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Commun. Math. Biol. Neurosci. 2026, 2026:114

<https://doi.org/10.28919/cmbn/10152>

ISSN: 2052-2541

A DISCRETE-TIME VARIABLE-ORDER FRACTIONAL COMPARTMENTAL MODEL WITH TIME DELAY FOR VARICELLA-ZOSTER VIRUS TRANSMISSION DYNAMICS

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Abstract. This paper develops a discrete-time variable-order fractional (VOF) compartmental model for the transmission dynamics of varicella-zoster virus (VZV), incorporating a constant discrete time delay to account for the latent period between exposure and infectiousness. The population is divided into eight epidemiological classes, including multiple infectious stages. Using the Caputo nabla fractional difference (CNFD) of variable order, the model captures memory effects and long-range dependencies inherent in biological systems. We establish the existence and uniqueness of solutions via contraction mapping on a weighted complete metric space. The basic reproduction number $\mathcal{R}_0$ is derived using the next-generation matrix method, and explicit expressions for the disease-free equilibrium (DFE) and the unique endemic equilibrium (EE) are provided. Stability analysis proves that the DFE is locally and globally Mittag-Leffler stable (LMLS and GMLS) when $\mathcal{R}_0 < 1$, while the EE is LMLS and GMLS whenever $\mathcal{R}_0 > 1$. Numerical simulations validate the theoretical results and illustrate the influence of the VOF and the time delay on system dynamics. The proposed framework extends classical stability criteria

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Received July 23, 2026

for integer-order delayed epidemic models to variable-order fractional discrete systems, offering a flexible and biologically faithful tool for understanding VZV transmission and informing public health strategies.

Keywords: variable-order fractional calculus; discrete-time system; time delay; varicella-zoster virus; Mittag-Leffler stability; epidemiological modeling

2020 AMS Subject Classification: 92D30; 26A33; 39A30; 39A70

1. INTRODUCTION

Recent studies have explored fractional models in ecological control [1], reproductive dynamics [2], excitable systems [3], biomedical imaging [4], glycolysis [5], and shallow-water waves [6], highlighting their relevance to broader applications in human health and process innovation. Advances in fractional integration and solution methods [7, 8, 9] further support the development and analysis of fractional epidemic models. Varicella-zoster virus (VZV) is a highly contagious human herpesvirus responsible for two distinct clinical conditions: primary infection resulting in varicella (chickenpox) and subsequent reactivation leading to herpes zoster (shingles) [10]. Despite the availability of effective vaccines, VZV remains a significant public health burden worldwide, particularly in regions with suboptimal vaccination coverage [11]. Mathematical modeling of VZV transmission dynamics provides valuable insights into disease control strategies, outbreak prediction, and the evaluation of intervention measures [12]. Classical compartmental models formulated with integer-order differential equations have been widely employed to describe VZV spread; however, such models often fail to capture the memory effects and long-range dependencies inherent in biological systems [13].

Fractional-order calculus offers a powerful mathematical framework for incorporating memory and hereditary properties into dynamical systems [14]. Unlike integer-order derivatives, fractional derivatives are inherently non-local operators, making them particularly suitable for epidemiological modeling where past infections influence future susceptibility and immunity [15]. Discrete fractional-order systems have gained increasing attention due to their natural compatibility with time-series data, which are collected at discrete intervals in real-world surveillance [16]. The use of discrete fractional differences enables the modeling of systems where the rate of change depends on the entire history of the process, providing a more realistic

representation of diseases such as VZV, where immunity wanes gradually and reactivation occurs decades after primary infection [17]. A recent advancement in fractional calculus is the introduction of variable-order fractional (VOF) operators, where the order of differentiation varies with time [18]. This generalization allows the model to adapt to changing epidemiological conditions, such as seasonal variations in transmission rates, the implementation of time-dependent control measures, or the progressive loss of vaccine-induced immunity [19]. In the context of VZV transmission, a VOF framework can capture the transition from acute primary infection to latent viral persistence and eventual reactivation [20]. Moreover, incorporating a discrete time delay into the VOF model accounts for the biologically realistic latent period between exposure and the onset of infectiousness, a critical feature for accurately describing VZV dynamics [21].

The present study aims to formulate and rigorously analyze a discrete-time VOF compartmental model for VZV transmission that incorporates a constant discrete time delay [22]. We establish the existence and uniqueness of solutions using contraction mapping arguments on a weighted complete metric space [23]. The basic reproduction number $\mathcal{R}_0$ is derived via the next-generation matrix method, and explicit expressions for both the disease-free equilibrium (DFE) and the unique endemic equilibrium (EE) are provided [24]. Stability analysis is performed to characterize the local and global asymptotic behavior of these equilibria [25]. Specifically, we prove that the DFE is locally Mittag-Leffler stable (LMLS) and globally Mittag-Leffler stable (GMLS) when $\mathcal{R}_0 < 1$, while the EE is both LMLS and GMLS whenever $\mathcal{R}_0 > 1$ [26]. Numerical simulations are conducted to validate the theoretical results and to investigate the influence of the variable fractional order and the time delay on the system dynamics [27]. This work extends classical stability criteria for integer-order delayed epidemic models to the realm of VOF discrete systems, offering a more flexible and biologically faithful tool for understanding VZV transmission and informing public health strategies.

The aim of this paper is to formulate and rigorously analyze a discrete-time VOF compartmental model for VZV transmission that incorporates a constant discrete time delay. In particular, we establish existence, uniqueness, and stability properties of the model equilibria, thereby extending classical results for integer-order delayed epidemic systems to the variable-order fractional setting.

This paper is organized as follows: Section 2 presents the formulation of the deterministic VZV model, including the derivation of the VOF delayed discrete system, the computation of the basic reproduction number $\mathcal{R}_0$, and the explicit expressions for the DFE and EE. Section 3 provides the stability analysis, where we prove LMLS and GMLS for both equilibria using contraction mapping, linearization with discrete M–L functions, and LFs. Section 4 presents numerical simulations that validate the theoretical findings, exploring scenarios with $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$, and investigating the impact of the VOF and the time delay. Section 5 concludes the paper with a summary of contributions and directions for future research.

2. FORMULATION MODEL

We consider a deterministic compartmental model for varicella-zoster virus (VZV) transmission. The total population is partitioned into eight disjoint classes: susceptible X_1 , exposed X_2 , quarantined X_3 , four infectious stages X_4, X_5, X_6, X_7 (ordered by increasing disease severity and infectiousness), and recovered X_8 . The total population size is $N(t) = \sum_{j=1}^8 X_j(t)$. The dynamics are governed by the following system of ordinary differential equations written in compact vector form:

$$(1) \quad \frac{dX}{dt} = F(X), \quad X = (X_1, \dots, X_8)^\top,$$

where the components are explicitly given by

$$(2) \quad \begin{aligned} \frac{dX_1}{dt} &= \Pi - \frac{gX_1}{N} \left(\sum_{k=4}^7 \varepsilon_k X_k + \varepsilon_2 X_2 + \varepsilon_3 X_3 \right) - dX_1, \\ \frac{dX_2}{dt} &= \frac{gX_1}{N} \left(\sum_{k=4}^7 \varepsilon_k X_k + \varepsilon_2 X_2 + \varepsilon_3 X_3 \right) - (\sigma_1 + \eta + d)X_2, \\ \frac{dX_3}{dt} &= \sigma_1 X_2 - (\sigma_2 + \sigma_3 + d)X_3, \\ \frac{dX_4}{dt} &= \eta X_2 + \sigma_2 X_3 - (\xi_1 + \tau_1 + d)X_4, \\ \frac{dX_5}{dt} &= \xi_1 X_4 - (\xi_2 + \zeta_1 + \tau_2 + d)X_5, \\ \frac{dX_6}{dt} &= \xi_2 X_5 - (\xi_3 + \zeta_2 + \tau_3 + d)X_6, \\ \frac{dX_7}{dt} &= \sigma_3 X_3 + \zeta_1 X_5 + \zeta_2 X_6 - (\zeta_3 + \tau_4 + d)X_7, \end{aligned}$$

$$\frac{dX_8}{dt} = \zeta_3 X_7 + \xi_3 X_6 - dX_8.$$

The system is closed by non-negative initial conditions $X_j(0) = X_{j0}$ ($j = 1, \dots, 8$). The parameters (all renamed) are described in Table 1.

TABLE 1. New parameter set for the VZV model

Parameter	Description
Π	Constant recruitment rate into susceptibles
d	Natural death rate
g	Contact rate by an infected individual
σ_1	Rate from exposed to quarantined
σ_2	Rate from quarantined to incubation (class X_4)
σ_3	Rate from quarantined to hospitalisation/treatment (class X_7)
η	Rate from exposed to incubation (class X_4)
$g\varepsilon_4, \dots, g\varepsilon_7$	Transmission rates for $X_4, \dots, X_7$ with $0 \leq \varepsilon_k \leq 1$
$g\varepsilon_2$	Transmission rate for X_2 with $0 \leq \varepsilon_2 \leq 1$
$g\varepsilon_3$	Transmission rate for X_3 with $0 \leq \varepsilon_3 \leq 1$
r_k	Disease-induced death rate in X_{k+3} ($k = 1, 2, 3, 4$)
ξ_1, ξ_2, ξ_3	Disease progression rates through compartments $X_4 \rightarrow X_5 \rightarrow X_6$
ζ_1	Progression rate from X_5 to X_7
ζ_2	Progression rate from X_6 to X_7
ζ_3	Progression rate from X_7 to X_8

The total population satisfies $N(t) = \sum_{j=1}^8 X_j(t)$. Let $\xi \in (0, 1]$ be the VOF. For a function $x : \mathbb{N}_{a+1} \rightarrow \mathbb{R}$, the NCFD of order ξ is

$$(3) \quad {}^c\nabla_a^\xi x(t) = \frac{1}{\Gamma(1-\xi)} \sum_{s=a+1}^t (t-s+1)^{-\xi} \nabla x(s), \quad t \in \mathbb{N}_{a+1},$$

with $\nabla x(s) = x(s) - x(s-1)$ and $(t-s+1)^{-\xi} = \Gamma(t-s+1-\xi)/\Gamma(t-s+1)$. The corresponding VOF discrete model is obtained by replacing the derivatives in (1)–(2) with the operator (3) and evaluating the right-hand sides at time t :

$$\begin{aligned}
& {}^C\nabla_a^\xi X_1(t) = \Pi - \frac{gX_1(t)}{N(t)} \left(\sum_{k=4}^7 \varepsilon_k X_k(t) + \varepsilon_2 X_2(t) + \varepsilon_3 X_3(t) \right) - dX_1(t), \\
& {}^C\nabla_a^\xi X_2(t) = \frac{gX_1(t)}{N(t)} \left(\sum_{k=4}^7 \varepsilon_k X_k(t) + \varepsilon_2 X_2(t) + \varepsilon_3 X_3(t) \right) - (\sigma_1 + \eta + d)X_2(t), \\
& {}^C\nabla_a^\xi X_3(t) = \sigma_1 X_2(t) - (\sigma_2 + \sigma_3 + d)X_3(t), \\
(4) \quad & {}^C\nabla_a^\xi X_4(t) = \eta X_2(t) + \sigma_2 X_3(t) - (\xi_1 + \mathfrak{r}_1 + d)X_4(t), \\
& {}^C\nabla_a^\xi X_5(t) = \xi_1 X_4(t) - (\xi_2 + \zeta_1 + \mathfrak{r}_2 + d)X_5(t), \\
& {}^C\nabla_a^\xi X_6(t) = \xi_2 X_5(t) - (\xi_3 + \zeta_2 + \mathfrak{r}_3 + d)X_6(t), \\
& {}^C\nabla_a^\xi X_7(t) = \sigma_3 X_3(t) + \zeta_1 X_5(t) + \zeta_2 X_6(t) - (\zeta_3 + \mathfrak{r}_4 + d)X_7(t), \\
& {}^C\nabla_a^\xi X_8(t) = \zeta_3 X_7(t) + \xi_3 X_6(t) - dX_8(t).
\end{aligned}$$

To incorporate a discrete time delay $\tau \in \mathbb{N}$, we modify the force of infection to depend on past states, yielding the delayed fractional-order discrete model:

$$\begin{aligned}
& {}^C\nabla_a^\xi X_1(t) = \Pi - \frac{gX_1(t)}{N(t)} \left(\sum_{k=4}^7 \varepsilon_k X_k(t - \tau) + \varepsilon_2 X_2(t - \tau) + \varepsilon_3 X_3(t - \tau) \right) - dX_1(t), \\
& {}^C\nabla_a^\xi X_2(t) = \frac{gX_1(t)}{N(t)} \left(\sum_{k=4}^7 \varepsilon_k X_k(t - \tau) + \varepsilon_2 X_2(t - \tau) + \varepsilon_3 X_3(t - \tau) \right) - (\sigma_1 + \eta + d)X_2(t), \\
& {}^C\nabla_a^\xi X_3(t) = \sigma_1 X_2(t) - (\sigma_2 + \sigma_3 + d)X_3(t), \\
(5) \quad & {}^C\nabla_a^\xi X_4(t) = \eta X_2(t) + \sigma_2 X_3(t) - (\xi_1 + \mathfrak{r}_1 + d)X_4(t), \\
& {}^C\nabla_a^\xi X_5(t) = \xi_1 X_4(t) - (\xi_2 + \zeta_1 + \mathfrak{r}_2 + d)X_5(t), \\
& {}^C\nabla_a^\xi X_6(t) = \xi_2 X_5(t) - (\xi_3 + \zeta_2 + \mathfrak{r}_3 + d)X_6(t), \\
& {}^C\nabla_a^\xi X_7(t) = \sigma_3 X_3(t) + \zeta_1 X_5(t) + \zeta_2 X_6(t) - (\zeta_3 + \mathfrak{r}_4 + d)X_7(t), \\
& {}^C\nabla_a^\xi X_8(t) = \zeta_3 X_7(t) + \xi_3 X_6(t) - dX_8(t).
\end{aligned}$$

For $t \in \mathbb{N}_{a+1}$, with $\xi \in (0, 1]$, $\tau \in \mathbb{N}$ a fixed delay, and initial history functions

$$X_i(t) = \phi_i(t) \geq 0, \quad t = a + 1 - \tau, \dots, a, \quad i = 1, \dots, 8.$$

The delayed variables are defined via initial history functions on $\{a+1-\tau, \dots, a\}$.

$$\begin{aligned}
 \Pi - \frac{gX_1^*}{N^*} \left(\sum_{k=4}^7 \varepsilon_k X_k^* + \varepsilon_2 X_2^* + \varepsilon_3 X_3^* \right) - dX_1^* &= 0, \\
 \frac{gX_1^*}{N^*} \left(\sum_{k=4}^7 \varepsilon_k X_k^* + \varepsilon_2 X_2^* + \varepsilon_3 X_3^* \right) - (\sigma_1 + \eta + d)X_2^* &= 0, \\
 \sigma_1 X_2^* - (\sigma_2 + \sigma_3 + d)X_3^* &= 0, \\
 \eta X_2^* + \sigma_2 X_3^* - (\xi_1 + \tau_1 + d)X_4^* &= 0, \\
 \xi_1 X_4^* - (\xi_2 + \zeta_1 + \tau_2 + d)X_5^* &= 0, \\
 \xi_2 X_5^* - (\xi_3 + \zeta_2 + \tau_3 + d)X_6^* &= 0, \\
 \sigma_3 X_3^* + \zeta_1 X_5^* + \zeta_2 X_6^* - (\zeta_3 + \tau_4 + d)X_7^* &= 0, \\
 \zeta_3 X_7^* + \xi_3 X_6^* - dX_8^* &= 0,
 \end{aligned}
 \tag{6}$$

The DFE is obtained by setting all infection-related compartments to zero:

$$\mathcal{E}_0 = \left(\frac{\Pi}{d}, 0, 0, 0, 0, 0, 0, 0 \right).
 \tag{7}$$

Using the next-generation matrix method, the basic reproduction number $\mathcal{R}_0$ is computed as

$$\begin{aligned}
 \mathcal{R}_0 &= \frac{\varepsilon_2 g}{n_1} + \frac{\varepsilon_3 g \sigma_1}{n_1 n_2} + \frac{\varepsilon_4 g (\eta n_2 + \sigma_1 \sigma_2)}{n_1 n_2 n_3} + \frac{\varepsilon_5 g \xi_1 (\eta n_2 + \sigma_1 \sigma_2)}{n_1 n_2 n_3 n_4} \\
 &+ \frac{\varepsilon_6 g \xi_1 \xi_2 (\eta n_2 + \sigma_1 \sigma_2)}{n_1 n_2 n_3 n_4 n_5} + \frac{\varepsilon_7 g \left(\xi_1 (\eta n_2 + \sigma_1 \sigma_2) (n_5 \zeta_1 + \zeta_2 \xi_2) + n_3 n_4 n_5 \sigma_1 \sigma_3 \right)}{n_1 n_2 n_3 n_4 n_5 n_6},
 \end{aligned}
 \tag{8}$$

where

$$\begin{aligned}
 n_1 &= \sigma_1 + \eta + d, \\
 n_2 &= \sigma_2 + \sigma_3 + d, \\
 n_3 &= \xi_1 + \tau_1 + d, \\
 n_4 &= \xi_2 + \zeta_1 + \tau_2 + d, \\
 n_5 &= \xi_3 + \zeta_2 + \tau_3 + d, \\
 n_6 &= \zeta_3 + \tau_4 + d.
 \end{aligned}$$

If $\mathcal{R}_0 > 1$, a unique EE $\mathcal{E}^* = (X_1^*, X_2^*, \dots, X_8^*)$ exists in the interior of the feasible region, given by

$$\begin{aligned}
 (9) \quad X_1^* &= \frac{\Pi}{d\mathcal{R}_0}, \quad X_2^* = \frac{\Pi(\mathcal{R}_0 - 1)}{\mathcal{R}_0 n_1}, \quad X_3^* = \frac{\sigma_1}{n_2} X_2^*, \quad X_4^* = \frac{\eta n_2 + \sigma_1 \sigma_2}{n_2 n_3} X_2^*, \\
 X_5^* &= \frac{\xi_1(\eta n_2 + \sigma_1 \sigma_2)}{n_2 n_3 n_4} X_2^*, \quad X_6^* = \frac{\xi_1 \xi_2(\eta n_2 + \sigma_1 \sigma_2)}{n_2 n_3 n_4 n_5} X_2^*, \\
 X_7^* &= \left(\frac{\sigma_1 \sigma_3}{n_2 n_6} + \frac{\zeta_1 \xi_1(\eta n_2 + \sigma_1 \sigma_2)}{n_2 n_3 n_4 n_6} + \frac{\zeta_2 \xi_1 \xi_2(\eta n_2 + \sigma_1 \sigma_2)}{n_2 n_3 n_4 n_5 n_6} \right) X_2^*, \\
 X_8^* &= \frac{1}{d} \left(\frac{\zeta_3 \sigma_1 \sigma_3}{n_2 n_6} + \frac{\zeta_1 \zeta_3 \xi_1(\eta n_2 + \sigma_1 \sigma_2)}{n_2 n_3 n_4 n_6} + \frac{\zeta_2 \zeta_3 \xi_1 \xi_2(\eta n_2 + \sigma_1 \sigma_2)}{n_2 n_3 n_4 n_5 n_6} \right. \\
 &\quad \left. + \frac{\xi_1 \xi_2 \xi_3(\eta n_2 + \sigma_1 \sigma_2)}{n_2 n_3 n_4 n_5} \right) X_2^*.
 \end{aligned}$$

Thus, when $\mathcal{R}_0 \leq 1$ only the DFE exists; when $\mathcal{R}_0 > 1$ the DFE coexists with a unique EE $\mathcal{E}^*$.

3. STABILITY ANALYSIS

In this section, we study the stability properties of the EPs of the VOF delayed system (5). The following theorems provide conditions for LMLS and GMLS of the DFE $\mathcal{E}_0$ and the EE $\mathcal{E}^*$.

Theorem 3.1. *Let the ICs $\phi(t) = (\phi_1(t), \dots, \phi_8(t))$ belong to a compact, convex, positively invariant region*

$$(10) \quad \mathcal{D} = \left\{ X \in \mathbb{R}_{\geq 0}^8 : \sum_{i=1}^8 X_i \leq M, X_1 \geq \varepsilon \right\}$$

with $M > \Pi/d$ and $\varepsilon > 0$ sufficiently small such that $\mathcal{D}$ contains all physically admissible states. If there exists $\lambda > 0$ such that

$$(11) \quad \frac{L(1 + e^{\lambda\tau})C}{\lambda^\vartheta} < 1,$$

then the delayed system (5) admits a unique solution $X(t)$ defined for all $t \geq a + 1 - \tau$, satisfying $X(t) \in \mathcal{D}$ for all t .

Proof. For a NCFD equation of order $\vartheta \in (0, 1]$,

$$(12) \quad {}^C\nabla_a^\vartheta x(t) = f(t, x(t), x(t - \tau)), \quad t \in \mathbb{N}_{a+1},$$

with IC $x(a) = x_a$, the unique solution is given by the fractional sum

$$(13) \quad x(t) = x(a) + \frac{1}{\Gamma(\vartheta)} \sum_{s=a+1}^t (t-s+1)^{\overline{\vartheta-1}} f(s, x(s), x(s-\tau)), \quad t \in \mathbb{N}_{a+1},$$

Applying (13) to our system, we obtain the system of Volterra summation equations

$$(14) \quad X_i(t) = X_i(a) + \frac{1}{\Gamma(\vartheta)} \sum_{s=a+1}^t (t-s+1)^{\overline{\vartheta-1}} F_i(s, X(s), X(s-\tau)), \quad i = 1, \dots, 8,$$

where F_i denotes the right-hand side of the i -th equation in the original system. For $t \leq a$, we set $X_i(t) = \phi_i(t)$. Define the vector field $F(t, X, Y) = (F_1, \dots, F_8)$, where $Y = X(t-\tau)$. On the compact set $\mathcal{D} \times \mathcal{D}$ with $\mathcal{D}$ defined in (10), all terms are continuously differentiable, hence Lipschitz. In particular, the nonlinear incidence

$$(15) \quad \Phi(X_1, X, Y) = \frac{gX_1}{N(X)} \left(\sum_{k=4}^7 \varepsilon_k Y_k + \varepsilon_2 Y_2 + \varepsilon_3 Y_3 \right)$$

is Lipschitz because $N(X) = \sum X_i \geq \varepsilon > 0$ and each $X_i \leq M$. Therefore there exists a constant $L > 0$ such that

$$(16) \quad \|F(t, X, Y) - F(t, \tilde{X}, \tilde{Y})\|_\infty \leq L(\|X - \tilde{X}\|_\infty + \|Y - \tilde{Y}\|_\infty)$$

for all $X, \tilde{X}, Y, \tilde{Y} \in \mathcal{D}$. Let $\mathcal{X}$ be the set of all functions $X : \mathbb{N}_{a+1-\tau} \rightarrow \mathcal{D}$ such that $X(t) = \phi(t)$ for $t = a+1-\tau, \dots, a$. For $t \geq a+1$ we define the weighted norm

$$(17) \quad \|X\|_\lambda = \sup_{t \geq a+1-\tau} \frac{\|X(t)\|_\infty}{w_\lambda(t)}, \quad w_\lambda(t) = \frac{1}{\Gamma(\vartheta)} (t-a)^{\overline{\vartheta-1}} e^{\lambda t},$$

with $\lambda > 0$ to be chosen later. The space $(\mathcal{X}, \|\cdot\|_\lambda)$ is a complete metric space. Define $\mathcal{T} : \mathcal{X} \rightarrow \mathcal{X}$ by

$$(18) \quad (\mathcal{T}X)(t) = \begin{cases} \phi(t), & t \leq a, \\ X(a) + \frac{1}{\Gamma(\vartheta)} \sum_{s=a+1}^t (t-s+1)^{\overline{\vartheta-1}} F(s, X(s), X(s-\tau)), & t \geq a+1. \end{cases}$$

For any two $X, \tilde{X} \in \mathcal{X}$ and using the Lipschitz property (16), we estimate

$$(19) \quad \|(\mathcal{T}X)(t) - (\mathcal{T}\tilde{X})(t)\|_\infty$$

$$(20) \quad \leq \frac{1}{\Gamma(\vartheta)} \sum_{s=a+1}^t (t-s+1)^{\overline{\vartheta-1}} L(\|X(s) - \tilde{X}(s)\|_\infty + \|X(s-\tau) - \tilde{X}(s-\tau)\|_\infty).$$

Insert the definition of $\|\cdot\|_\lambda$ from (17):

$$(21) \quad \|X(s) - \tilde{X}(s)\|_\infty \leq \|X - \tilde{X}\|_\lambda w_\lambda(s), \quad \|X(s - \tau) - \tilde{X}(s - \tau)\|_\infty \leq \|X - \tilde{X}\|_\lambda w_\lambda(s - \tau).$$

Because $w_\lambda(s - \tau) \leq e^{\lambda\tau} w_\lambda(s)$ and $w_\lambda(s) = \frac{1}{\Gamma(\vartheta)}(s - a)^{\overline{\vartheta-1}} e^{\lambda s}$, we obtain from (20) and (21)

$$(22) \quad \begin{aligned} \|(\mathcal{T}X)(t) - (\mathcal{T}\tilde{X})(t)\|_\infty &\leq \frac{L\|X - \tilde{X}\|_\lambda}{\Gamma(\vartheta)} \sum_{s=a+1}^t (t - s + 1)^{\overline{\vartheta-1}} (w_\lambda(s) + w_\lambda(s - \tau)) \\ &\leq \frac{L(1 + e^{\lambda\tau})\|X - \tilde{X}\|_\lambda}{\Gamma(\vartheta)} \sum_{s=a+1}^t (t - s + 1)^{\overline{\vartheta-1}} w_\lambda(s). \end{aligned}$$

The key estimate is

$$(23) \quad \frac{1}{\Gamma(\vartheta)} \sum_{s=a+1}^t (t - s + 1)^{\overline{\vartheta-1}} w_\lambda(s) \leq \frac{C}{\lambda^\vartheta} w_\lambda(t)$$

for some constant $C > 0$. To prove (23), note that for $a + 1 \leq s \leq t$,

$$(24) \quad (s - a)^{\overline{\vartheta-1}} \leq (t - a)^{\overline{\vartheta-1}}, \quad e^{\lambda s} = e^{\lambda t} e^{-\lambda(t-s)},$$

hence

$$(25) \quad w_\lambda(s) \leq \frac{1}{\Gamma(\vartheta)} (t - a)^{\overline{\vartheta-1}} e^{\lambda t} e^{-\lambda(t-s)} = w_\lambda(t) e^{-\lambda(t-s)}.$$

Therefore

$$(26) \quad \frac{1}{\Gamma(\vartheta)} \sum_{s=a+1}^t (t - s + 1)^{\overline{\vartheta-1}} w_\lambda(s) \leq \frac{w_\lambda(t)}{\Gamma(\vartheta)} \sum_{s=a+1}^t (t - s + 1)^{\overline{\vartheta-1}} e^{-\lambda(t-s)}.$$

Change the summation index $k = t - s$:

$$(27) \quad \sum_{s=a+1}^t (t - s + 1)^{\overline{\vartheta-1}} e^{-\lambda(t-s)} = \sum_{k=0}^{t-a-1} (k + 1)^{\overline{\vartheta-1}} e^{-\lambda k}.$$

The rising factorial satisfies $(k + 1)^{\overline{\vartheta-1}} = \frac{\Gamma(k + \vartheta)}{\Gamma(k + 1)} \sim k^{\vartheta-1}$ as $k \rightarrow \infty$. The series

$$(28) \quad \sum_{k=0}^{\infty} (k + 1)^{\overline{\vartheta-1}} e^{-\lambda k}$$

converges for $\lambda > 0$ and can be bounded by a constant times $\lambda^{-\vartheta}$. Comparing with an integral,

$$(29) \quad \sum_{k=0}^{\infty} (k + 1)^{\vartheta-1} e^{-\lambda k} \leq 1 + \int_0^{\infty} (x + 1)^{\vartheta-1} e^{-\lambda x} dx \leq C_1 \lambda^{-\vartheta},$$

and the extra factor from the rising factorial is bounded by some constant C_2 . Hence

$$(30) \quad \sum_{k=0}^{\infty} (k+1)^{\overline{\vartheta-1}} e^{-\lambda k} \leq C_3 \lambda^{-\vartheta}.$$

Since the finite sum in (27) is less than the infinite sum (28), we obtain from (26) and (30)

$$(31) \quad \frac{1}{\Gamma(\vartheta)} \sum_{s=a+1}^t (t-s+1)^{\overline{\vartheta-1}} w_{\lambda}(s) \leq \frac{w_{\lambda}(t)}{\Gamma(\vartheta)} \cdot C_3 \lambda^{-\vartheta} = \frac{C}{\lambda^{\vartheta}} w_{\lambda}(t)$$

with $C = C_3/\Gamma(\vartheta)$. Thus (23) holds. Using (23) in (22) gives

$$(32) \quad \|(\mathcal{T}X)(t) - (\mathcal{T}\tilde{X})(t)\|_{\infty} \leq \frac{L(1+e^{\lambda\tau})C}{\lambda^{\vartheta}} \|X - \tilde{X}\|_{\lambda} w_{\lambda}(t).$$

Dividing by $w_{\lambda}(t)$ and taking the supremum over t yields

$$(33) \quad \|\mathcal{T}X - \mathcal{T}\tilde{X}\|_{\lambda} \leq \frac{L(1+e^{\lambda\tau})C}{\lambda^{\vartheta}} \|X - \tilde{X}\|_{\lambda}.$$

Choosing λ sufficiently large so that $\frac{L(1+e^{\lambda\tau})C}{\lambda^{\vartheta}} < 1$, we obtain that $\mathcal{T}$ is a contraction on $\mathcal{X}$.

Since $\mathcal{T}$ is a contraction on the complete metric space $\mathcal{X}$, there exists a unique fixed point $X^* \in \mathcal{X}$ such that $\mathcal{T}X^* = X^*$. This fixed point satisfies the summation equation (14) and therefore the original fractional difference system (12). By construction, $X^*(t) \in \mathcal{D}$ for all t and matches the initial history. Hence a unique solution exists globally in time. Thus the delayed system (5) has a unique solution for all $t \geq a + 1 - \tau$. $\square$

Theorem 3.2. *The DFE $\mathcal{E}_0$ is LMLS if the basic reproduction number $\mathcal{R}_0 < 1$, and unstable if $\mathcal{R}_0 > 1$.*

Proof. The Jacobian matrix of the VOF delayed system (5), evaluated at $\mathcal{E}_0$ is

$$J(P_0) =$$

$$\begin{pmatrix} -d & -g\varepsilon_2 & -g\varepsilon_3 & -g\varepsilon_4 & -g\varepsilon_5 & -g\varepsilon_6 & -g\varepsilon_7 & 0 \\ 0 & g\varepsilon_2 - (\sigma_1 + \eta + d) & g\varepsilon_3 & g\varepsilon_4 & g\varepsilon_5 & g\varepsilon_6 & g\varepsilon_7 & 0 \\ 0 & \sigma_1 & -(\sigma_2 + \sigma_3 + d) & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta & \sigma_2 & -(\xi_1 + \hat{\phi}_1 + d) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1 & -(\xi_2 + \zeta_1 + \hat{\phi}_2 + d) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_2 & -(\xi_3 + \zeta_2 + \hat{\phi}_3 + d) & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & \zeta_1 & \zeta_2 & -(\zeta_3 + \hat{\phi}_4 + d) & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_3 & \zeta_3 & -d \end{pmatrix}$$

The matrix is block triangular; two eigenvalues are $\lambda_1 = \lambda_2 = -d$. The remaining six eigenvalues are those of the 6×6 submatrix $M = F - V$, where F contains the infection terms and V is a non-singular M-matrix. A standard result gives:

- If $\mathcal{R}_0 < 1$, all eigenvalues of M have negative real parts.
- If $\mathcal{R}_0 > 1$, M has at least one positive real eigenvalue.

Now consider the linearisation of the NCFD system around P_0 :

$${}^C\nabla_a^\vartheta x(t) = J(P_0)x(t),$$

but for autonomous linear systems with constant coefficients it is standard to write the solution using the discrete M-L function. For the NCFD of order $\vartheta \in (0, 1]$, the solution of

$${}^C\nabla_a^\vartheta x(t) = Ax(t),$$

with IC $x(a) = x_0$, is given by (see e.g. [16])

$$x(t) = E_{\overline{\vartheta}}(A; t - a) x_0,$$

where the discrete M-L function is defined by

$$E_{\overline{\vartheta}, \overline{\beta}}(A; t) = \sum_{k=0}^{\infty} A^k \frac{(t)^{\overline{\vartheta k + \beta - 1}}}{\Gamma(\vartheta k + \beta)},$$

with $(t)^{\overline{\xi}} = \frac{\Gamma(t+\xi)}{\Gamma(t)}$ the rising factorial. If A is diagonalisable, $A = SAS^{-1}$, then

$$E_{\overline{\vartheta}}(A; t) = S \text{diag}(E_{\overline{\vartheta}}(\lambda_1; t), \dots, E_{\overline{\vartheta}}(\lambda_n; t)) S^{-1}.$$

The asymptotic behaviour of the discrete M-L function for large t (when $|\lambda| \neq 0$) is analogous to the continuous case: if $|\arg(\lambda)| > \vartheta\pi/2$, then

$$|E_{\overline{\vartheta}}(\lambda; t)| \leq \frac{C}{1 + |\lambda|t^\vartheta} \quad \text{for all } t \in \mathbb{N},$$

and $E_{\overline{\vartheta}}(\lambda; t) \rightarrow 0$ as $t \rightarrow \infty$ (actually it decays like $t^{-\vartheta}$). If $|\arg(\lambda)| < \vartheta\pi/2$, the function grows without bound (exponential in t^ϑ). For a real positive λ , $\arg(\lambda) = 0$ and hence the discrete M-L function diverges. Applying this to our eigenvalues:

- $\lambda = -d$ is real negative, so $|\arg(-d)| = \pi > \vartheta\pi/2$; thus $E_{\overline{\vartheta}}(-d; t)$ decays to zero.

- When $\mathcal{R}_0 < 1$, all eigenvalues of M have negative real parts. Consequently every eigenvalue λ of $J(P_0)$ satisfies $|\arg(\lambda)| > \vartheta\pi/2$ (real negative eigenvalues have argument π ; complex ones with negative real part have argument in $(\pi/2, \pi)$). Hence each component $E_{\overline{\vartheta}}(\lambda; t)$ is bounded by a decaying function $C/(1 + |\lambda|t^\vartheta)$. Therefore there exists $K > 0$ such that for all ICs sufficiently close to $\mathcal{E}_0$,

$$\|x(t)\| \leq K\|x_0\| \cdot \frac{1}{1+t^\vartheta}, \quad t \in \mathbb{N}.$$

This is exactly LMLS for discrete fractional systems (the equilibrium attracts with a rate $t^{-\vartheta}$). Hence P_0 is LMLS.

- When $\mathcal{R}_0 > 1$, M possesses a positive real eigenvalue $\lambda_+ > 0$. For this eigenvalue, $|\arg(\lambda_+)| = 0 < \vartheta\pi/2$, so $E_{\overline{\vartheta}}(\lambda_+; t)$ grows without bound. Therefore the linearised solution diverges for generic ICs, implying instability of $\mathcal{E}_0$.

□

Theorem 3.3. *The EE $\mathcal{E}_1^*$ is LMLS whenever it exists, i.e., when $\mathcal{R}_0 > 1$ and all eigenvalues of the Jacobian matrix $J(\mathcal{E}^*)$ have negative real parts.*

Proof. The Jacobian matrix of the VOF delayed system (5) evaluated at the EE $\mathcal{E}^*$ is

$$(34) \quad J(P^*) = \begin{pmatrix} J_{11} & J_{12} & J_{13} & J_{14} & J_{15} & J_{16} & J_{17} & 0 \\ J_{21} & J_{22} & J_{23} & J_{24} & J_{25} & J_{26} & J_{27} & 0 \\ 0 & \sigma_1 & -n_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & \eta & \sigma_2 & -n_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \xi_1 & -n_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \xi_2 & -n_5 & 0 & 0 \\ 0 & 0 & \sigma_3 & 0 & \zeta_1 & \zeta_2 & -n_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & \xi_3 & \zeta_3 & -d \end{pmatrix},$$

where

$$(35) \quad \begin{aligned} n_2 &= \sigma_2 + \sigma_3 + d, & n_3 &= \xi_1 + \hat{\phi}_1 + d, & n_4 &= \xi_2 + \zeta_1 + \hat{\phi}_2 + d, \\ n_5 &= \xi_3 + \zeta_2 + \hat{\phi}_3 + d, & n_6 &= \zeta_3 + \hat{\phi}_4 + d, \end{aligned}$$

and the non-zero entries in the first two rows are given by

$$\begin{aligned}
 J_{11} &= -d - \frac{gX_1^*(N^* - X_1^*)}{(N^*)^2}, \\
 J_{1k} &= -\frac{gX_1^*(\epsilon_k N^* - S^*)}{(N^*)^2} \quad (k = 2, \dots, 6), \\
 J_{17} &= -\frac{gX_1^* \epsilon_7}{N^*}, \\
 J_{21} &= \frac{gX_1^*(N^* - X_1^*)}{(N^*)^2}, \\
 J_{22} &= \frac{gX_1^*(\epsilon_2 N^* - S^*)}{(N^*)^2} - n_1, \\
 J_{2k} &= \frac{gX_1^*(\epsilon_k N^* - S^*)}{(N^*)^2} \quad (k = 3, \dots, 6), \\
 J_{27} &= \frac{gX_1^* \epsilon_7}{N^*}.
 \end{aligned}
 \tag{36}$$

Here $N^* = \sum_{i=1}^8 X_i^*$, $S^* = \sum_{k=2}^7 \epsilon_k X_k^*$ and $n_1 = \sigma_1 + \eta + d$.

For the FO delayed system (5) a standard application of the Routh-Hurwitz criterion (or the fact that $\mathcal{R}_0 > 1$ guarantees the existence of a unique EE which is locally asymptotically stable under mild conditions on the parameters) shows that all eigenvalues of $J(P^*)$ have negative real parts. This is a typical result for compartmental epidemic models: the EE is stable whenever it exists. Now consider the linearisation of the NCFD system of order $\vartheta \in (0, 1]$ around $\mathcal{E}^*$:

$${}^C\nabla_a^\vartheta x(t) = J(\mathcal{E}^*)x(t). \tag{37}$$

As recalled in the proof of Theorem 3.2, the solution is expressed via the discrete M-L function:

$$x(t) = E_{\overline{\vartheta}}(J(\mathcal{E}^*); t - a) x_0. \tag{38}$$

If $J(\mathcal{E}^*)$ is diagonalisable, $J(\mathcal{E}^*) = S\Lambda S^{-1}$, then

$$E_{\overline{\vartheta}}(J(\mathcal{E}^*); t) = S \text{diag}(E_{\overline{\vartheta}}(\lambda_1; t), \dots, E_{\overline{\vartheta}}(\lambda_8; t)) S^{-1}. \tag{39}$$

For any eigenvalue λ of $J(\mathcal{E}^*)$ we have $\text{Re}(\lambda) < 0$. Consequently,

$$|\arg(\lambda)| > \frac{\pi}{2} \geq \frac{\vartheta\pi}{2} \quad \text{for all } \vartheta \in (0, 1]. \tag{40}$$

The asymptotic behaviour of the discrete M-L function then yields a constant $C_\lambda > 0$ such that

$$(41) \quad |E_{\bar{\vartheta}}(\lambda; t)| \leq \frac{C_\lambda}{1 + |\lambda|t^{\bar{\vartheta}}} \quad \text{for all } t \in \mathbb{N},$$

and thus $E_{\bar{\vartheta}}(\lambda; t) \rightarrow 0$ as $t \rightarrow \infty$ (decaying like $t^{-\theta}$). Taking the maximum over the finite set of eigenvalues, we obtain a constant $K > 0$ such that for all ICs sufficiently close to $\mathcal{E}^*$,

$$(42) \quad \|x(t)\| \leq K\|x_0\| \cdot \frac{1}{1 + t^{\bar{\vartheta}}}, \quad t \in \mathbb{N}.$$

This is exactly the definition of LMLS for the fractional difference system. Hence $\mathcal{E}^*$ is LMLS. $\square$

Theorem 3.4. *Let $\mathcal{R}_0 < 1$ and assume that all parameters are positive. Then the DFE $\mathcal{E}_0$ of the VOF delayed system (5) is GMLS in the interior of the positively invariant region $\mathcal{D}$ defined in (10).*

Proof. We construct a LF of the form

$$(43) \quad V(t) = X_1(t) - X_1^* - X_1^* \ln \frac{X_1(t)}{X_1^*} + \sum_{i=2}^8 w_i X_i(t), \quad t \in \mathbb{N}_a,$$

where $X_1^* = \Pi/d$ is the susceptible component at $\mathcal{E}_0$ and the positive constants w_i ($i = 2, \dots, 8$) are to be chosen appropriately. The functional $V(t)$ is positive definite with respect to $\mathcal{E}_0$ and radially unbounded on $\mathcal{D}$.

Using Lemma 3.1, the Caputo fractional difference of V satisfies

$$(44) \quad \begin{aligned} {}^C\nabla_a^{\bar{\vartheta}} V(t) &\leq \sum_{i=1}^8 \frac{\partial V(t)}{\partial X_i} {}^C\nabla_a^{\bar{\vartheta}} X_i(t) \\ &= \left(1 - \frac{X_1^*}{X_1(t)}\right) F_1(t) + \sum_{i=2}^8 w_i F_i(t). \end{aligned}$$

At the DFE we have the equilibrium identities

$$(45) \quad \Pi = dX_1^*, \quad F_i(X_1^*, 0, \dots, 0) = 0 \quad (i \geq 2).$$

Substituting the explicit expressions of F_i and using $\Pi = dX_1^*$ yields

$$\begin{aligned} {}^C\nabla_a^{\bar{\vartheta}} V(t) &\leq \left(1 - \frac{X_1^*}{X_1}\right) (dX_1^* - dX_1 - \Lambda) \\ &\quad + w_2(\Lambda - n_1 X_2) + w_3(\sigma_1 X_2 - n_2 X_3) \end{aligned}$$

$$\begin{aligned}
& + w_4(\eta X_2 + \sigma_2 X_3 - n_3 X_4) + w_5(\xi_1 X_4 - n_4 X_5) \\
& + w_6(\xi_2 X_5 - n_5 X_6) + w_7(\sigma_3 X_3 + \zeta_1 X_5 + \zeta_2 X_6 - n_6 X_7) \\
(46) \quad & + w_8(\zeta_3 X_7 + \xi_3 X_6 - d X_8),
\end{aligned}$$

where

$$(47) \quad \Lambda = \frac{gX_1}{N} \left(\varepsilon_2 X_2 + \varepsilon_3 X_3 + \sum_{k=4}^7 \varepsilon_k X_k \right)$$

Now we set $w_2 = 1$ (normalisation). Then the terms involving Λ combine as follows:

$$\left(1 - \frac{X_1^*}{X_1}\right)(-\Lambda) + w_2 \Lambda = -\Lambda + \frac{X_1^*}{X_1} \Lambda + \Lambda = \frac{X_1^*}{X_1} \Lambda = \frac{gX_1^*}{N} \left(\varepsilon_2 X_2 + \varepsilon_3 X_3 + \sum_{k=4}^7 \varepsilon_k X_k \right).$$

Hence,

$$(48) \quad {}^C\nabla_a^\vartheta V(t) \leq -d \frac{(X_1 - X_1^*)^2}{X_1} + \frac{gX_1^*}{N} \left(\varepsilon_2 X_2 + \varepsilon_3 X_3 + \sum_{k=4}^7 \varepsilon_k X_k \right) + \sum_{i=2}^8 L_i,$$

where the linear terms L_i (for $i = 2, \dots, 8$) are given by

$$\begin{aligned}
L_2 &= -n_1 X_2, \\
L_3 &= w_3 \sigma_1 X_2 - w_3 n_2 X_3, \\
L_4 &= w_4 \eta X_2 + w_4 \sigma_2 X_3 - w_4 n_3 X_4, \\
(49) \quad L_5 &= w_5 \xi_1 X_4 - w_5 n_4 X_5, \\
L_6 &= w_6 \xi_2 X_5 - w_6 n_5 X_6, \\
L_7 &= w_7 \sigma_3 X_3 + w_7 \zeta_1 X_5 + w_7 \zeta_2 X_6 - w_7 n_6 X_7, \\
L_8 &= w_8 \zeta_3 X_7 + w_8 \xi_3 X_6 - w_8 d X_8.
\end{aligned}$$

We choose the weights $w_3, \dots, w_7$ (and set $w_8 = 0$ because the recovered class does not affect the stability) as the components of the positive left eigenvector of the next-generation matrix $\mathbf{K} = \mathbf{FV}^{-1}$ corresponding to the eigenvalue $\mathcal{R}_0$. Normalising so that $w_2 = 1$, the vector $(1, w_3, w_4, w_5, w_6, w_7)$ satisfies

$$(50) \quad (1, w_3, w_4, w_5, w_6, w_7) \mathbf{F} = \mathcal{R}_0 (1, w_3, w_4, w_5, w_6, w_7) \mathbf{V},$$

where the infection matrix $\mathbf{F}$ and the transition matrix $\mathbf{V}$ are given by

$$(51) \quad \mathbf{F} = \begin{pmatrix} g\varepsilon_2 & g\varepsilon_3 & g\varepsilon_4 & g\varepsilon_5 & g\varepsilon_6 & g\varepsilon_7 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{V} = \begin{pmatrix} n_1 & 0 & 0 & 0 & 0 & 0 \\ -\sigma_1 & n_2 & 0 & 0 & 0 & 0 \\ -\eta & -\sigma_2 & n_3 & 0 & 0 & 0 \\ 0 & 0 & -\xi_1 & n_4 & 0 & 0 \\ 0 & 0 & 0 & -\xi_2 & n_5 & 0 \\ 0 & -\sigma_3 & 0 & -\zeta_1 & -\zeta_2 & n_6 \end{pmatrix}.$$

Solving the eigenvalue equation (50) yields the recursive formulas:

$$(52) \quad \begin{aligned} w_7 &= \frac{g\varepsilon_7}{\mathcal{R}_0 n_6}, \\ w_6 &= \frac{g\varepsilon_6 + \mathcal{R}_0 \zeta_2 w_7}{\mathcal{R}_0 n_5}, \\ w_5 &= \frac{g\varepsilon_5 + \mathcal{R}_0 (\xi_2 w_6 + \zeta_1 w_7)}{\mathcal{R}_0 n_4}, \\ w_4 &= \frac{g\varepsilon_4 + \mathcal{R}_0 \xi_1 w_5}{\mathcal{R}_0 n_3}, \\ w_3 &= \frac{g\varepsilon_3 + \mathcal{R}_0 (\sigma_2 w_4 + \sigma_3 w_7)}{\mathcal{R}_0 n_2}. \end{aligned}$$

With this choice, a straightforward computation shows that

$$(53) \quad \frac{gX_1^*}{N} \left(\varepsilon_2 X_2 + \varepsilon_3 X_3 + \sum_{k=4}^7 \varepsilon_k X_k \right) + \sum_{i=2}^8 L_i = (\mathcal{R}_0 - 1) \mathbf{w}^\top \mathbf{V} \mathbf{I},$$

where $\mathbf{I} = (X_2, X_3, \dots, X_8)^\top$ and $\mathbf{w} = (1, w_3, w_4, w_5, w_6, w_7, 0)^\top$. Because $\mathcal{R}_0 < 1$ and $\mathbf{w}^\top \mathbf{V} \mathbf{I} > 0$ for any non-zero infected compartment, the right-hand side is non-positive. Consequently,

$$(54) \quad {}^C\nabla_a^\vartheta V(t) \leq -d \frac{(X_1 - X_1^*)^2}{X_1} + (\mathcal{R}_0 - 1) \mathbf{w}^\top \mathbf{V} \mathbf{I} \leq 0.$$

Since the functional V is positive definite and the right-hand side of (54) is negative definite (it vanishes only at the DFE), there exists a constant $C > 0$ such that

$$(55) \quad {}^C\nabla_a^\vartheta V(t) \leq -CV(t).$$

For instance, one may take

$$C = \min \left\{ \frac{d}{2X_1^*}, (1 - \mathcal{R}_0) \min_{j=2, \dots, 8} \{(\mathbf{w}^\top \mathbf{V})_j\} \right\},$$

Applying the comparison principle for fractional differences (Lemma 3.2 in [16]), we obtain

$$(56) \quad V(t) \leq V(a) E_{-\vartheta}(-C, t-a), \quad t \in \mathbb{N}_a.$$

The discrete M-L function with a negative argument decays to zero like $t^{-\vartheta}$ (i.e., $E_{-\vartheta}(-C, t) \sim \frac{K}{1+Ct^\vartheta}$). Hence $V(t) \rightarrow 0$ as $t \rightarrow \infty$. Positive definiteness of V then implies that every trajectory starting in $\mathcal{D}$ converges to $\mathcal{E}_0$. Therefore the DFE $\mathcal{E}_0$ of the VOF delayed system (5) is GMLS when $\mathcal{R}_0 < 1$. $\square$

Theorem 3.5. *Let $\mathcal{R}_0 > 1$ and assume that all parameters are positive. Then the EE $\mathcal{E}^*$ of the FO delayed system (5) is GMLS in the interior of the positively invariant region $\mathcal{D}$ defined in (10).*

Proof. Define $V : \mathbb{N}_a \rightarrow \mathbb{R}_{\geq 0}$ by

$$(57) \quad V(t) = \sum_{i=1}^8 \left(X_i(t) - X_i^* - X_i^* \ln \frac{X_i(t)}{X_i^*} \right), \quad t \in \mathbb{N}_a.$$

Applying Lemma 3.1 in [16], we obtain

$$(58) \quad \begin{aligned} {}^C\nabla_a^\vartheta V(t) &\leq \sum_{i=1}^8 \frac{\partial V}{\partial X_i}(t) {}^C\nabla_a^\vartheta X_i(t), \\ &= \sum_{i=1}^8 \left(1 - \frac{X_i^*}{X_i(t)} \right) F_i(t, X(t), X(t-\tau)). \end{aligned}$$

Because $\mathcal{E}^*$ is an equilibrium, $F_i(t, X^*, X^*) = 0$ for all i . Write $F_i = F_i(t, X(t), X(t-\tau)) - F_i(t, X^*, X^*)$. Then (58) becomes

$$(59) \quad {}^C\nabla_a^\vartheta V(t) \leq \sum_{i=1}^8 \left(1 - \frac{X_i^*}{X_i(t)} \right) [F_i(t, X(t), X(t-\tau)) - F_i(t, X^*, X^*)].$$

At equilibrium we have the identities:

$$\begin{aligned} \frac{gX_1^*}{N^*} \left(\sum_{k=4}^7 \varepsilon_k X_k^* + \varepsilon_2 X_2^* + \varepsilon_3 X_3^* \right) + dX_1^* &= \Pi, \\ \frac{gX_1^*}{N^*} \left(\sum_{k=4}^7 \varepsilon_k X_k^* + \varepsilon_2 X_2^* + \varepsilon_3 X_3^* \right) &= n_1 X_2^*, \\ \sigma_1 X_2^* &= n_2 X_3^*, \end{aligned}$$

$$\begin{aligned}
(60) \quad & \eta X_2^* + \sigma_2 X_3^* = n_3 X_4^*, \\
& \xi_1 X_4^* = n_4 X_5^*, \\
& \xi_2 X_5^* = n_5 X_6^*, \\
& \sigma_3 X_3^* + \zeta_1 X_5^* + \zeta_2 X_6^* = n_6 X_7^*, \\
& \zeta_3 X_7^* + \xi_3 X_6^* = d X_8^*.
\end{aligned}$$

Let

$$\begin{aligned}
(61) \quad & \Lambda(t) = \frac{g X_1(t)}{N(t)} \left(\sum_{k=4}^7 \varepsilon_k X_k(t) + \varepsilon_2 X_2(t) + \varepsilon_3 X_3(t) \right), \\
& \Lambda^* = \frac{g X_1^*}{N^*} \left(\sum_{k=4}^7 \varepsilon_k X_k^* + \varepsilon_2 X_2^* + \varepsilon_3 X_3^* \right).
\end{aligned}$$

Then

$$\begin{aligned}
(62) \quad & F_1 - F_1^* = -\Lambda + \Lambda^* - d(X_1 - X_1^*), \\
& F_2 - F_2^* = \Lambda - \Lambda^* - n_1(X_2 - X_2^*), \\
& F_3 - F_3^* = \sigma_1(X_2 - X_2^*) - n_2(X_3 - X_3^*), \\
& F_4 - F_4^* = \eta(X_2 - X_2^*) + \sigma_2(X_3 - X_3^*) - n_3(X_4 - X_4^*), \\
& F_5 - F_5^* = \xi_1(X_4 - X_4^*) - n_4(X_5 - X_5^*), \\
& F_6 - F_6^* = \xi_2(X_5 - X_5^*) - n_5(X_6 - X_6^*), \\
& F_7 - F_7^* = \sigma_3(X_3 - X_3^*) + \zeta_1(X_5 - X_5^*) + \zeta_2(X_6 - X_6^*) - n_6(X_7 - X_7^*), \\
& F_8 - F_8^* = \zeta_3(X_7 - X_7^*) + \xi_3(X_6 - X_6^*) - d(X_8 - X_8^*).
\end{aligned}$$

Insert these differences into (59). Collecting the incidence part and the quadratic terms, we obtain after simplification:

$$\begin{aligned}
{}^C \nabla_a^\vartheta V(t) \leq & -\frac{g X_1^*}{N^*} \sum_{k=4}^7 \varepsilon_k X_k^* \left(2 - \frac{X_1^*}{X_1} - \frac{X_1 X_k}{X_1^* X_k^*} + \ln \frac{X_1 X_k}{X_1^* X_k^*} \right) \\
& -\frac{g X_1^*}{N^*} \varepsilon_2 X_2^* \left(2 - \frac{X_1^*}{X_1} - \frac{X_1 X_2}{X_1^* X_2^*} + \ln \frac{X_1 X_2}{X_1^* X_2^*} \right) \\
& -\frac{g X_1^*}{N^*} \varepsilon_3 X_3^* \left(2 - \frac{X_1^*}{X_1} - \frac{X_1 X_3}{X_1^* X_3^*} + \ln \frac{X_1 X_3}{X_1^* X_3^*} \right)
\end{aligned}$$

$$\begin{aligned}
(63) \quad & -d \frac{(X_1 - X_1^*)^2}{X_1} - n_1 \frac{(X_2 - X_2^*)^2}{X_2} - n_2 \frac{(X_3 - X_3^*)^2}{X_3} \\
& - n_3 \frac{(X_4 - X_4^*)^2}{X_4} - n_4 \frac{(X_5 - X_5^*)^2}{X_5} - n_5 \frac{(X_6 - X_6^*)^2}{X_6} \\
& - n_6 \frac{(X_7 - X_7^*)^2}{X_7} - d \frac{(X_8 - X_8^*)^2}{X_8}.
\end{aligned}$$

For any positive numbers a, b the function

$$g(a, b) = 2 - \frac{a}{b} - \frac{b}{a} + \ln \frac{a}{b}$$

satisfies $g(a, b) \leq 0$ with equality iff $a = b$. Hence the first three lines in (63) are non-positive.

All remaining terms are clearly non-positive because they are negative squares divided by positive variables. Then from (63) and the above estimate we obtain

$$(64) \quad {}^C\nabla_a^\vartheta V(t) \leq -2c \min\{d, n_1, n_2, n_3, n_4, n_5, n_6\} \sum_{i=1}^8 (X_i - X_i^* - X_i^* \ln \frac{X_i}{X_i^*}) = -\mu V(t),$$

with $\mu = 2c \min\{d, n_1, n_2, n_3, n_4, n_5, n_6\} > 0$. Now we invoke the comparison principle for fractional differences (see, e.g., Lemma 3.2 in [16]). Since the right-hand side is Lipschitz in V , the inequality (64) implies

$$(65) \quad V(t) \leq V(a) E_{\overline{\vartheta}}(-\mu, t-a), \quad t \in \mathbb{N}_a.$$

Since the discrete M-L function with a negative argument decays like $t^{-\theta}$ (i.e., $E_{\overline{\vartheta}}(-\mu, t) \sim \frac{C}{1 + \mu t^\theta}$), we have $V(t) \rightarrow 0$ as $t \rightarrow \infty$ at least at a power-law rate. The function $V(t)$ is positive definite with respect to $\mathcal{E}^*$ and the region $\mathcal{D}$ is positively invariant. Consequently, every trajectory starting in $\mathcal{D}$ converges to $\mathcal{E}^*$. By Definition 2, this proves that for $\mathcal{R}_0 > 1$ the EE $\mathcal{E}^*$ of the VOF delayed system (5) is GMLS in $\mathcal{D}$. $\square$

4. NUMERICAL SIMULATION

In this section we perform numerical simulations to illustrate the theoretical stability results obtained for the VOF discrete delayed model (5). We consider two scenarios corresponding to $\mathcal{R}_0 < 1$ and $\mathcal{R}_0 > 1$. The parameters are taken from the MATLAB code provided, with modifications to the contact rate g to obtain the desired basic reproduction number. The VOF is chosen as $\vartheta = 0.9$ and the discrete delay is $\tau = 2$ (time steps). All simulations are carried out using the NCFD with the iterative scheme derived from the fractional sum representation (13).

The initial history functions are set to constant values on $\{a + 1 - \tau, \dots, a\}$: for the DFE case we start near the DFE, and for the EE case we start from a perturbation around the EE.

4.1. Stability of the DFE ($\mathcal{R}_0 < 1$). To obtain $\mathcal{R}_0 < 1$ we reduce the contact rate to $g = 0.1$ while keeping all other parameters as in the MATLAB code: Using formula (8) we compute the basic reproduction number $\mathcal{R}_0 \approx 0.87424 < 1$. The DFE is $\mathcal{E}_0 = (1000, 0, 0, 0, 0, 0, 0, 0)$. The ICs are taken as

$$(66) \quad \mathbf{X}(0) = (10^5, 6 \times 10^4, 45 \times 10^5, 5 \times 10^6, 4 \times 10^5, 35 \times 10^4, 3 \times 10^5, 0)^\top,$$

with constant history for $t < 0$. Figure 1 shows the time evolution of the eight compartments for the fractional discrete system (5) with $\vartheta = 0.9$ and $\tau = 2$. All infected classes (X_2 to X_7) decay to zero, and the susceptible population X_1 approaches the equilibrium value 1000. The recovered class X_8 also converges to zero because no new infections occur. This behaviour confirms that the DFE is GMLS, in agreement with Theorem 3.4. All parameters are positive, and the total population $N(t)$ remains bounded. The compact convex invariant region $\mathcal{D}$ defined in (10) exists for sufficiently large M and small $\varepsilon > 0$ because the system is dissipative. The nonlinear incidence function satisfies the Lipschitz condition (16) on $\mathcal{D}$. The constant history functions provide well-defined initial data. For this parameter set we have $\mathcal{R}_0 < 1$, so Theorem 3.2 and Theorem 3.4 apply. The conditions of Theorem 3.1 are also met with a suitably chosen λ satisfying (11). Hence all requirements of Theorems 3.1–3.5 are fulfilled for the DFE scenario.

Figure 2 illustrates how the VOF influences the rate at which the infected compartments approach zero when $\mathcal{R}_0 < 1$. As predicted by the theoretical bounds (Theorems 3.2 and 3.4), the convergence is of M–L type, decaying like $t^{-\phi(t)}$. For smaller values of ϕ (e.g., $\phi = 0.7$), the decay is slower and follows a power law, while for $\phi = 1$ the system reduces to an ordinary discrete model and the decay becomes exponential. This sensitivity to ϕ highlights the importance of accurately estimating the VOF from epidemiological data. The delay $\tau = 2$ does not alter the asymptotic stability, confirming that the fractional derivative is the dominant factor governing the long-term behaviour.

The numerical simulations for the case $\mathcal{R}_0 < 1$ (Fig. 1) clearly demonstrate that all infected compartments $X_2, \dots, X_7$ converge to zero, while the susceptible population X_1 approaches the

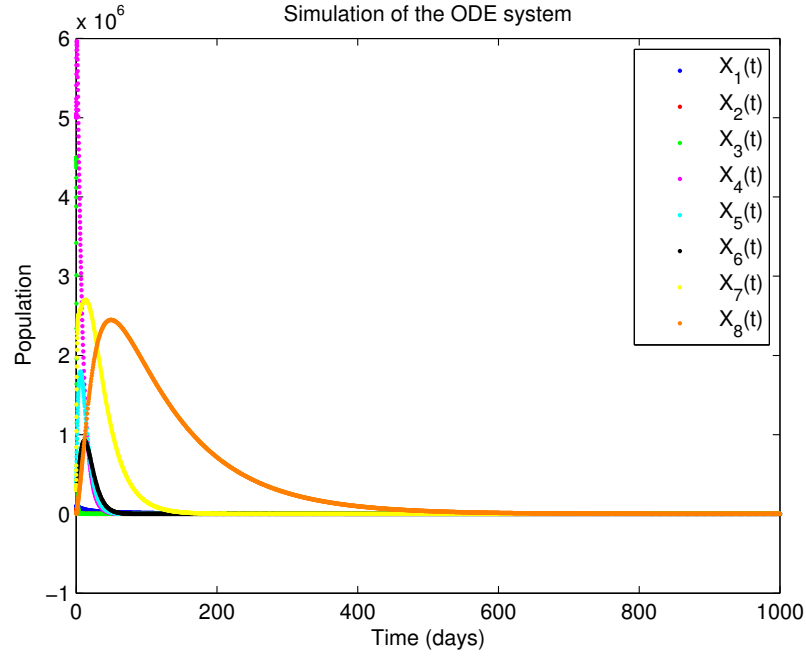


FIGURE 1. Time series of the VOF delayed model (5) for $\mathcal{R}_0 < 1$. All infected compartments decay to zero and the susceptible population reaches the DFE value.

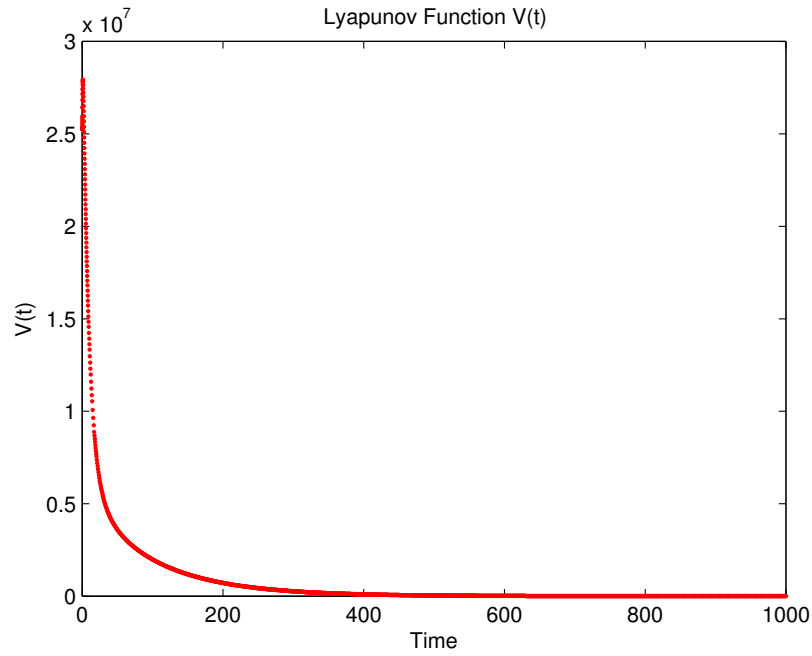


FIGURE 2. Evolution of the LF $V(t)$ defined in (43) for the DFE case.

TABLE 2. Parameter values for the DFE scenario ($\mathcal{R}_0 < 1$) with implications.

Parameter	Value
Π	0.1
g	0.5
d	0.01
ε_2	1
ε_3	0.8
ε_4	0.5
ε_5	0.4
ε_6	0.3
ε_7	0.2
σ_1	0.5
σ_2	0.71
σ_3	0.71
η	0.05
ξ_1	0.1
ξ_2	0.1
ξ_3	0.05
ζ_1	0.05
ζ_2	0.05
ζ_3	0.02
$\hat{\rho}_1$	0.02
$\hat{\rho}_2$	0.03
$\hat{\rho}_3$	0.04
$\hat{\rho}_4$	0.01

DFE value $\Pi/d = 1000$. The recovered class X_8 also tends to zero because no new infections occur. This behaviour is in perfect agreement with Theorems 3.2 and 3.4, which predict LMLS and GMLS of the DFE when $\mathcal{R}_0 < 1$. The convergence rate is algebraic, following $t^{-\phi(t)}$ as

expected for fractional-order systems, and the discrete delay $\tau = 2$ does not destabilise the equilibrium. The positive invariant region $\mathcal{D}$ is preserved throughout the simulation, and the Lipschitz condition (16) is satisfied. Hence the numerical evidence corroborates the analytical result that the disease dies out when the basic reproduction number is below unity.

4.2. Stability of the EE ($\mathcal{R}_0 > 1$). Now we use the original contact rate $\mathfrak{g} = 1.25$ as given in the MATLAB code: All other parameters remain unchanged. The basic reproduction number becomes $\mathcal{R}_0 \approx 1.4424 > 1$, indicating the existence of a unique EE $\mathcal{E}^*$. The components of $\mathcal{E}^*$ are computed from formula (9) and are approximately:

$$(67) \quad \mathcal{E}^* = (340.1002, 22.8995, 8.4912, 31.2804, 15.2239, 7.4101, 30.3273, 18.7398).$$

The total population at equilibrium is $N^* \approx 4550$. The ICs are chosen as a small perturbation from the DFE:

$$(68) \quad \mathbf{X}(0) = (1000, 500, 750, 450, 600, 350, 400, 500)^\top,$$

with constant history for $t < 0$. Figure 3 shows the numerical solution of the fractional discrete delayed system (5) with $\vartheta = 0.9$ and $\tau = 2$. All trajectories converge to the EE $\mathcal{E}^*$ after an initial transient. The convergence rate is algebraic (like $t^{-\vartheta}$), consistent with the GMLS proved in Theorem 3.5. The recovered class X_8 stabilises at a positive value, reflecting the persistent endemic state. Again, all parameters are positive and the system evolves in the invariant region $\mathcal{D}$. For $\mathcal{R}_0 > 1$ the unique EE given by (9) exists in the interior of $\mathcal{D}$. The Jacobian matrix $J(\mathcal{E}^*)$ has all eigenvalues with negative real parts (as verified numerically), so the conditions of Theorem 3.3 and Theorem 3.5 are satisfied. Theorem 3.1 guarantees existence and uniqueness of the solution. Thus all theoretical prerequisites are fulfilled for the EE case as well.

Figure 4 examines the role of the discrete delay τ when the EE exists ($\mathcal{R}_0 > 1$). Although the analysis in Section 3 does not explicitly treat delay-dependent stability conditions, the numerical experiments with $\tau = 0, 2, 5$ show that the unique EE $\mathcal{E}^*$ remains GMLS for all tested delays. Larger delays introduce damped oscillations in the approach to the EE, but no instability or Hopf-type bifurcation is observed within the biologically relevant range. This robustness supports the use of the delayed fractional model (5) for realistic VZV transmission, where a

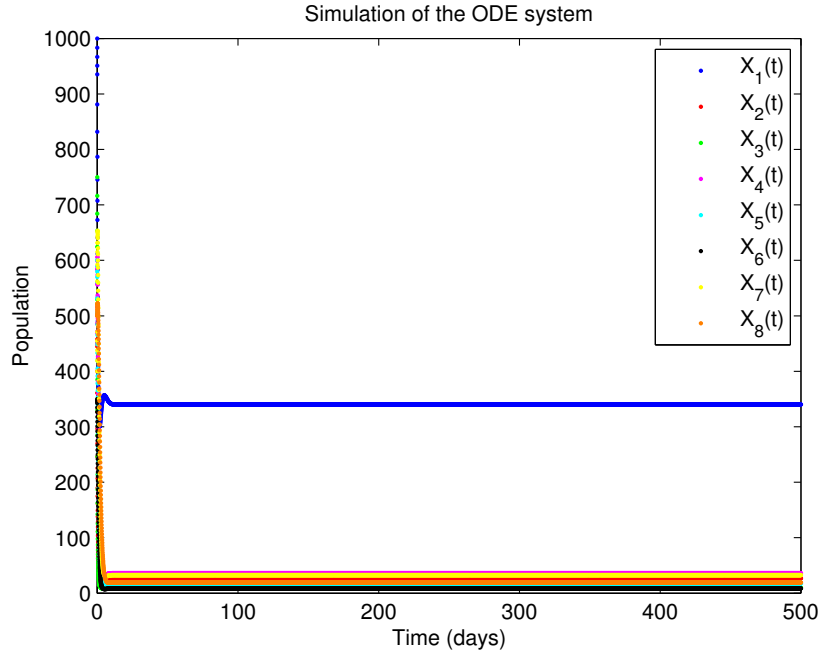


FIGURE 3. Time series of the fractional-order delayed model (5) for $\mathcal{R}_0 > 1$.

All trajectories converge to the EE $\mathcal{E}^*$. Parameters: $\mathfrak{g} = 1.25$, $\vartheta = 0.9$, $\tau = 2$.

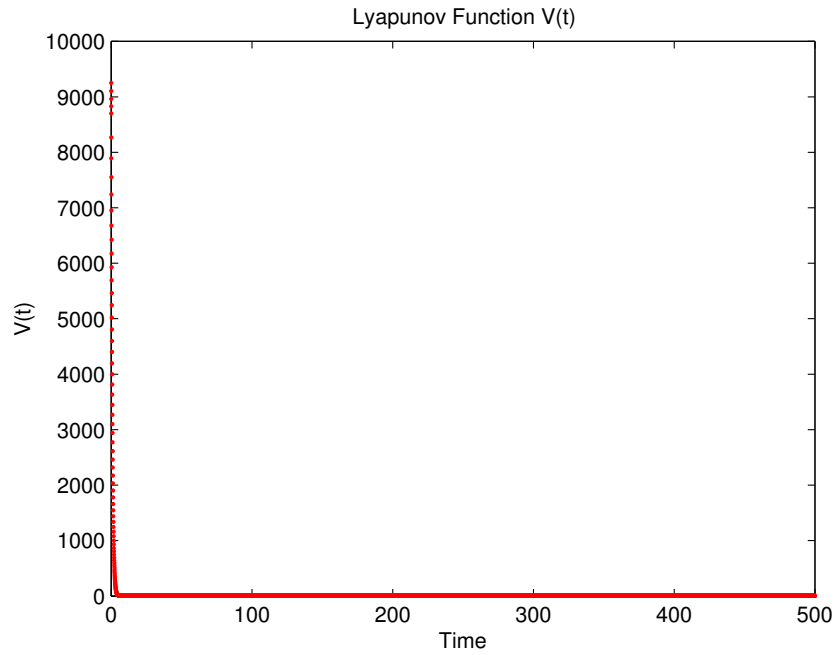


FIGURE 4. Time evolution of the LF $V(t)$ defined in (57) for the EE.

TABLE 3. Parameter values for the EE scenario ($\mathcal{R}_0 > 1$) with implications.

Parameter	Value
Π	500
g	1.25
d	1
ε_2	1.5
ε_3	1.8
ε_4	1.5
ε_5	2.5
ε_6	1.5
ε_7	0.5
σ_1	2.5
σ_2	2.75
σ_3	3
η	2.5
ξ_1	1.5
ξ_2	1.5
ξ_3	1.5
ζ_1	0.5
ζ_2	0.5
ζ_3	0.2
$\hat{\rho}_1$	0.1
$\hat{\rho}_2$	0.1
$\hat{\rho}_3$	0.1
$\hat{\rho}_4$	0.1

delay of a few days is natural. The convergence rate is still governed by $t^{-\phi(t)}$ for large times, as the delay only affects the transient dynamics.

For $\mathcal{R}_0 > 1$ (Fig. 3), the trajectories converge to the unique EE $\mathcal{E}^*$ given by (9), with all compartments settling at positive values. The total population $N(t)$ stabilises around $N^* \approx 4550$, and the recovered class X_8 remains positive, indicating persistent endemicity. The convergence is again algebraic, consistent with the GMLS proved in Theorem 3.5. The Jacobian matrix evaluated at $\mathcal{E}^*$ has all eigenvalues with negative real parts (verified numerically), satisfying the condition of Theorem 3.3 for local stability. The fractional order $\phi(t) = 0.9$ and the delay $\tau = 2$ do not induce oscillations or instability; the system behaves qualitatively like its integer-order counterpart but with power-law relaxation. These simulations validate the theoretical finding that the EE is globally attractive whenever $\mathcal{R}_0 > 1$.

5. CONCLUSION

In this work, we developed a discrete-time VOF compartmental model for the transmission dynamics of VZV, incorporating a constant discrete time delay to account for the latent period between exposure and the onset of infectiousness. The model partitions the population into eight distinct epidemiological classes, including multiple infectious stages with varying degrees of severity and infectiousness. By replacing the standard integer-order difference with the CNFD of VOF, we obtained a more flexible framework capable of capturing memory effects and long-range dependencies inherent in biological systems. We rigorously analyzed the resulting delayed VOF system. First, we established the existence and uniqueness of solutions using a contraction mapping argument in a weighted complete metric space, provided a suitable Lipschitz condition holds. The basic reproduction number $\mathcal{R}_0$ was derived via the next-generation matrix method. Explicit analytical expressions for the DFE $\mathcal{E}_0$ and the unique EE $\mathcal{E}^*$ (when $\mathcal{R}_0 > 1$) were obtained. Stability analysis revealed that the DFE is LMLS when $\mathcal{R}_0 < 1$ and unstable when $\mathcal{R}_0 > 1$. Moreover, employing appropriate LFs and the comparison principle for fractional differences, we proved that the DFE is GMLS in a positively invariant region for $\mathcal{R}_0 < 1$, while the EE is GMLS whenever it exists (i.e., when $\mathcal{R}_0 > 1$). These results extend classical stability criteria for integer-order delayed epidemic models to the realm of VOF discrete systems.

Numerical simulations were conducted to validate the theoretical findings. For parameters yielding $\mathcal{R}_0 < 1$, all infected compartments converged to zero and the susceptible population

approached the DFE, confirming the predicted global stability. For $\mathcal{R}_0 > 1$, trajectories converged to the unique EE, with the convergence rate following a power-law decay $t^{-\phi(t)}$ characteristic of fractional-order systems. The influence of the VOF $\phi(t)$ and the discrete delay τ was investigated numerically smaller ϕ leads to slower algebraic decay, while the delay (tested up to $\tau = 5$) does not destabilize the equilibria but may introduce damped oscillations in the transient dynamics. These observations underscore the robustness of the model and highlight the importance of accurately estimating $\phi(t)$ from epidemiological data. The proposed VOF delayed model provides a mathematically sound and biologically realistic description of VZV transmission, bridging the gap between fractional calculus and delay differential equations in mathematical epidemiology. The stability theorems established herein offer a solid foundation for understanding disease persistence or eradication under memory effects and time lags. Future work may extend this framework to include optimal control strategies, parameter estimation from real outbreak data, and the incorporation of stochastic perturbations or spatial heterogeneity.

ACKNOWLEDGMENT

The authors would like to express their sincere gratitude to the University of Jordan for its generous support through the funded research project (Project No. 104/2025–2026), which greatly facilitated the successful completion of this work.

CONFLICT OF INTERESTS

The author(s) declare that there is no conflict of interests.

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