1+f_1\tilde{x}_2)^2} + \frac{b_1(c_1-\tilde{x}_1^2)}{(c_1+\tilde{x}_1^2)^2} - \right. \right. \\ &\quad \left. \left. 2\rho_1\tilde{x}_2 \right) \tilde{t}_1(-\tilde{t}_2) + \left(a_1\tilde{x}_1 \left(1 - \frac{\tilde{x}_1}{k_1} \right) \frac{f_1^2}{(1+f_1\tilde{x}_2)^3} - \rho_1\tilde{x}_1 \right) (-\tilde{t}_2)^2 \right] (\tilde{\Omega}_4^{[6]})^2, \\ \tilde{K}'_2 &= 2 \left[\frac{g_1\tilde{x}_1\tilde{x}_2(\tilde{x}_1^2-3c_1)}{(c_1+\tilde{x}_1^2)^3} (\tilde{t}_1)^2 + \left(\frac{g_1(c_1-\tilde{x}_1^2)}{(c_1+\tilde{x}_1^2)^2} - \eta + 2e_1\rho_1\tilde{x}_2 \right) \tilde{t}_1(-\tilde{t}_2) + \left(\frac{-a_2}{k_2(1+f_2\tilde{x}_s)} + e_1\rho_1\tilde{x}_1 - \delta_2 + \right. \right. \\ &\quad \left. \left. \frac{b_2\gamma_1\tilde{x}_2\tilde{x}_s[3c_2-\gamma_1\tilde{x}_2^2]}{(c_2+\gamma_1\tilde{x}_2^2)^3} \right) (\tilde{t}_2)^2 + \left(\left(1 - \frac{2\tilde{x}_2}{k_2} \right) \frac{-a_2f_2}{(1+f_2\tilde{x}_s)^2} - 2\rho_2\tilde{x}_s - \frac{b_2(c_2-\gamma_1\tilde{x}_2^2)}{(c_2+\gamma_1\tilde{x}_2^2)^2} \right) \tilde{t}_2\tilde{t}_3 + \left(a_2\tilde{x}_2 \left(1 - \right. \right. \\ &\quad \left. \left. \frac{\tilde{x}_2}{k_2} \right) \frac{f_2^2}{(1+f_1\tilde{x}_s)^3} - \rho_2\tilde{x}_2 \right) (\tilde{t}_3)^2 \right] (\tilde{\Omega}_4^{[6]})^2, \\ \tilde{K}'_3 &= 2 \left[\frac{g_2\gamma_1\tilde{x}_2\tilde{x}_s(\gamma_1\tilde{x}_2^2-3c_2)}{(c_2+\gamma_1\tilde{x}_2^2)^3} (\tilde{t}_1)^2 + \left(\frac{g_2(c_2-\gamma_1\tilde{x}_2^2)}{(c_2+\gamma_1\tilde{x}_2^2)^2} + 2e_2\rho_2\tilde{x}_s \right) \tilde{t}_2\tilde{t}_3 + 2e_2\rho_2\tilde{x}_2(\tilde{t}_3)^2 + 2\beta\tilde{t}_3 \right] (\tilde{\Omega}_4^{[6]})^2, \\ \tilde{K}'_4 &= -2\beta\tilde{t}_3(\tilde{\Omega}_4^{[6]})^2, \end{aligned}$$

Hence, it obtains that

$$(\tilde{\mathcal{U}}^{[6]})^T [D^2 f(M_6, \tilde{d}_4)(\tilde{\Omega}^{[6]}, \tilde{\Omega}^{[6]})] = -2\beta\tilde{t}_3(\tilde{\Omega}_4^{[6]})^2\tilde{\mathcal{U}}_4^{[6]} \neq 0.$$

As stated in "Sotomayor's theorem," system (1) at M_1 displays a (TB) at $\tilde{d}_4 = d_4$, because $\tilde{t}_3$ always $\tilde{t}_3 \neq 0$. Similarly for the (EP) M_5 .

Theorem (7): Suppose that conditions (20.4 – 20.8) in [24] when another condition are fulfilled:

$$\frac{g_1\tilde{x}_2}{c_1} + e_1\rho_1\tilde{x}_2^2 > \eta\tilde{x}_2, \quad (17)$$

$$\frac{b_1\tilde{t}_1}{c_1} \neq \frac{a_1}{k_1(1+f_1\tilde{x}_2)} + \left(\frac{a_1f_1}{(1+f_1\tilde{x}_2)^2} + 2\rho_1\tilde{x}_2 \right) \tilde{t}_1, \quad (18)$$

$$\frac{a_1f_1\tilde{t}_1}{k_1(1+f_1\tilde{x}_2)^2} + \frac{a_1f_1^2(\tilde{t}_1)^2}{(1+f_1\tilde{x}_2)^3} \neq \frac{b_1\tilde{x}_2}{c_1^2} + \delta_1 + \rho_1(\tilde{t}_1)^2, \quad (19)$$

then system (1) at the (EP) $M_8 = (0, \check{x}_2, \check{x}_s, \check{x}_l)$ with $\check{a}_1 = a_1 = (1 + f_1 \check{x}_2) \left(\frac{b_1 \check{x}_2}{c_1} + \rho_1 \check{x}_2^2 + d_1 \right)$, has (TB) or (PB) could happen, but (SNB) cannot be done.

Proof: By employing (JM) J_8 , as specified in Eq. (20.1) in [24] the eigenvalue ($\lambda_{8x_1} = 0$) of system (1) at $\check{a}_1 = a_1 = (1 + f_1 \check{x}_2) \left(\frac{b_1 \check{x}_2}{c_1} + \rho_1 \check{x}_2^2 + d_1 \right)$, then J_8 with $\check{a}_1 = a_1$ becomes

$$\check{J}_8 = J_8(M_8, \check{a}_1) = [\check{u}_{ij}]_{4 \times 4}.$$

where $\check{u}_{ij} = u_{ij}$, $i, j = 1, 2, 3, 4$ as specified in Eq. (20.1) in [24] accept $\check{u}_{11} = 0$.

When the eigenvalue $\lambda_{8x_1} = 0$ then assume the eigenvector be written in $\check{\Omega}^{[8]} = (\check{\Omega}_1^{[8]}, \check{\Omega}_2^{[8]}, \check{\Omega}_3^{[8]}, \check{\Omega}_4^{[8]})^T$.

Hence $(\check{J}_8 - \lambda_{8x_1} I) \check{\Omega}^{[6]} = 0$, by condition (20.4 – 20.8) in [24] and condition (17) that gives

$$\check{\Omega}^{[8]} = (\check{\Omega}_1^{[8]}, \check{\tau}_1 \check{\Omega}_1^{[8]}, \check{\tau}_2 \check{\Omega}_1^{[8]}, \check{\tau}_3 \check{\Omega}_1^{[8]})^T \text{ where } \check{\tau}_1 = \frac{\check{u}_{21}(\check{u}_{33}\check{u}_{44} - \check{u}_{34}\check{u}_{43})}{\check{u}_{22}(\check{u}_{34}\check{u}_{43} - \check{u}_{33}\check{u}_{44}) + \check{u}_{23}\check{u}_{32}\check{u}_{44}}, \check{\tau}_2 = \frac{\check{u}_{32}\check{u}_{44}\check{\tau}_1}{\check{u}_{34}\check{u}_{43} - \check{u}_{33}\check{u}_{44}},$$

$$\check{\tau}_3 = \frac{-\check{u}_{43}\check{\tau}_2}{\check{u}_{44}} \text{ and the real constant } \check{\Omega}_1^{[8]} \neq 0.$$

For $\lambda_{8x_1} = 0$, the eigenvector of $\check{J}_8^T$ is $\check{\Upsilon}^{[8]} = (\check{\Upsilon}_1^{[8]}, \check{\Upsilon}_2^{[8]}, \check{\Upsilon}_3^{[8]}, \check{\Upsilon}_4^{[8]})^T$.

We find $(\check{J}_8^T - \lambda_{8x_1} I) \check{\Upsilon}^{[8]} = 0$, we solving this equation for $\check{\Upsilon}^{[6]}$ and by condition (20.4) in [24] and

condition (17) we obtain $\check{\Upsilon}^{[8]} = (\check{\Upsilon}_1^{[8]}, 0, 0, 0)^T$, consider $\check{\Upsilon}_1^{[8]} \neq 0$ real number.

$$\begin{aligned} \text{Currently, suppose } \frac{\partial f}{\partial a_1} &= f_{a_1}(M, a_1) = \left(\frac{\partial f_1}{\partial a_1}, \frac{\partial f_2}{\partial a_1}, \frac{\partial f_3}{\partial a_1}, \frac{\partial f_4}{\partial a_1} \right)^T \\ &= \left(x_1 \left(1 - \frac{x_1}{k_1} \right) \left(\frac{1}{1 + f_1 x_2} \right), 0, 0, 0 \right)^T. \end{aligned}$$

So, $f_{a_1}(M_8, \check{a}_1) = (0, 0, 0, 0)^T$ then $(\check{\Upsilon}^{[8]})^T f_{a_1}(M_8, \check{a}_1) = 0$.

As a result, using "Sotomayor's theorem," we infer that the prerequisites for a (SNB) are not met.

The first criterion for (TB) is met.

Right now,

$$Df_{a_1}(M, a_1) = \begin{bmatrix} \left(1 - \frac{2x_1}{k_1} \right) \left(\frac{1}{1 + f_1 x_2} \right) & \left(1 - \frac{x_1}{k_1} \right) \left(\frac{-f_1 x_1}{(1 + f_1 x_2)^2} \right) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where $Df_{a_1}(M, a_1)$ is the derivative of $f_{a_1}(M, a_1)$ with respect $M = (x_1, x_2, x_s, x_l)^T$.

Further, it is observed that

$$Df_{a_1}(M_8, \check{a}_1) \check{\Omega}^{[8]} = \begin{bmatrix} \frac{1}{1+f_1\check{x}_2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \check{\Omega}_1^{[8]} \\ \check{\tau}_1 \check{\Omega}_1^{[8]} \\ \check{\tau}_2 \check{\Omega}_1^{[8]} \\ \check{\tau}_3 \check{\Omega}_1^{[8]} \end{bmatrix} = \begin{bmatrix} \frac{\check{\Omega}_1^{[8]}}{1+f_1\check{x}_2} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \text{ so}$$

$$(\check{\mathcal{U}}^{[8]})^T [Df_{a_1}(M_8, \check{a}_1) \check{\Omega}^{[8]}] = (\check{\mathcal{U}}_1^{[8]}, 0, 0, 0) \left(\frac{\check{\Omega}_1^{[8]}}{1+f_1\check{x}_2}, 0, 0, 0 \right)^T = \frac{\check{\Omega}_1^{[8]} \check{\mathcal{U}}_1^{[8]}}{1+f_1\check{x}_2} \neq 0.$$

Moreover, by substituting $\check{\Omega}^{[8]}$ in (2) we get:

$$D^2 f(M_8, \check{a}_1) (\check{\Omega}^{[8]}, \check{\Omega}^{[8]}) = \begin{bmatrix} \check{K}'_1 \\ \check{K}'_2 \\ \check{K}'_3 \\ 2\beta \check{\tau}_2 \check{\tau}_3 (\check{\Omega}_1^{[8]})^2 \end{bmatrix}, \text{ where}$$

$$\check{K}'_1 = 2 \left[\frac{-a_1}{k_1(1+f_1\check{x}_2)} + \left(\frac{-a_1 f_1}{(1+f_1\check{x}_2)^2} + \frac{b_1}{c_1} - 2\rho_1 \check{x}_2 \right) \check{\tau}_1 \right] (\check{\Omega}_1^{[8]})^2,$$

$$\check{K}'_2 = 2 \left[\left(\frac{g_1}{c_1} - \eta + 2e_1 \rho_1 \check{x}_2 \right) \check{\tau}_1 + \left(\frac{-a_2}{k_2(1+f_2\check{x}_s)} - \delta_2 + \frac{b_2 \gamma_1 \check{x}_2 \check{x}_s [3c_2 - \gamma_1 \check{x}_2^2]}{(c_2 + \gamma_1 \check{x}_2^2)^3} \right) (\check{\tau}_1)^2 + \left(\left(1 - \frac{2\check{x}_2}{k_2} \right) \frac{-a_2 f_2}{(1+f_2\check{x}_s)^2} - \right. \right. \\ \left. \left. 2\rho_2 \check{x}_s - \frac{b_2(c_2 - \gamma_1 \check{x}_2^2)}{(c_2 + \gamma_1 \check{x}_2^2)^2} \right) \check{\tau}_1 \check{\tau}_2 + \left(a_2 \check{x}_2 \left(1 - \frac{\check{x}_2}{k_2} \right) \frac{f_2^2}{(1+f_1\check{x}_s)^3} - \rho_2 \check{x}_2 \right) (\check{\tau}_2)^2 \right] (\check{\Omega}_1^{[8]})^2,$$

$$\check{K}'_3 = 2 \left[\frac{g_2 \gamma_1 \check{x}_2 \check{x}_s (\gamma_1 \check{x}_2^2 - 3c_2)}{(c_2 + \gamma_1 \check{x}_2^2)^3} (\check{\tau}_1)^2 + \left(\frac{g_2(c_2 - \gamma_1 \check{x}_2^2)}{(c_2 + \gamma_1 \check{x}_2^2)^2} + 2e_2 \rho_2 \check{x}_s \right) \check{\tau}_1 \check{\tau}_2 + 2e_2 \rho_2 \check{x}_2 (\check{\tau}_2)^2 - 2\beta \check{\tau}_2 \check{\tau}_3 \right] (\check{\Omega}_1^{[8]})^2,$$

Hence, it obtains that by condition (18)

$$(\check{\mathcal{U}}^{[8]})^T [D^2 f(M_8, \check{a}_1) (\check{\Omega}^{[8]}, \check{\Omega}^{[8]})] = 2 \left[\frac{-a_1}{k_1(1+f_1\check{x}_2)} + \left(\frac{-a_1 f_1}{(1+f_1\check{x}_2)^2} + \frac{b_1}{c_1} - 2\rho_1 \check{x}_2 \right) \check{\tau}_1 \right] (\check{\Omega}_1^{[8]})^2 \check{\mathcal{U}}_1^{[8]} \neq 0.$$

As stated in "Sotomayor's theorem," system (1) at M_8 displays a (TB) at $\check{a}_1 = a_1$.

Nonetheless, (TB) does not occur if condition (18) is not satisfied, and by employing $\check{\Omega}^{[8]}$ in Eq.

(3), we get:

$$D^3F(M_8, \check{a}_1)(\check{\Omega}^{[8]}, \check{\Omega}^{[8]}, \check{\Omega}^{[8]}) = \begin{bmatrix} \check{R}'_1 \\ \check{R}'_2 \\ \check{R}'_3 \\ 0 \end{bmatrix}, \text{ where}$$

$$\begin{aligned} \check{R}'_1 &= 6 \left[\left(\frac{-b_1 \check{x}_2}{c_1^2} - \delta_1 \right) + \frac{a_1 f_1 \check{x}_1}{k_1 (1+f_1 \check{x}_2)^2} + \left(\frac{a_1 f_1^2}{(1+f_1 \check{x}_2)^3} - \rho_1 \right) (\check{t}_1)^2 \right] (\check{\Omega}_1^{[8]})^3 \\ \check{R}'_2 &= 6 \left[\frac{-g_1 \check{x}_2}{c_2^2} + e_1 \rho_1 (\check{t}_1)^2 + b_2 \gamma_1 \check{x}_s \frac{c_2 (c_2 - 6\gamma_1 \check{x}_2^2) + \gamma_1^2 \check{x}_2^4}{(c_2 + \gamma_1 \check{x}_2^2)^4} (\check{t}_1)^3 + \left(\frac{2a_2 f_2}{k_2 (1+f_2 \check{x}_s)^2} + \right. \right. \\ &\quad \left. \left. b_2 \gamma_1 \check{x}_2 \frac{(3c_1 - \gamma_1 \check{x}_2^2)}{(c_2 + \gamma_1 \check{x}_2^2)^3} \right) (\check{t}_1)^2 \check{t}_2 + \left(\left(1 - \frac{2\check{x}_2}{k_2} \right) \frac{a_2 f_2^2}{(1+f_2 \check{x}_s)^3} - \rho_2 \right) \check{t}_1 (\check{t}_2)^2 - 6 \left(1 - \frac{\check{x}_2}{k_2} \right) \frac{a_2 f_2^3 \check{x}_2}{(1+f_2 \check{x}_s)^4} (\check{t}_2)^3 \right] (\check{\Omega}_1^{[8]})^3, \\ \check{R}'_3 &= 6 \left[g_2 \gamma_1 \check{x}_s \frac{c_2 (c_2 - 6\gamma_1 \check{x}_2^2) + \gamma_1^2 \check{x}_2^4}{(c_2 + \gamma_1 \check{x}_2^2)^4} (\check{t}_1)^3 - g_2 \gamma_1 \check{x}_2 \frac{3c_2 - \gamma_1 \check{x}_2^2}{(c_2 + \gamma_1 \check{x}_2^2)^3} (\check{t}_1)^2 \check{t}_2 + e_2 \rho_2 (\check{t}_1)^2 \check{t}_2 \right] (\check{\Omega}_1^{[8]})^3, \end{aligned}$$

Hence, it obtains that by condition (19)

$$\begin{aligned} (\check{\mathcal{U}}^{[8]})^T [D^3F(M_8, \check{a}_1)(\check{\Omega}^{[8]}, \check{\Omega}^{[8]}, \check{\Omega}^{[8]})] &= 6 \left[\left(\frac{-b_1 \check{x}_2}{c_1^2} - \delta_1 \right) + \frac{a_1 f_1 \check{x}_1}{k_1 (1+f_1 \check{x}_2)^2} + \right. \\ &\quad \left. \left(\frac{a_1 f_1^2}{(1+f_1 \check{x}_2)^3} - \rho_1 \right) (\check{t}_1)^2 \right] (\check{\Omega}_1^{[8]})^3 \check{\mathcal{U}}_1^{[4]} \neq 0. \end{aligned}$$

Therefore, as stated in "Sotomayor's theorem," system (1) at M_8 displays a (PB) at $\check{a}_1 = a_1$, but (SNB) cannot be displayed.

Theorem (8): Suppose that conditions (21.3 – 21.11) in [24] when another condition are fulfilled:

$$\beta \dot{x}_s > \frac{(\dot{z}_{11}\dot{z}_{22} - \dot{z}_{12}\dot{z}_{21})(\dot{z}_{33}\dot{z}_{44} - \dot{z}_{34}\dot{z}_{43})}{\dot{z}_{11}\dot{z}_{23}\dot{z}_{32}}, \quad (20)$$

$$\dot{t}_7 \neq \dot{t}_8, \quad (21)$$

where

$$\begin{aligned} \dot{t}_1 &= \frac{\dot{z}_{12}\dot{t}_2}{\dot{z}_{11}}, \quad \dot{t}_2 = \frac{\dot{z}_{11}\dot{z}_{23}\dot{t}_3}{\dot{z}_{11}\dot{z}_{22} - \dot{z}_{12}\dot{z}_{21}}, \quad \dot{t}_3 = \frac{\dot{z}_{44}}{\beta \dot{x}_1}, \quad \dot{t}_4 = \frac{-\dot{z}_{21}\dot{t}_5}{\dot{z}_{11}}, \quad \dot{t}_5 = \frac{\dot{z}_{11}\dot{z}_{32}\dot{t}_6}{\dot{z}_{12}\dot{z}_{21} - \dot{z}_{11}\dot{z}_{22}}, \quad \dot{t}_6 = \frac{\dot{z}_{44}}{\beta \dot{x}_s}, \\ \dot{t}_7 &= \left[\left(\frac{a_1}{k_1 (1+f_1 \check{x}_2)} + \frac{b_1 \check{x}_1 \check{x}_2 (3c_1 - \check{x}_1^2)}{(c_1 + \check{x}_1^2)^3} + 3\delta_1 \check{x}_1 \right) \dot{t}_1^2 \dot{t}_4 + \frac{b_1 (c_1 - \check{x}_1^2)}{(c_1 + \check{x}_1^2)^2} \dot{t}_1 \dot{t}_2 \dot{t}_4 + \rho_1 \check{x}_1 \dot{t}_2^2 \dot{t}_4 + \frac{g_1 \check{x}_1 \check{x}_2 (3c_1 - \check{x}_1^2)}{(c_1 + \check{x}_1^2)^3} \dot{t}_1^2 \dot{t}_5 + \right. \\ &\quad \left(\frac{g_1 (c_1 - \check{x}_1^2)}{(c_1 + \check{x}_1^2)^2} + 2e_1 \rho_1 \check{x}_2 \right) \dot{t}_1 \dot{t}_2 \dot{t}_5 + \left(\frac{a_2}{k_2 (1+f_2 \check{x}_s)} + \delta_2 \right) \dot{t}_2 \dot{t}_5 + \rho_2 \check{x}_2 \dot{t}_3^2 \dot{t}_5 + \frac{g_2 \gamma_1 \check{x}_2 \check{x}_s (3c_2 - \gamma_1 \check{x}_2^2)}{(c_2 + \gamma_1 \check{x}_2^2)^3} \dot{t}_2^2 \dot{t}_6 + \\ &\quad \left. e_2 \rho_2 \check{x}_2 \dot{t}_3^2 \dot{t}_6 + \beta \dot{t}_3 \dot{t}_6 \right], \end{aligned}$$

$$\begin{aligned} \dot{t}_8 = & \left[\left(\left(1 - \frac{2\dot{x}_1}{k_1} \right) \frac{a_1 f_1}{(1+k_1\dot{x}_2)^2} + 2\rho_1\dot{x}_2 \right) \dot{t}_1\dot{t}_2\dot{t}_4 + \left(1 - \frac{\dot{x}_1}{k_1} \right) \frac{a_1 f_1^2 \dot{x}_1}{(1+f_1\dot{x}_2)^3} \tau_2\tau_4 + \eta\dot{t}_1\dot{t}_2\dot{t}_5 + \left(e_1\rho_1\dot{x}_1 + \right. \right. \\ & \left. \frac{b_2\gamma_1\dot{x}_2\dot{x}_s(3c_2-\gamma_1x_2^2)}{(c_2+\gamma_1\dot{x}_2^2)^3} \right) \dot{t}_2^2\tau_5 + \left(\left(1 - \frac{2\dot{x}_2}{k_2} \right) \frac{a_2 f_2}{(1+f_2\dot{x}_s)^2} + 2\rho_2\dot{x}_s + \frac{b_2(c_2-\gamma_1x_2^2)}{(c_2+\gamma_1\dot{x}_2^2)^2} \right) \dot{t}_2\dot{t}_3\dot{t}_5 + a_2\dot{x}_2 \left(1 - \right. \\ & \left. \frac{\dot{x}_2}{k_2} \right) \frac{f_2^2}{(1+f_2\dot{x}_s)^3} \tau_3^2\dot{t}_5 + \left(\frac{g_2(c_2-\gamma_1\dot{x}_2^2)}{(c_2+\gamma_1\dot{x}_2)^2} + 2e_2\rho_2\dot{x}_s \right) \dot{t}_2\dot{t}_3\dot{t}_6 + B\dot{t}_3]. \end{aligned}$$

Then system (1) at the (EP) $M_9 = (\dot{x}_1, \dot{x}_2, \dot{x}_s, \dot{x}_I)$ with $\dot{d}_4 = d_4 = \beta\dot{x}_s - \frac{(\dot{z}_{11}\dot{z}_{22}-\dot{z}_{12}\dot{z}_{21})(\dot{z}_{33}\dot{z}_{44}-\dot{z}_{34}\dot{z}_{43})}{\dot{z}_{11}\dot{z}_{23}\dot{z}_{32}}$ has (SNB) could happen, but (TB) or (PB) cannot be done.

Proof: From the characteristic equation $\lambda^4 + \dot{\phi}_1\lambda^3 + \dot{\phi}_2\lambda^2 + \dot{\phi}_3\lambda + \dot{\phi}_4 = 0$, of (JM) J_9 which is specified in Eq.(21.2) in [24], system (1) at the (EP) M_9 possess eigenvalue say $(\lambda_{9x_I} = 0) \Leftrightarrow \dot{\phi}_4 = 0$ then M_9 will be non-hyperbolic (EP) so J_9 with $\dot{d}_4 = d_4 = \beta\dot{x}_s - \frac{(\dot{z}_{11}\dot{z}_{22}-\dot{z}_{12}\dot{z}_{21})(\dot{z}_{33}\dot{z}_{44}-\dot{z}_{34}\dot{z}_{43})}{\dot{z}_{11}\dot{z}_{23}\dot{z}_{32}}$, and condition (20) becomes

$$j_9 = J_9(M_9, \dot{d}) = [\dot{z}_{ij}]_{4 \times 4},$$

where $\dot{z}_{ij} = z_{ij}$, $i, j = 1, 2, 3, 4$ as specified in Eq. (21.1) in [24], accept $\dot{z}_{44} = \frac{(\dot{z}_{11}\dot{z}_{22}-\dot{z}_{12}\dot{z}_{21})(\dot{z}_{33}\dot{z}_{44}-\dot{z}_{34}\dot{z}_{43})}{\dot{z}_{11}\dot{z}_{23}\dot{z}_{32}}$.

When the eigenvalue $\lambda_{9x_I} = 0$ then assume the eigenvector be written in $\dot{\Omega}^{[9]} = (\dot{\Omega}_1^{[8]}, \dot{\Omega}_2^{[8]}, \dot{\Omega}_3^{[8]}, \dot{\Omega}_4^{[8]})^T$.

Hence $(j_9 - \lambda_{9x_I}I)\tilde{\Omega}^{[6]} = 0$, by condition (21.3 – 21.11) in [24] that gives $\dot{\Omega}^{[9]} = (-\dot{t}_1\dot{\Omega}_4^{[9]}, \dot{t}_2\dot{\Omega}_4^{[9]}, -\dot{t}_3\dot{\Omega}_4^{[9]}, \dot{\Omega}_4^{[9]})^T$ since $\dot{\Omega}_4^{[9]} \neq 0$ real number.

The eigenvector is $\dot{\mathcal{U}}^{[9]} = (\dot{\mathcal{U}}_1^{[9]}, \dot{\mathcal{U}}_2^{[9]}, \dot{\mathcal{U}}_3^{[9]}, \dot{\mathcal{U}}_4^{[9]})^T$ of j_9^T is, when $\lambda_{9x_I} = 0$.

We find $(j_9^T - \lambda_{9x_I}I)\dot{\mathcal{U}}^{[9]} = 0$, we solving this equation for $\dot{\mathcal{U}}^{[9]}$ and by condition (21.4) in [24] we obtain $\dot{\mathcal{U}}^{[9]} = (-\dot{t}_4\dot{\mathcal{U}}_4^{[9]}, -\dot{t}_5\dot{\mathcal{U}}_4^{[9]}, \dot{t}_6\dot{\mathcal{U}}_4^{[9]}, \dot{\mathcal{U}}_4^{[9]})^T$, since $\dot{\mathcal{U}}_4^{[9]} \neq 0$ real number.

Right now, assume $\frac{\partial f}{\partial d_4} = f_{d_4}(M, d_4) = \left(\frac{\partial f_1}{\partial d_4}, \frac{\partial f_2}{\partial d_4}, \frac{\partial f_3}{\partial d_4}, \frac{\partial f_4}{\partial d_4} \right)^T = (0, 0, 0, -x_I)^T$.

So, $f_{d_4}(M_9, \dot{d}_4) = (0, 0, 0, -\dot{x}_I)^T$,

then $(\ddot{U}^{[9]})^T f_{d_4}(M_9, \dot{d}_4) = -\dot{x}_I \ddot{U}_4^{[9]} \neq 0$.

Moreover, by substituting $\dot{\Omega}^{[9]}$ in (2) we get:

$$D^2 f(M_9, \dot{d}_4)(\dot{\Omega}^{[9]}, \dot{\Omega}^{[9]}) = \begin{bmatrix} \dot{K}'_1 \\ \dot{K}'_2 \\ \dot{K}'_3 \\ -2\beta \dot{t}_3 (\dot{\Omega}_4^{[9]})^2 \end{bmatrix}, \text{ where}$$

$$\dot{K}'_1 = 2 \left[\left(\frac{-a_1}{k_1(1+f_1\dot{x}_2)} - \frac{b_1\dot{x}_1\dot{x}_2(3c_1-\dot{x}_1^2)}{(c_1+\dot{x}_1^2)^3} - 3\delta_1\dot{x}_1 \right) (\dot{t}_1)^2 + \left(\left(1 - \frac{2\dot{x}_1}{k_1} \right) \frac{-a_1f_1}{(1+f_1\dot{x}_2)^2} + \frac{b_1(c_1-\dot{x}_1^2)}{(c_1+\dot{x}_1^2)^2} - 2\rho_1\dot{x}_2 \right) (-\dot{t}_1\dot{t}_2) + \left(a_1\dot{x}_1 \left(1 - \frac{\dot{x}_1}{k_1} \right) \frac{f_1^2}{(1+f_1\dot{x}_2)^3} - \rho_1\dot{x}_1 \right) (\dot{t}_2)^2 \right],$$

$$\dot{K}'_2 = 2 \left[\frac{g_1\dot{x}_1\dot{x}_2(\dot{x}_1^2-3c_1)}{(c_1+\dot{x}_1^2)^3} (\dot{t}_1)^2 + \left(\frac{g_1(c_1-\dot{x}_1^2)}{(c_1+\dot{x}_1^2)^2} - \eta + 2e_1\rho_1\dot{x}_2 \right) (-\dot{t}_1\dot{t}_2) + \left(\frac{-a_2}{k_2(1+f_2\dot{x}_s)} + e_1\rho_1\dot{x}_1 - \delta_2 + \frac{b_2\gamma_1\dot{x}_2\dot{x}_s[3c_2-\gamma_1\dot{x}_2^2]}{(c_2+\gamma_1\dot{x}_2^2)^3} \right) (\dot{t}_2)^2 + \left(\left(1 - \frac{2\dot{x}_2}{k_2} \right) \frac{-a_2f_2}{(1+f_2\dot{x}_s)^2} - 2\rho_2\dot{x}_s - \frac{b_2(c_2-\gamma_1\dot{x}_2^2)}{(c_2+\gamma_1\dot{x}_2^2)^2} \right) (-\dot{t}_2\dot{t}_3) + \left(a_2\dot{x}_2 \left(1 - \frac{\dot{x}_2}{k_2} \right) \frac{f_2^2}{(1+f_1\dot{x}_s)^3} - \rho_2\dot{x}_2 \right) (\dot{t}_3)^2 \right] (\dot{\Omega}_4^{[9]})^2,$$

$$\dot{K}'_3 = 2 \left[\frac{g_2\gamma_1\dot{x}_2\dot{x}_s(\gamma_1\dot{x}_2^2-3c_2)}{(c_2+\gamma_1\dot{x}_2^2)^3} (\dot{t}_1)^2 + \left(\frac{g_2(c_2-\gamma_1\dot{x}_2^2)}{(c_2+\gamma_1\dot{x}_2^2)^2} + 2e_2\rho_2\dot{x}_s \right) (-\dot{t}_2\dot{t}_3) + 2e_2\rho_2\dot{x}_2(\dot{t}_3)^2 + 2\beta\dot{t}_3 \right] (\dot{\Omega}_4^{[9]})^2,$$

Hence, it obtains that by condition (21)

$$(\ddot{U}^{[9]})^T [D^2 f(M_9, \dot{d}_4)(\dot{\Omega}^{[9]}, \dot{\Omega}^{[9]})] = 2(\dot{t}_7 - \dot{t}_8)(\dot{\Omega}_4^{[9]})^2 \ddot{U}_4^{[9]} \neq 0.$$

Therefore, as stated in ‘‘Sotomayor’s theorem’’ system (1) at M_9 displays (SNB) at $\dot{d}_4 = d_4$ when that conditions (21.3 – 21.11) in [24] are fulfilled, but (TB) and (PB) cannot be displayed. Similarly, for the (EP) M_{10} and M_{11} .

4. HOPF BIFURCATION ANALYSIS

This section presents a theorem demonstrating that the occurrence of a Hopf bifurcation (HB) at the positive (EP) M_9 of system (1) renders the application of the (HB) theory [25] suitable for (LBA).

Theorem (9): Suppose that conditions (21.3 – 21.11) in [24] and a further condition are met:

$$A\mathcal{G} < K \quad (22)$$

$$A^2 + \mathcal{F} > \mathcal{G}, \quad (23)$$

$$A\mathcal{H} + (A^2 + \mathcal{F} - \mathcal{G})\mathcal{G} > 2AK + \mathcal{L}, \quad (24)$$

$$\mathcal{H} > A\mathcal{F}, \quad (25)$$

$$(A^2 + \mathcal{F} - \mathcal{G})\mathcal{H} < (\mathcal{H} - A\mathcal{F})\mathcal{G} + A(AK + 2\mathcal{L}), \quad (26)$$

$$\begin{aligned} & \frac{\dot{\phi}_3(d_4^*)}{\dot{\phi}_1(d_4^*)} (-z_{11} - (z_{22} + z_{33})\dot{\phi}_3(d_4^*) + ((z_{11} + z_{22})z_{33} - z_{23}z_{32} - (z_{21}z_{12} + z_{22}z_{11}))) + \\ & z_{11}z_{32}z_{23}\dot{\phi}_3(d_4^*) + \dot{\phi}_2(d_4^*) + 2\dot{\phi}_3(d_4^*) \neq (z_{12}z_{21} - z_{11}z_{22})z_{33}\dot{\phi}_3(d_4^*) + \left(\frac{\dot{\phi}_3(d_4^*)}{\dot{\phi}_1(d_4^*)}\right)^2 ((z_{11} + z_{22})z_{33} - \\ & z_{23}z_{32} - (z_{21}z_{12} + z_{22}z_{11})), \end{aligned} \quad (27)$$

where

$$A = -(z_{11} + z_{22} + z_{33}), \mathcal{F} = z_{33}(z_{11} + z_{22}) + z_{22}z_{11} - z_{34}z_{43} - z_{23}z_{32} - z_{12}z_{21},$$

$$\mathcal{G} = z_{33}(z_{11} + z_{22}) - z_{32}z_{23} - z_{21}z_{12} + z_{11}z_{22},$$

$$\mathcal{H} = z_{34}z_{43}(z_{11} + z_{22}) + z_{23}z_{11}z_{32} + z_{11}(z_{12}z_{21} - z_{11}z_{22}),$$

$$K = z_{33}(z_{12}z_{21} - z_{11}z_{22}) + z_{11}z_{23}z_{32}, \mathcal{L} = -(z_{11}z_{22} - z_{12}z_{21})z_{34}z_{43},$$

$\dot{\phi}_1, \dot{\phi}_2$ and $\dot{\phi}_3$ as present in Eq. (21.2) in [24] the value of the parameters $d_4^* = d_4 = \beta\dot{x}_s$ system (1) possesses a (HB) near the positive (EP) M_9 .

Proof: To verify that the necessary and sufficient criteria for the occurrence of (HB) are satisfied, we need to select a parameter, say (d_4^*) , according to the (HB) theorem for $n = 4$, and apply it to the characteristic equation of system (1) as specified in [24] at M_9 .

$$\dot{\phi}_i(d_4^*) > 0, i = 1, 3, 4,$$

$$\Delta_1(d_4^*) = \dot{\phi}_1\dot{\phi}_2 - \dot{\phi}_3 > 0, \dot{\phi}_1^3 - 4\Delta_1 > 0 \text{ and}$$

$$\Delta_2(d_4^*) = (\dot{\phi}_1\dot{\phi}_2 - \dot{\phi}_3)\dot{\phi}_3 - \dot{\phi}_2^2\dot{\phi}_4 = 0.$$

Then $\dot{\phi}_i(d_4^*) > 0; i = 1, 3, 4, \Delta_1(d_4^*) > 0$ and $\dot{\phi}_1^3 - 4\Delta_1 > 0$ under conditions (21.3 – 21.10) in [24] are fulfilled.

In contrast, it is noted that $\Delta_2 = 0$ yields

$$-\mathcal{M}_1(d_4^*)^3 + \mathcal{M}_2(d_4^*)^2 + \mathcal{M}_3 d_4^* + \mathcal{M}_4 = 0, \quad (28)$$

where

$$\mathcal{M}_1 = A\mathcal{G} - K,$$

$$\mathcal{M}_2 = -[-A\mathcal{H} - (A^2 + \mathcal{F} - \mathcal{G})\mathcal{G} + 2AK + \mathcal{L}],$$

$$\mathcal{M}_3 = -[(A^2 + \mathcal{F} - \mathcal{G})\mathcal{H} - (\mathcal{H} - A\mathcal{F})\mathcal{G} - A(AK + 2\mathcal{L})],$$

$$\mathcal{M}_4 = -[(\mathcal{H} - A\mathcal{F})\mathcal{H} + A^2\mathcal{L}].$$

Using Descartes' rule of signs, we can see that Eq. (28), if the conditions (21.3 – 21.10) in [24] with conditions (22 – 26) are met, has a single positive root (d_4^*).

Now, at $d = d_4^*$ the characteristic Eq.(21.2) in [24] can be written as

$$\left(\lambda_9^2 + \frac{\dot{\phi}_3}{\dot{\phi}_1}\right)\left(\lambda_9^2 + \dot{\phi}_1\lambda_9 + \frac{\Delta_1}{\dot{\phi}_1}\right) = 0,$$

which has four roots

$$\lambda_{9x_1, x_2} = \pm i\sqrt{\frac{\dot{\phi}_3}{\dot{\phi}_1}} \text{ and } \lambda_{9x_s, x_l} = \frac{1}{2}\left(-\dot{\phi}_1 \pm \sqrt{\dot{\phi}_1^2 - 4\frac{\Delta_1}{\dot{\phi}_1}}\right).$$

Clearly, at $d_4 = d_4^*$ Two eigenvalues (λ_{9x_1} and λ_{9x_2}) are purely imaginary, while the other two (λ_{9x_s} and λ_{9x_l}) are real and negative. For each value of d in the vicinity of d^* , the roots often assume the following form.

$$\lambda_{9x_1, x_2} = \varsigma_1 \pm i\varsigma_2, \quad \lambda_{9x_s, x_l} = \frac{1}{2}\left(-\dot{\phi}_1 \pm \sqrt{\dot{\phi}_1^2 - 4\frac{\Delta_1}{\dot{\phi}_1}}\right).$$

Clearly, $\text{Re}(\lambda_{9x_1, x_2}(d_4)) \big|_{d_4=d_4^*} = \varsigma_1(d_4^*) = 0$, This signifies that when $d_4 = d_4^*$, the initial necessary and sufficient criterion for (HB) is satisfied.

To validate the transversality requirement, we must establish that

$$\Theta^*(d_4^*)\Psi^*(d_4^*) + \Gamma^*(d_4^*)\Phi^*(d_4^*) \neq 0,$$

Be advised that for $d_4 = d_4^*$ we possess $\varsigma_1 = 0$ and $\varsigma_2 = \sqrt{\frac{\dot{\phi}_3}{\dot{\phi}_1}}$, substituting the value of ς_2 gives the following simplifications:

$$\Psi^*(d_4^*) = -2\dot{\phi}_3(d_4^*),$$

$$\Phi^*(d_4^*) = \frac{2}{\dot{\phi}_1(d_4^*)}\sqrt{\frac{\dot{\phi}_3}{\dot{\phi}_1}}(\dot{\phi}_1(d_4^*)\dot{\phi}_2(d_4^*) - 2\dot{\phi}_3(d_4^*)),$$

$$\begin{aligned}\Theta^*(d_4^*) &= \dot{\phi}_4'(d_4^*) - \dot{\phi}_2'(d_4^*) \frac{\dot{\phi}_3(d_4^*)}{\dot{\phi}_1(d_4^*)}, \\ \Gamma^*(d_4^*) &= \sqrt{\frac{\dot{\phi}_3}{\dot{\phi}_1}}(d_4^*) \left(\dot{\phi}_3'(d_4^*) - \dot{\phi}_1'(d_4^*) \frac{\dot{\phi}_3(d_4^*)}{\dot{\phi}_1(d_4^*)} \right),\end{aligned}$$

where

$$\begin{aligned}\dot{\phi}_1' &= \frac{\partial \phi_1}{\partial d_4} \Big|_{d_4=d_4^*} = 1, \\ \dot{\phi}_2' &= \frac{\partial \phi_2}{\partial d_4} \Big|_{d_4=d_4^*} = -z_{11} - (z_{22} + z_{33}), \\ \dot{\phi}_3' &= \frac{\partial \phi_3}{\partial d_4} \Big|_{d_4=d_4^*} = (z_{11} + z_{22})z_{33} - z_{23}z_{32} - (z_{12}z_{21} + z_{11}z_{22}), \\ \dot{\phi}_4' &= \frac{\partial \phi_4}{\partial d_4} \Big|_{d_4=d_4^*} = -(z_{11}z_{22} - z_{12}z_{21})z_{33} + z_{11}z_{32}z_{23}. \\ \Theta^*(d_4^*)\Psi^*(d_4^*) + \Gamma^*(d_4^*)\Phi^*(d_4^*) &= 2 \left[\frac{\dot{\phi}_3(d_4^*)}{\dot{\phi}_1(d_4^*)} (-z_{11} - (z_{22} + z_{33})\dot{\phi}_3(d_4^*) + ((z_{11} + z_{22})z_{33} - z_{23}z_{32} - \right. \\ & (z_{21}z_{12} + z_{11}z_{22})) + 2\dot{\phi}_3(d_4^*) + ((z_{11}z_{22} - z_{12}z_{21})z_{33} + z_{11}z_{32}z_{23})\dot{\phi}_3(d_4^*) - \left. \left(\frac{\dot{\phi}_3(d_4^*)}{\dot{\phi}_1(d_4^*)} \right)^2 ((z_{11} + \right. \\ & \left. z_{22})z_{33} - z_{32}z_{23} - (z_{12}z_{21} + z_{11}z_{22})) + \dot{\phi}_2(d_4^*) \right] \neq 0,\end{aligned}$$

under conditions (21.3 – 21.11) in [24] and (27), Thus, we conclude that the (HB) occurs at the parameter $d_4 = d_4^*$ in proximity to the (EP) M_9 . Similarly with (EPs) M_{10} and M_{11} .

5. NUMERICAL SIMULATION:

In this section, the behavior of dynamics of system (1) is examined numerically by mathematical methods. The purpose of this study is to verify the analytical results we have obtained and to examine the effect of varying the given Sets for all parameters affecting the system's dynamical behavior. It has been noted that system (1) has a (GAS) positive (EP), As illustrated in Figure (1), the subsequent set of theoretical parameters that fulfill the stability requirements of the positive (EP).

$$\left. \begin{aligned}a_1=2.7, \quad k_1=3.5, \quad f_1=0.3, \quad b_1=0.6, \quad c_1=5.0, \quad \rho_1=0.05, \quad \delta_1=0.04, \\ d_1=0.02, \quad a_2=1.8, \quad k_2=2.8, \quad f_2=0.2, \quad g_1=0.25, \quad b_2=0.7, \quad c_2=3.5, \\ \gamma_1=0.15, \quad \eta=0.04, \quad e_1=0.03, \quad \rho_2=0.15, \quad \delta_2=0.04, \quad d_2=0.02, \\ g_2=0.25, \quad e_2=0.01, \quad \delta_3=0.04, \quad \beta=0.04, \quad d_3=0.02, \quad d_4=0.02.\end{aligned} \right\} \quad (29)$$

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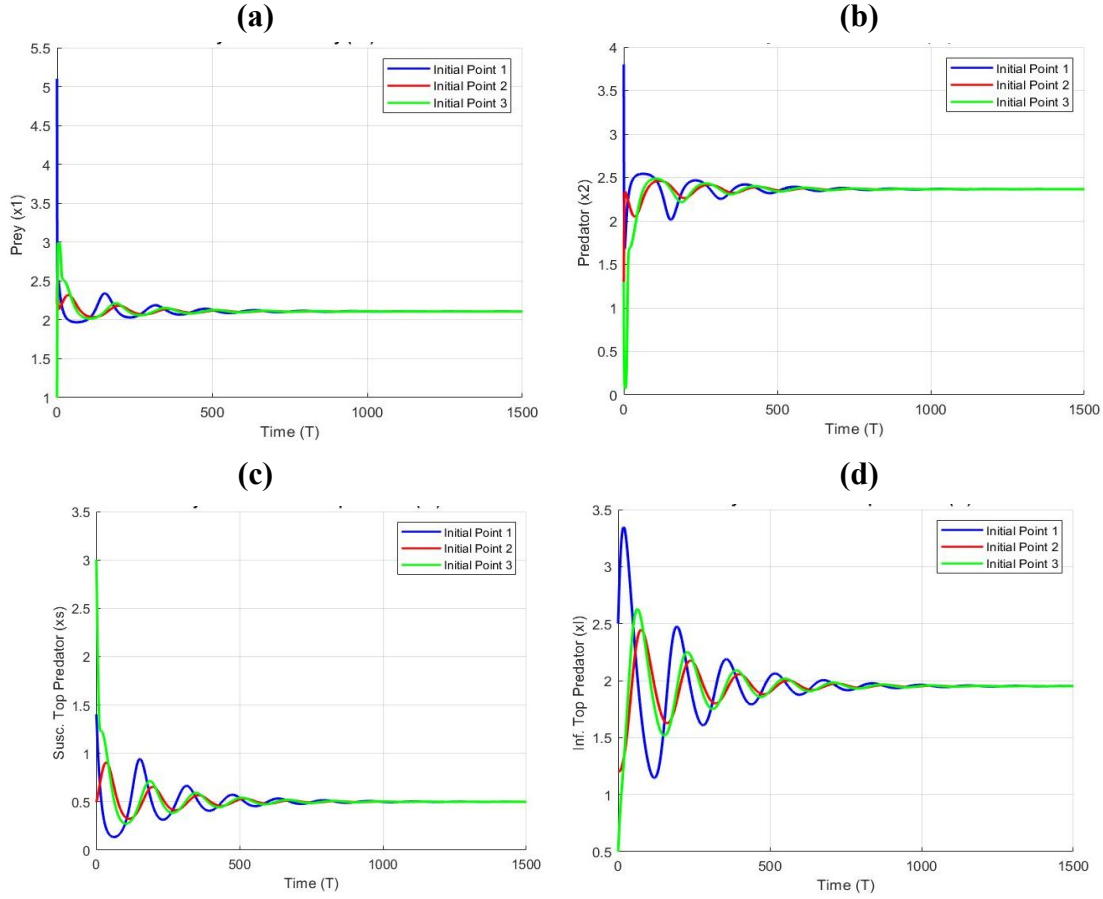


Figure -1 The curves followed by system (1), when began at three different starting points $(5.1, 3.8, 1.4, 2.5)$, $(2.2, 1.2, 0.5, 1.2)$, and $(1.0, 1.3, 3.0, 0.5)$, for the data in (29). (a) the path of x_1 over time, (b) the path of x_2 over time, (c) the path of x_s over time, and (d) the path of x_l over time, approach to $M_9 = (2.108, 2.367, 0.5, 1.952)$.

As the positive equilibrium point of system (1) is asymptotically approached, the generalized approximation of the solution (GAS) of system (1) is clear in Figure (1). $M_9 = (2.108, 2.367, 0.5, 1.952)$.

To examine the influence of parameter values on the system's dynamic behavior, we will alter one parameter at a time while maintaining the other parameters as specified in the data (29) with the beginning point $(2.2, 1.8, 1.0, 0.8)$, and the results are shown in Table 2.

Table 2: The dynamic behavior of system (1) at every system parameter

Parameter Range	Stable (EP)	Bifurcation	Persistence
$0.0001 \leq a_1 < 1.0237$,	M_8		Not Persists
$1.0237 \leq a_1 < 5$,	M_9	1.0237	Persists
$0.1 \leq k_1 < 5$.	M_9		Persists
$0.00001 \leq f_1 < 1.4755$,	M_9	1.4755	Persists
$1.4755 \leq f_1 < 5$.	M_8		Not Persists
$0.251 \leq b_1 < 2.632$,	M_9	2.632	Persists
$2.632 \leq b_1 < 5$	M_8		Not Persists
$0 \leq c_1 < 1.14$,	M_8		Not Persists
$1.14 \leq c_1 < 5$.	M_9	1.14	Persists
$0.00001 \leq \rho_1 < 0.22$,	M_9	0.22	Persists
$0.22 \leq \rho_1 < 5$.	M_8		Not Persists
$0.00001 \leq \delta_1 < 5$.	M_9		Persists
$0.00001 \leq d_1 < 1$.	M_9		Persists
$0.00001 \leq a_2 < 0.0843$,	M_1		Not Persists
$0.0843 \leq a_2 < 0.1558$,	M_3	0.0843	Not Persists
$0.1558 \leq a_2 < 0.329$,	M_6	0.1558	Not Persists
$0.329 \leq a_2 < 5$,	M_9	0.329	Persists
$0.00001 \leq a_2 < 0.01879, a_1 = 0.01$	M_0		Not Persists
$0.01879 \leq a_2 < 0.07936, a_1 = 0.01$	M_2	0.01879	Not Persists
$0.07936 \leq a_2 < 0.4, a_1 = 0.01$.	M_4	0.07936	Not Persists
$0.1 \leq k_2 < 0.8758$,	M_3		Not Persists
$0.8758 \leq k_2 < 1.0109$,	M_6	0.8758	Not Persists
$1.0109 \leq k_2 < 5$.	M_9	1.0109	Persists
$0.0001 < f_2 < 5$.	M_9		Persists
$0.00001 \leq g_1 < 0.59$.	M_9		Persists
$0.251 \leq b_2 < 5$,	M_9		Persists
$0.00001 \leq c_2 < 0.0002$,	M_6		Not Persists
$0.0002 \leq c_2 < 0.215$,	M_1	0.0002	Not Persists
$0.215 \leq c_2 < 0.4$.	M_9	0.215	Persists
$0.00001 \leq \gamma_1 < 1.0147$,	M_9		Persists
$1.0147 \leq \gamma_1 < 1.2666$,	M_6	1.0147	Not Persists
$1.2666 \leq \gamma_1 < 5$.	M_3	1.2666	Not Persists

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$0.00001 \leq \eta < 0.372,$	M_9		Persists
$0.372 \leq \eta < 0.4495,$	M_6	0.372	Not Persists
$0.4495 \leq \eta < 0.6103,$	M_3	0.4495	Not Persists
$0.6103 \leq \eta < 5.$	M_1	0.6103	Not Persists
$0 < e_1 < 1.$	M_9		Persists
$0.00001 \leq \rho_2 < 5,$	M_9		Persists
$0.00001 \leq \delta_2 < 1.1009,$	M_9		Persists
$1.1009 \leq \delta_2 < 1.4508,$	M_6	1.1009	Not Persists
$1.4508 \leq \delta_2 < 5.$	M_3	1.4508	Not Persists
$0.00001 \leq d_2 < 0.955,$	M_9		Persists
$0.955 \leq d_2 < 1.$	M_6	0.955	Not Persists
$0 \leq g_2 < 0.1,$	M_3		Not Persists
$0.1 \leq g_2 < 0.1095,$	M_6	0.1	Not Persists
$0.1095 \leq g_2 < 0.7$	M_9	0.1095	Persists
$0 < e_2 < 1.$	M_9		Persists
$0.00001 \leq \delta_3 < 0.1182,$	M_9		Persists
$0.1182 \leq \delta_3 < 0.1265,$	M_6	0.1182	Not Persists
$0.1265 \leq \delta_3 < 5.$	M_3	0.1265	Not Persists
$0.00001 \leq \beta < 0.0088,$	M_6		Not Persists
$0.0088 \leq \beta < 0.9,$	M_9	0.0088	Persists
$0.9 \leq \beta \leq 1.$	periodic		Not Persists
$0.00001 \leq d_3 < 0.0982,$	M_9		Persists
$0.0982 \leq d_3 < 0.1065,$	M_6	0.0982	Not Persists
$0.1065 \leq d_3 < 1.$	M_3	0.1065	Not Persists
$0.00001 \leq d_4 < 0.0752,$	M_9		Persists
$0.0752 \leq d_4 < 0.05429.$	M_6	0.0752	Not Persists

The influence of changing the growth rate of prey within the value $0.0001 \leq a_1 < 1.0237$ was investigated, it has been noted that system (1) continues to toward M_8 , Nonetheless, extending this parameter to its extremes further $1.0237 \leq a_1 < 5$ the solution toward to M_9 , as seen in Figure (2).

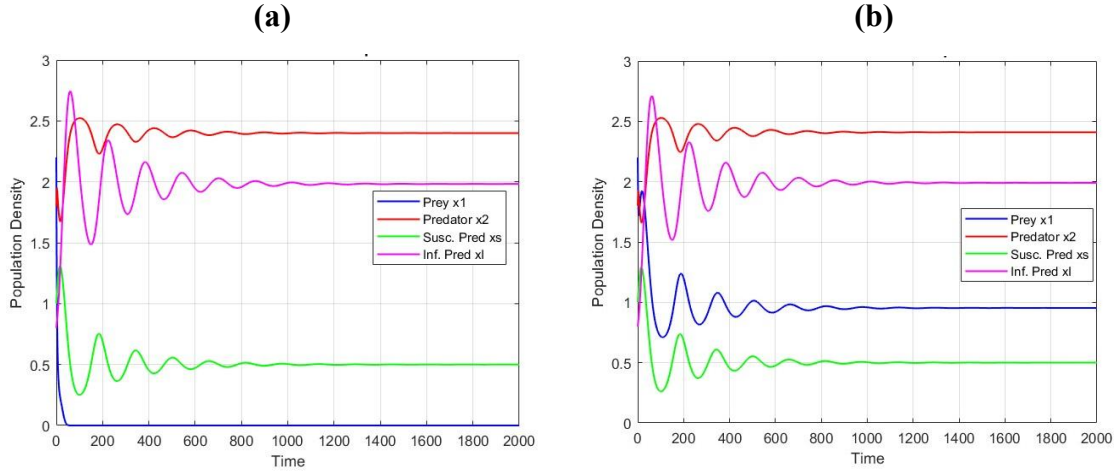
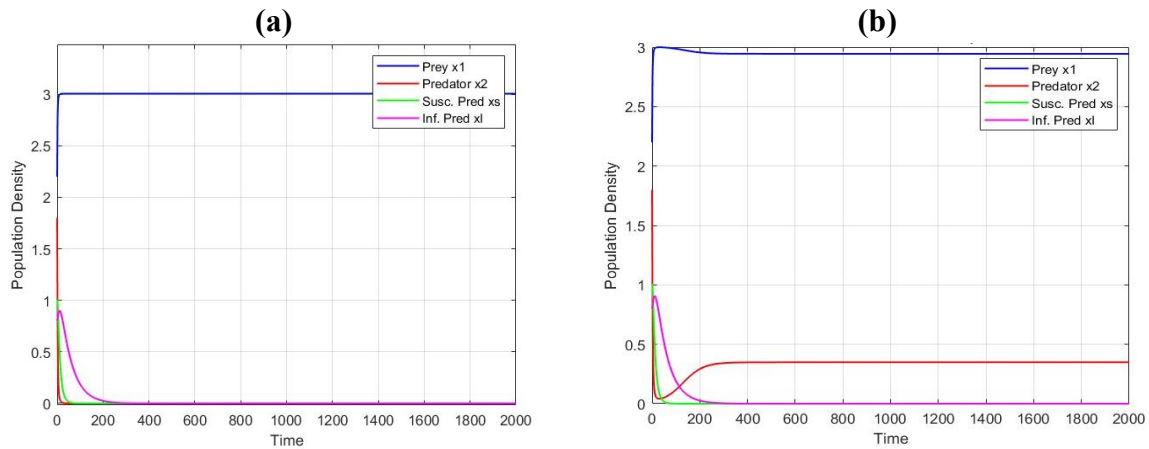


Figure -2 (a) the solution time series for system (1), which approaches to $M_8 = (0, 2.4, 0.5, 1.982)$ when $a_1 = 0.5$, (b) the solution time series for system (1) which approach to $M_9 = (0.953, 2.409, 0.5, 1.991)$ when $a_1 = 1.4$.

The study examines the predator population's response Regarding fluctuations in the predator population growth rate, with a range of $0.00001 \leq a_2 < 0.0843$, It is observed that system (1) is close to M_1 , but the system reaches M_3 when the parameter is increased by $0.0843 \leq a_2 < 0.1558$. When the parameter is increased by $0.1558 \leq a_2 < 0.329$, the system is close to M_6 , but the system reaches M_9 when the parameter is increased by $0.329 \leq a_2 < 5$, also increasing this parameter further $0.00001 \leq a_2 < 0.01879$, when $a_1 = 0.01$, the system reaches M_0 , when increasing this parameter further $0.01879 \leq a_2 < 0.07936$, when $a_1 = 0.01$, the system reaches M_2 , finally upping this parameter even further $0.07936 \leq a_2 < 0.4$, when $a_1 = 0.01$, the system is close to M_4 , as seen in Figure (3).



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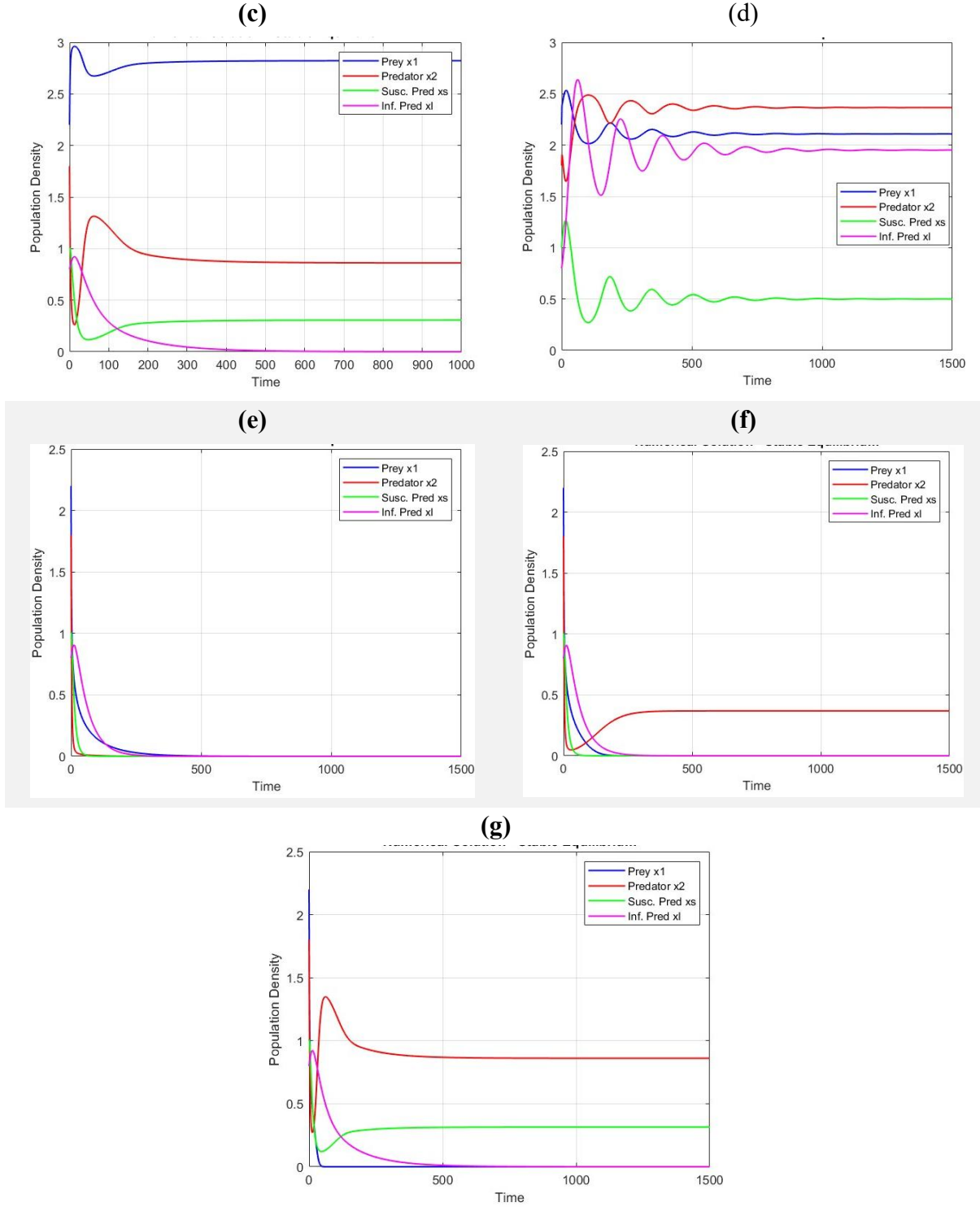


Figure -3 (a) The solution time series for system (1) which approach to $M_1 = (3.01, 0, 0, 0)$ when $a_2 = 0.004$, (b) the solution time series for system (1) which approach to $M_3 = (2.944, 0.3483, 0, 0)$ when $a_2 = 0.11$, (c) the solution time series for system (1) which approach to $M_6 = (2.822, 0.861, 0.306, 0)$ when $a_2 = 0.28$, (d) the solution time series for system (1) which approach to $M_9 = (2.108, 2.366, 0.5, 1.952)$ when $a_2 = 1.8$, (e) the solution time series for system (1) which approach to $M_0 = (0, 0, 0, 0)$ when $a_2 = 0.003$ and $a_1 = 0.01$, (f) the solution time

series for system (1) which approach to $M_2 = (0, 0.368, 0, 0)$ when $a_2 = 0.04$ and $a_1 = 0.01$, (g) the solution time series for system (1) which approach to $M_4 = (0, 0.861, 0.315, 0)$ when $a_2 = 0.2$ and $a_1 = 0.01$.

The impact of switching up the amount of fear of prey inside the range $0.00001 \leq f_1 < 1.4755$, It was found that system (1) is still getting closer to M_9 asymptotically throughout the course of the research, regardless of finding this parameter was raised much further $1.4755 \leq f_1 < 5$, the probability of the infected apex predator becoming extinct increases, and the system is close to M_8 as seen in Figure (4).

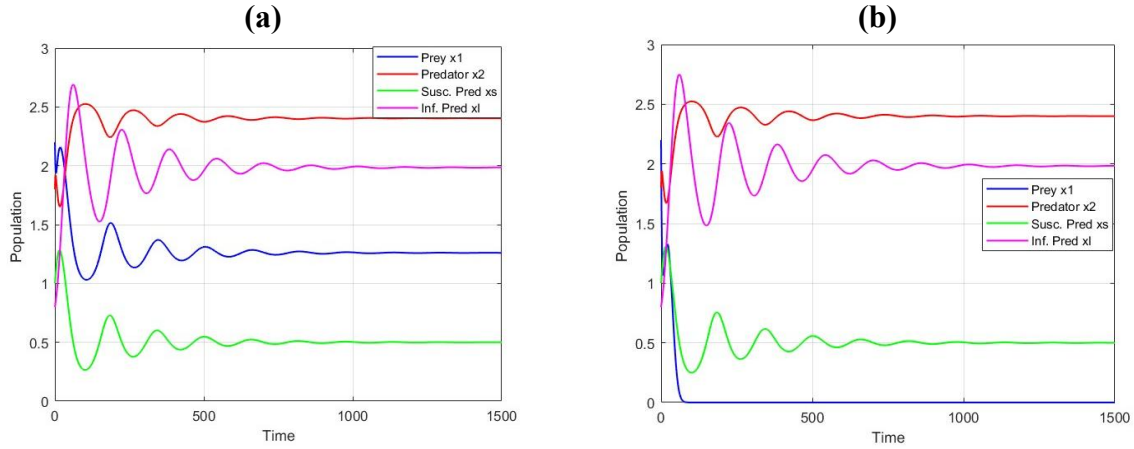


Figure -4 (a) The solution time series for system (1) which approach to $M_9 = (1.26, 2.403, 0.5, 1.986)$ when $f_1 = 0.8$, (b) the solution time series for system (1) which approach to $M_8 = (0, 2.4, 0.5, 1.984)$ when $f_1 = 2$.

The impact of switching up the amount of fear of predator inside the range $0.0001 < f_2 < 5$. the investigation revealed that system (1) continued to asymptotically approach the positive equilibrium point defined by M_9 , as seen in Figure (5).

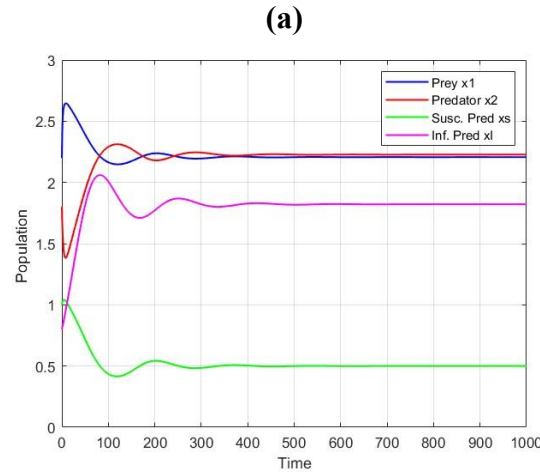


Figure -5 (a) The solution time series for system (1) which approach to $M_9 = (2.207, 2.227, 0.5, 1.821)$ when $f_2 =$

0.9.

When the hunting cooperation in predator the impact of this difference in the range $0.00001 \leq \rho_1 < 0.22$, was investigated, it has been noted that system (1) continues to approach M_9 , regardless of finding this parameter was raised much further $0.22 \leq \rho_1 < 5$ and the system is close to M_8 , as seen in Figure (6).

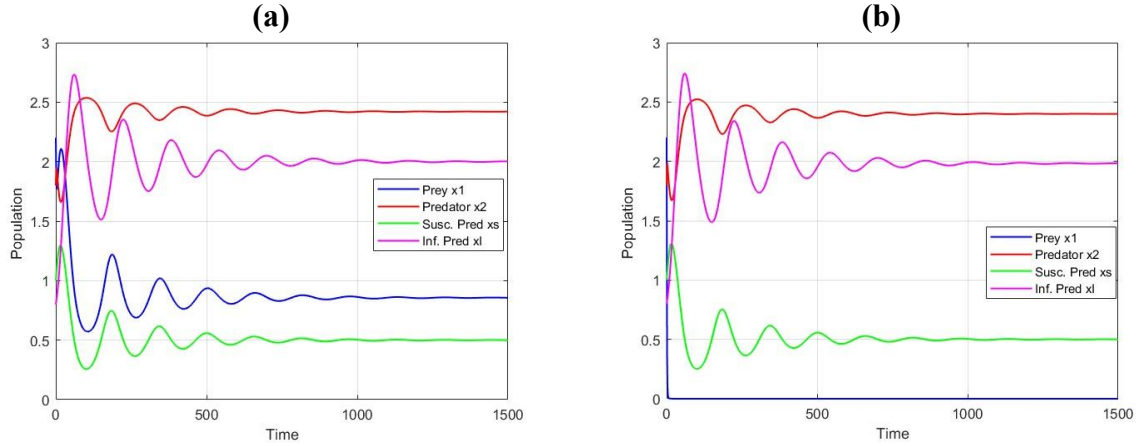


Figure -6 (a) System (1)'s solution time series, which toward to $M_9 = (0.856, 2.421, 0.5, 2)$ when $\rho_1 = 0.2$, (b) the solution time series for system (1) which approach to $M_8 = (0, 2.4, 0.5, 1.984)$ when $\rho_1 = 0.6$.

When the hunting cooperation in susceptible top predator the impact of this difference in the range $0.00001 \leq \rho_2 < 5$ was investigated, it has been noted that system (1) continues to approach M_9 , as seen in Figure (7).

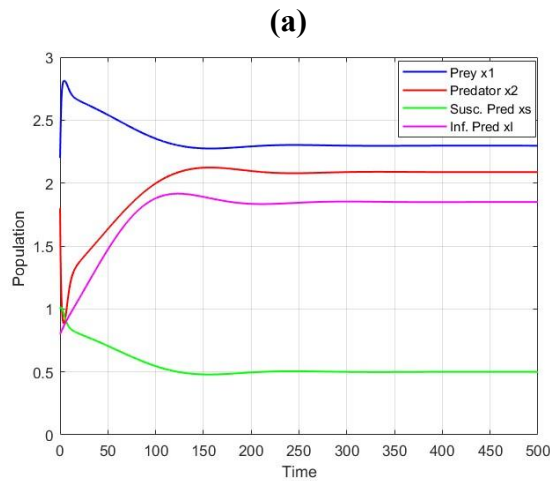


Figure -7 (a) System (1)'s solution time series, which toward to $M_9 = (2.3, 2.087, 0.5, 1.85)$ when $\rho_2 = 0.8$.

This research investigated the toxin rate of the predator and the effects of variations within the range of $0.00001 \leq \delta_2 < 1.1009$, it has been noted that system (1) continues to approach M_9 , the effect of this variation $1.1009 \leq \delta_2 < 1.4508$, was investigated, it has been noted that system (1) continues to approach M_6 . Nonetheless, extending this parameter to its extremes further $1.4508 \leq \delta_2 < 5$. the system is close to M_3 , as seen in Figure (8).

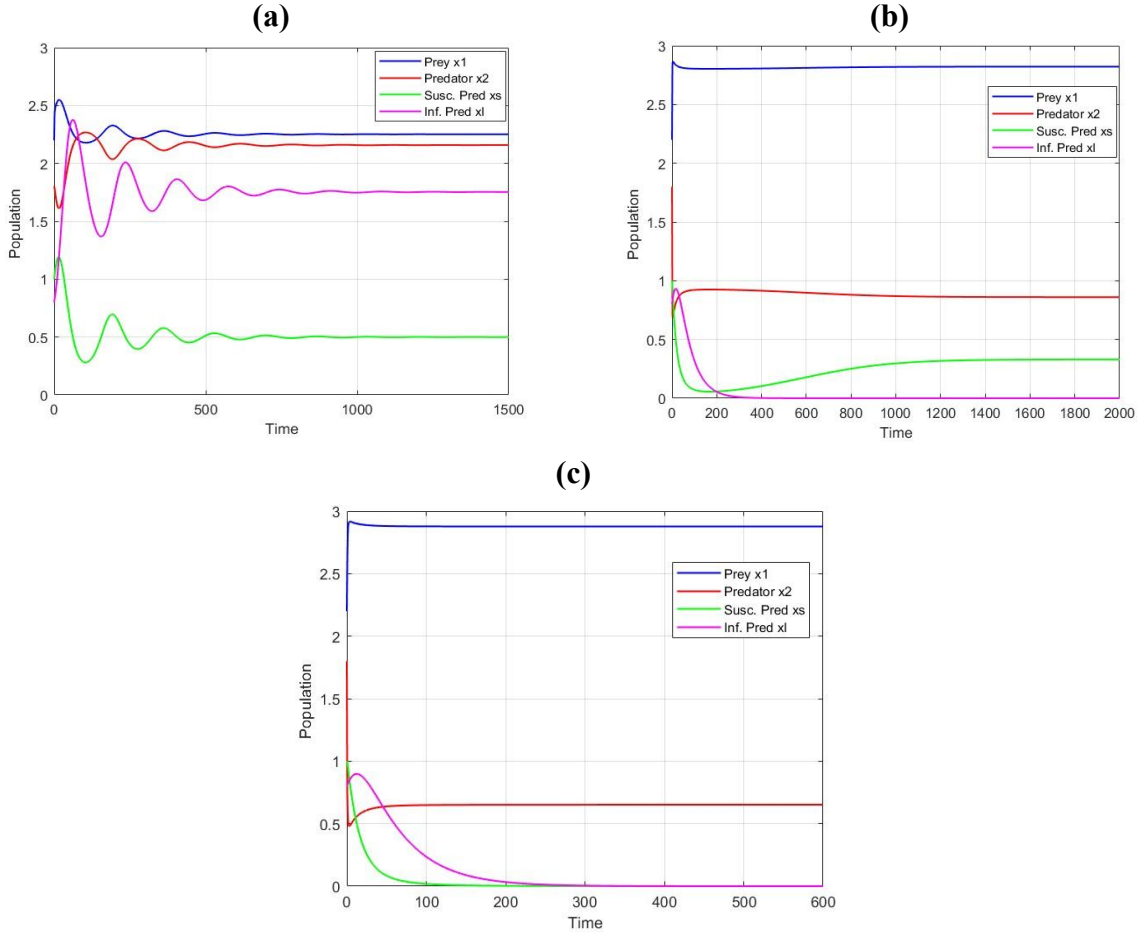


Figure-8 (a) System (1)'s solution time series, which toward to $M_9 = (2.252, 2.159, 0.5, 1.753)$ when $\delta_2=0.1$, (b)the solution time series for system (1) which approach to $M_6 = (2.822, 0.861, 0.33, 0)$ when $\delta_2 =1.2$, (c) the solution time series for system (1) which approach to $M_3 = (2.877, 0.652, 0, 0)$ when $\delta_2 =4$.

The number of predators in the range changes when the number of food that is protected by predators changes $0.00001 \leq \eta < 0.372$, it is observed that system (1) is close to M_9 , although the system reaches M_6 when the parameter is increased further by $0.372 \leq \eta < 0.4495$, and M_3 when the parameter is increased by $0.4495 \leq \eta < 0.6103$, finally, the system is close to M_1

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when the parameter is increased by $0.6103 \leq \eta < 5$, as seen in Figure (9).

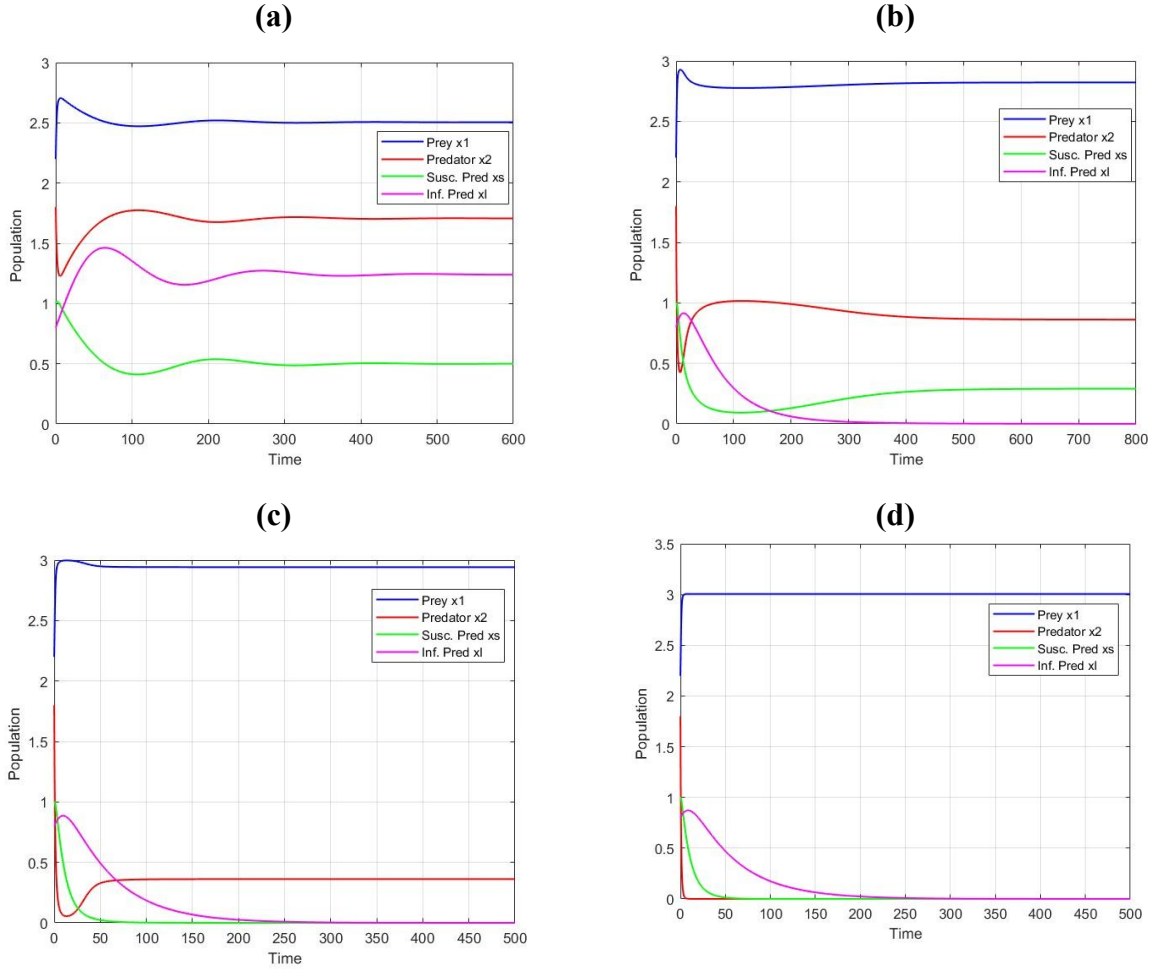


Figure -9 (a) The solution time series for system (1) which approach to $M_9 = (2.505, 1.706, 0.501, 1.24)$ when $\eta = 0.2$, (b) the solution time series for system (1) which approach to $M_6 = (2.822, 0.862, 0.29, 0)$ when $\eta = 0.4$, (c) the solution time series for system (1) which approach to $M_3 = (2.941, 0.362, 0, 0)$ when $\eta = 0.54$, (d) the solution time series for system (1) which approach to $M_1 = (3.006, 0, 0, 0)$ when $\eta = 75$.

The dynamical behavior of system (1) is examined in the value of variations in the disease transmission rate of the top predator $0.00001 \leq \beta < 0.0088$, it is observed that system (1) approaches M_6 . However, when this parameter is further increased to $0.0088 \leq \beta < 0.9$, the system reaches M_9 . Finally, when this parameter is increased to $0.9 \leq \beta \leq 1$, the system (1) approach to periodic solution, as seen in Figure (10).

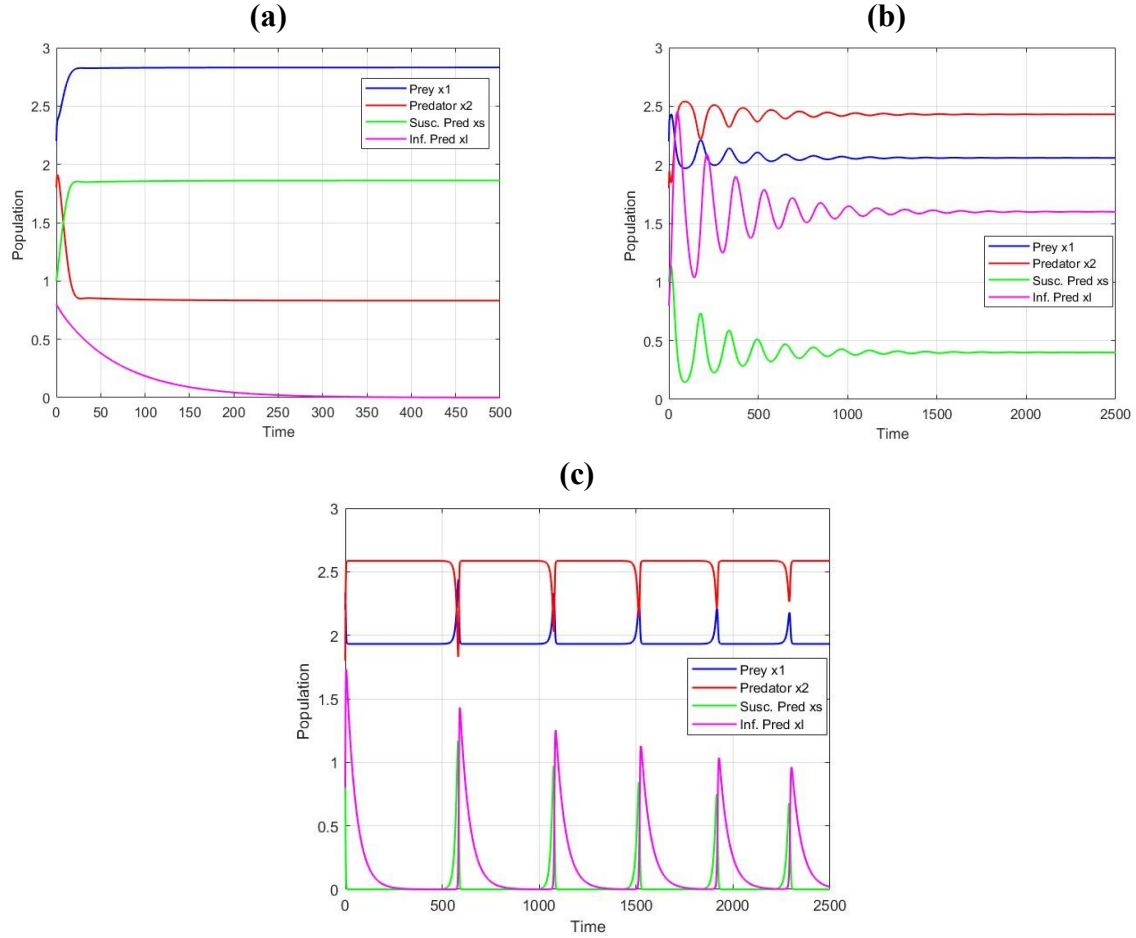


Figure -10 (a) System (1)'s solution time series, which toward to $M_6 = (2.831, 0.832, 1.861, 0)$ when $\beta=0.003$, (b) the solution time series for system (1) which approach to $M_9 = (2.059, 2.43, 0.4, 1.599)$ when $\beta=0.05$, (c) the solution of system (1) approach to periodic solution when $\beta = 0.95$.

6. RESULTS AND DISCUSSION

This work delineates the criteria for the emergence of local bifurcation in a prey-predator model of a food chain that includes the factors of fear, hunting cooperation, anti-predator, toxins and disease in top predator, with the use of Sotomayor's theorem. It is located near the (EP):

- At M_0, M_6 and M_7 system (1) exhibits (TB) at inherent growth rates of the prey (a_1^0), the natural mortality rate of infected apex predators ($\tilde{d}_4$) in order.
- At M_1, M_2, M_3, M_4, M_5 and M_8 system (1) exhibits (TB) and (PB) at inherent growth rates of the prey and the predator ($\bar{a}_1, \hat{a}_1, \check{a}_1, \bar{a}_2$), The rates at which food is converted into energy differ ($\check{g}_2$) in order.

- At M_9, M_{10} and M_{11} system (1) exhibits (SNB) at the mortality rate of infected apex predator $(\dot{d}_4)$ in order.

Additionally, examinations for (HB) at the natural mortality rate of vulnerable predators (d^*) in proximity to M_9, M_{10} and M_{11} are conducted. Conversely, numerical simulations were performed utilizing the MATLAB software for three separate beginning conditions and one theoretical dataset presented in (29). The findings indicated that:

1. The parameters $a_1, f_1, \rho_1, c_1, a_2, k_2, b_1, c_2, \gamma_1, \eta, \delta_2, d_2, g_2, \delta_3, \beta, d_3$ and d_4 have the greatest impact on regulating the system's stability (1).
2. The parameters $k_1, \delta_1, d_1, \rho_2, b_2, f_2, g_1, e_1$ and e_2 are unhelpful in preserving system stability (1) since the solutions keep getting closer to the positive (EP).

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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