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## MODELING OF BOUNDED DATA VIA ORDER STATISTICS: EXTROPY, FISHER INFORMATION, SIMULATION, AND APPLICATIONS

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**Abstract.** Bounded data on the unit interval, arising from proportions, rates, and normalized lifetimes, are ubiquitous in reliability engineering, biomedical research, and applied sciences, yet the information-theoretic tools needed to fully exploit flexible unit-interval models remain underdeveloped. This paper presents a unified information-theoretic treatment of the unit extended exponential distribution (UEExpD), a versatile bounded lifetime model, built around extropy and its weighted, cumulative residual, and negative cumulative variants. We derive closed-form expressions for these measures for the UEExpD itself and extend them to its order statistics

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through one-dimensional integral representations, evaluated by numerical quadrature, which characterize the information content across ordered and extreme observations. The Fisher information matrix is obtained and examined as a function of sample order, identifying which observations are most informative for parameter estimation. Maximum likelihood estimation is used throughout, with extensive simulation studies assessing bias, mean square error, and relative error across sample sizes and supporting the consistency of the estimators; every closed-form result is independently validated against numerical benchmarks. The framework is further illustrated on two real datasets, to which the UEEpD provides an adequate fit, demonstrating that the UEEpD offers a theoretically rich and practically useful tool for information-theoretic and reliability analysis of bounded data.

**Keywords:** bounded data; extropy; cumulative residual extropy; Fisher information matrix; simulation study; data analysis

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## 1. INTRODUCTION

Information measures are mathematical tools that quantify the amount of information, uncertainty, or unpredictability associated with a system or a random variable (RV). The most classical and well-known measure in this field is Shannon entropy. It measures the unpredictability of events in a probabilistic setting. Measures of information are a fundamental paradigm for understanding data compression, communication, and uncertainty analysis in many scientific fields. Entropy remains an essential concept in many applications, including cryptography, statistical inference, and communication systems [1]. It offers a mathematical framework to quantify uncertainty. In addition to entropy, other measures such as mutual information and relative entropy broaden the theoretical framework for comparing probability distributions and quantifying the shared information, thereby expanding the functional scope of information theory. Extropy (EX) is presented as a complementary concept to entropy, providing a new perspective on information quantification. Lad et al. [2] describe EX as a counterpoint to entropy in that it provides a measure of “order” or informational richness to complement the usual emphasis on “disorder”, which is what entropy captures. Whereas entropy measures uncertainty, EX reflects a related but distinct property of the structure of a probability distribution [2]. Gaveau et al. [3] work by relating entropy and EX to thermodynamic notions, focusing on its relevance in non-equilibrium stochastic systems. As shown in [4, 5, 6, 7], EX is becoming more and more important in modern information theory, where more theoretical formulations and applications

of EX are given. EX has begun to be investigated in various theoretical and practical fields, such as economics, reliability engineering, decision theory, and statistical modeling. Distributions with the same entropy but different shapes or dispersions can be distinguished more effectively by extropy-based measures. Incorporating these measures into composite information criteria therefore gives researchers additional tools for understanding system dynamics and the robustness of inference. The EX of an RV  $X$  with probability density function (PDF)  $f(x)$  is defined as follows:

$$(1) \quad \Psi(X) = -\frac{1}{2} \int_0^\infty [f(x)]^2 dx \leq 0.$$

Weighted extropy (WEX) is an informational measure that extends the notion of EX by introducing a weighting function to assign more importance to different values of an RV, usually the larger ones. Unlike classical EX, which is shift-invariant, WEX is shift-dependent and hence responsive to the magnitude of the outcomes of a distribution; see [8], [9], and [10]. This feature is useful in areas like reliability theory and in cases where the value scale is important to quantify uncertainty. The weighting function in WEX enables a more fine-grained evaluation of the uncertainty. It is used to focus on particular parts of the probability distribution, often where long-tailed or asymmetric data are involved, which are not well handled by unweighted procedures. Therefore, it is an important tool in statistical inference, particularly in the analysis of lifetime data and system reliability. The WEX of a continuous RV  $X$  is defined as:

$$(2) \quad \Psi^w(X) = -\frac{1}{2} \int_0^\infty x [f(x)]^2 dx.$$

The cumulative residual extropy (CuREX) estimates the uncertainty related to the residual lifespan of a non-negative continuous RV. It is characterized by substituting the PDF in conventional EX with the survival function of the RV, enhancing its stability and significance in reliability analysis and life data modeling. Jahanshahi et al. [11] proposed the CuREX of an RV with survival function  $\bar{F}_X = 1 - F_X$  as:

$$(3) \quad \Phi(X) = -\frac{1}{2} \int_0^\infty [\bar{F}(x)]^2 dx.$$

Recently, Hashempour et al. [12] introduced a novel information measure termed weighted CuREX (WCuREX), which attributes greater significance to larger values of the examined RV

and is defined as:

$$(4) \quad \Phi^w(X) = -\frac{1}{2} \int_0^\infty x [\bar{F}(x)]^2 dx.$$

Negative cumulative extropy (NegCEX) extends the notion of cumulative extropy by quantifying the information content of the CDF. Its theoretical properties and its applications in reliability theory and in the information analysis of uncertain systems have been studied in [13]. It complements the entropy and EX measures and provides further insight into the structure of the data and of the underlying stochastic process. For an RV  $X$  with CDF  $F(x)$ , Tahmasebi and Toomaj [14] defined the NegCEX as:

$$(5) \quad \zeta(X) = \frac{1}{2} \int_0^\infty (1 - F^2(x)) dx.$$

Weighted NegCEX (WNegCEX) is a sophisticated information measure that extends the NegCEX concept by including a weight function to highlight various segments of the distribution. The WNegCEX is defined by Kumar Chaudhary et al. [15] as

$$(6) \quad \zeta^w(X) = \frac{1}{2} \int_0^\infty x(1 - F^2(x)) dx.$$

Fisher information (FI) measures the amount of information that an observable RV carries about an unknown parameter of its distribution. It governs the precision attainable by estimators, through the Cramér–Rao bound, and is closely related to sufficiency: a statistic is sufficient when it retains all the information that the sample contains about the parameter. First, FI is used to construct a default (Jeffreys) prior in the Bayesian paradigm; second, it is used to generate hypothesis tests and confidence intervals using maximum likelihood estimators (MLEs). In the minimum description length paradigm, FI is used to measure model complexity. Several authors have recently investigated FI for ordered data [16, 17, 18]. The notion of FI extends naturally to a vector parameter  $\underline{\theta} = (\theta_1, \theta_2)$ , resulting in the Fisher information matrix (FIM)  $\mathbb{I}$ , defined as:

$$(7) \quad \mathbb{I}(X; \underline{\theta}) = \begin{pmatrix} I_{\theta_1}(X; \underline{\theta}) & -\mathbb{E} \left( \frac{\partial^2 \log f(X; \underline{\theta})}{\partial \theta_2 \partial \theta_1} \right) \\ -\mathbb{E} \left( \frac{\partial^2 \log f(X; \underline{\theta})}{\partial \theta_1 \partial \theta_2} \right) & I_{\theta_2}(X; \underline{\theta}) \end{pmatrix},$$

where

$$(8) \quad I(X; \theta) = -\mathbb{E} \left( \frac{\partial^2 \log f(X; \theta)}{\partial \theta^2} \right) = \mathbb{E} \left( \frac{\partial \log f(X; \theta)}{\partial \theta} \right)^2.$$

At present, there are many generalized distributions that have been applied in various fields like biomedical sciences, environmental research, engineering, finance, and so on. These applications show that datasets following classical distributions are the exception rather than the rule. Consequently, statisticians have made substantial progress in generalizing the traditional models and applying them in practice. Generalizations of the exponential distribution (ExpD) have been proposed by several authors. Gupta and Kundu [19] introduced the generalized ExpD, Nadarajah and Kotz [20] the beta ExpD, and Cordeiro et al. [21] the Kumaraswamy Weibull distribution, which contains the Kumaraswamy ExpD as a special case. Khan et al. [22] introduced the transmuted generalized ExpD, Mahdavi and Kundu [23] the alpha power ExpD, and Afify et al. [24] the odd exponentiated half-logistic-G family, which can generate further exponential-based models. Further extensions of the extended ExpD (EExpD) increase its flexibility through additional parameters. De Andrade et al. [25] introduced the exponentiated generalized EExpD, a four-parameter lifetime model with the exponential and exponentiated EExpDs as special cases; a related extension is the extended exponential Weibull distribution [26]. The hazard rate of the exponentiated generalized EExpD can be bathtub-shaped, inverted bathtub-shaped, increasing, decreasing or constant, which makes it well suited to reliability and survival data; its moments, quantile function and entropy measures have been derived, and its parameters are usually estimated by maximum likelihood. Recently, Ragab et al. [27] proposed the unit EExpD (UEExpD), a two-parameter distribution on the unit interval obtained from the EExpD. It is suitable for modeling proportions and probabilities, and its shape and scale parameters allow it to accommodate a variety of hazard-rate shapes [27]. The PDF of the UEExpD is

$$(9) \quad f(x; \lambda, \mu) = \frac{\lambda^2}{\lambda + \mu} (1 - \mu \log(x)) x^{\lambda-1}, \quad 0 < x < 1, \mu, \lambda > 0.$$

The corresponding CDF of UEExpD is given by

$$(10) \quad F(x; \lambda, \mu) = \left( 1 - \frac{\mu \lambda \log(x)}{\lambda + \mu} \right) x^\lambda.$$

The promising applicability of the UEEExpD in reliability and lifetime analysis motivates us to study its information-theoretic properties, especially when flexible models are becoming more and more important to capture complex real-world data. This work builds on Ragab et al. [27], who introduced the UEEExpD, and extends their results by providing theoretical results and practical tools that increase the usability of information measures in modern statistical analysis.

**Motivation.** Bounded data on the unit interval, such as proportions, rates, indices, and normalized lifetimes, arise pervasively in reliability engineering, biomedical survival studies, and economic and environmental measurement, yet classical lifetime distributions such as the exponential and Weibull families are not naturally suited to this domain. The unit extended exponential distribution (UEEExpD), introduced by Ragab et al. [27], was developed precisely to fill this gap, offering hazard shapes flexible enough to accommodate the varied failure behavior typical of bounded reliability data. However, its introduction has so far been confined to establishing the distribution itself; the information-theoretic machinery needed to make it a fully usable tool for modern statistical inference of extropy-based uncertainty measures and Fisher-information-based estimation theory has remained largely unexplored. This paper is motivated by closing that gap.

- While entropy-based analysis of bounded lifetime models is well developed, extropy, the increasingly influential complementary measure, and its weighted, cumulative residual, and negative cumulative variants have not been examined for the UEEExpD, despite their potential value in sharpening the discrimination between structurally similar distributions.
- The Fisher information matrix derived here is not merely a technical prerequisite for efficient maximum likelihood estimation; reframed as a function of the order statistic  $r$ , it reveals which observations carry the most information about each parameter, a direct and practically actionable insight for designing inspection or censoring schemes in reliability testing.
- Because reliability data are so often observed, censored, or acted upon in ordered form ( $k$ -out-of- $n$  systems, ranked-set sampling, progressive censoring), we extend the extropy framework to the order statistics of the UEEExpD, obtaining one-dimensional integral

representations, expressed through the parent CDF and PDF, that are evaluated accurately by numerical quadrature and allow the behavior of uncertainty across the ordered sample to be characterized systematically.

- Every closed-form result developed here is validated against direct numerical integration, and the adequacy of the model for real data is assessed through a goodness-of-fit test and graphical diagnostics, so the contribution rests on demonstrated, checkable evidence rather than assertion alone.

Taken together, these contributions move the UEExpD from a flexible distribution in search of tools to a distribution equipped with a substantially richer information-theoretic toolkit: closed-form extropy measures for the parent distribution and integral representations for its order statistics, an interpretable Fisher information structure, and estimation theory validated by both simulation and real-data analysis.

The subsequent sections of this work are structured as follows. In Section 2, we examine various EX measures for the UEExpD, encompassing the weighted, cumulative, and negative variations. Section 3 derives the Fisher information matrix of the UEExpD. Section 4 investigates the EX measures and FI of order statistics. Section 5 presents a simulation study to investigate the behavior of FI across different sample sizes and numerical simulations of EX measures to assess their precision and convergence characteristics. In Section 6, we evaluate the outcomes using order statistics after applying the suggested techniques to two real datasets. Section 7 concludes the paper by summarizing the principal findings.

## 2. EXTROPY-BASED INFORMATION MEASURES

This section derives explicit expressions for various EX measures, including EX, WEX, CuREX, WCuREX, NegCEX, and WNegCEX. These measures enhance the theoretical understanding of the UEExpD and offer practical tools for the analysis of reliability, survival, and bounded lifetime data.

**2.1. Extropy.** Let  $X$  be a continuous RV following the UEExpD. Then, from (1) and (9), for  $\lambda > \frac{1}{2}$  we have the EX as

$$\begin{aligned}
\Psi(X) &= -\frac{1}{2} \int_0^1 \left[ \frac{\lambda^2}{\lambda + \mu} (1 - \mu \log x) x^{\lambda-1} \right]^2 dx \\
(11) \quad &= -\frac{\lambda^4}{2(\lambda + \mu)^2} \left( \frac{1}{2\lambda - 1} + \frac{2\mu}{(2\lambda - 1)^2} + \frac{2\mu^2}{(2\lambda - 1)^3} \right).
\end{aligned}$$

The condition  $\lambda > \frac{1}{2}$  is necessary: since  $f^2(x) \sim \frac{\lambda^4 \mu^2}{(\lambda + \mu)^2} (\log x)^2 x^{2\lambda-2}$  as  $x \rightarrow 0^+$ , the integral in (11) diverges for  $\lambda \leq \frac{1}{2}$ .

**Proposition 1.** Fix  $\lambda > \frac{1}{2}$  and let  $\mu^* = (2\lambda - 1)(\lambda - 1)$ . Then

$$\frac{\partial \Psi(X)}{\partial \mu} = \frac{\lambda^4 (\mu^* - \mu)}{(2\lambda - 1)^3 (\lambda + \mu)^3}.$$

Consequently,

- (i) if  $\frac{1}{2} < \lambda \leq 1$ , then  $\Psi(X)$  is strictly decreasing in  $\mu \in (0, \infty)$ ;
- (ii) if  $\lambda > 1$ , then  $\Psi(X)$  is strictly increasing on  $(0, \mu^*]$  and strictly decreasing on  $[\mu^*, \infty)$ ;  
hence it attains its maximum at  $\mu = \mu^*$ .

*Proof.* Let  $a = 2\lambda - 1 > 0$  and  $N(\mu) = a^2 + 2a\mu + 2\mu^2$ . By (11),

$$(12) \quad \Psi(X) = -\frac{\lambda^4}{2a^3} \frac{N(\mu)}{(\lambda + \mu)^2}.$$

Since  $N'(\mu) = 2a + 4\mu$ ,  $\lambda - a = 1 - \lambda$  and  $2\lambda - a = 1$ ,

$$(\lambda + \mu)N'(\mu) - 2N(\mu) = 2a(\lambda - a) + 2\mu(2\lambda - a) = 2(\mu - \mu^*),$$

so that

$$\frac{\partial \Psi(X)}{\partial \mu} = \frac{\lambda^4 (\mu^* - \mu)}{a^3 (\lambda + \mu)^3}.$$

The sign of  $\partial \Psi(X)/\partial \mu$  is therefore that of  $\mu^* - \mu$ . If  $\frac{1}{2} < \lambda \leq 1$ , then  $\mu^* \leq 0 < \mu$ , which gives (i). If  $\lambda > 1$ , then  $\mu^* > 0$  and the derivative is positive on  $(0, \mu^*)$  and negative on  $(\mu^*, \infty)$ , which gives (ii).  $\square$

**Proposition 2.** For  $\lambda > \frac{1}{2}$ , the extropy  $\Psi(X)$  of the UEEExpD satisfies

- (i)  $\lim_{\mu \rightarrow 0^+} \Psi(X) = -\frac{\lambda^2}{2(2\lambda - 1)}$ ;
- (ii)  $\lim_{\mu \rightarrow \infty} \Psi(X) = -\frac{\lambda^4}{(2\lambda - 1)^3} < 0$ .

*Proof.* With  $a = 2\lambda - 1$ , (12) gives  $\Psi(X) = -\frac{\lambda^4}{2a^3}R(\mu)$ , where

$$R(\mu) = \frac{a^2 + 2a\mu + 2\mu^2}{(\lambda + \mu)^2}.$$

Since  $R(\mu) \rightarrow a^2/\lambda^2$  as  $\mu \rightarrow 0^+$  and  $R(\mu) \rightarrow 2$  as  $\mu \rightarrow \infty$ ,

$$\lim_{\mu \rightarrow 0^+} \Psi(X) = -\frac{\lambda^2}{2a}, \quad \lim_{\mu \rightarrow \infty} \Psi(X) = -\frac{\lambda^4}{a^3},$$

which are (i) and (ii); the second limit is negative because  $a > 0$ .  $\square$

**2.2. Weighted extropy.** Let  $X$  be a continuous RV following the UExpD. Then, from (2) and (9), the WEX is given by

$$\Psi^w(X) = -\frac{1}{2} \int_0^1 x \left[ \frac{\lambda^2}{\lambda + \mu} (1 - \mu \log x) x^{\lambda-1} \right]^2 dx = -\frac{\lambda^4}{2(\lambda + \mu)^2} \int_0^1 (1 - \mu \log x)^2 x^{2\lambda-1} dx.$$

Since  $\int_0^1 x^{c-1} (-\log x)^k dx = k!/c^{k+1}$  for  $c > 0$ , taking  $c = 2\lambda$  gives

$$\int_0^1 (1 - \mu \log x)^2 x^{2\lambda-1} dx = \frac{1}{2\lambda} + \frac{2\mu}{(2\lambda)^2} + \frac{2\mu^2}{(2\lambda)^3} = \frac{2\lambda^2 + 2\lambda\mu + \mu^2}{4\lambda^3}.$$

Thus the WEX simplifies to the compact form

$$(13) \quad \Psi^w(X) = -\frac{1}{8}\lambda \left( \frac{\lambda^2}{(\lambda + \mu)^2} + 1 \right).$$

**2.3. Cumulative residual extropy.** Using equations (3) and (10), the CuREX of the UExpD is given by

$$\Phi(X) = -\frac{1}{2} \int_0^1 \left( 1 - \left( 1 - \frac{\mu\lambda \log x}{\lambda + \mu} \right) x^\lambda \right)^2 dx.$$

Let  $A = \frac{\mu\lambda}{\lambda + \mu}$ . Then  $1 - \left( 1 - \frac{\mu\lambda \log x}{\lambda + \mu} \right) x^\lambda = 1 - x^\lambda + Ax^\lambda \log x$ . Thus the integral

$$\begin{aligned} \int_0^1 \left( 1 - \left( 1 - \frac{\mu\lambda \log x}{\lambda + \mu} \right) x^\lambda \right)^2 dx &= 1 - 2 \frac{1}{\lambda + 1} + \frac{1}{2\lambda + 1} \\ &\quad + A^2 \frac{2}{(2\lambda + 1)^3} + 2A \left( -\frac{1}{(\lambda + 1)^2} \right) - 2A \left( -\frac{1}{(2\lambda + 1)^2} \right) \\ &= 1 - \frac{2}{\lambda + 1} + \frac{1}{2\lambda + 1} + \frac{2A^2}{(2\lambda + 1)^3} - \frac{2A}{(\lambda + 1)^2} + \frac{2A}{(2\lambda + 1)^2}. \end{aligned}$$

Therefore, the CuREX of the UExpD is

$$\Phi(X) = -\frac{1}{2} \left[ 1 - \frac{2}{\lambda + 1} + \frac{1}{2\lambda + 1} + \frac{2}{(2\lambda + 1)^3} \left( \frac{\mu\lambda}{\lambda + \mu} \right)^2 \right]$$

$$(14) \quad - \frac{2}{(\lambda+1)^2} \left( \frac{\mu\lambda}{\lambda+\mu} \right) + \frac{2}{(2\lambda+1)^2} \left( \frac{\mu\lambda}{\lambda+\mu} \right) \Bigg].$$

**Proposition 3.** Let  $F(x) = \left(1 - \frac{\mu\lambda \log x}{\lambda+\mu}\right)x^\lambda$  be the CDF (10) of the UExpD. Then

$$\Phi(X) = -\frac{1}{2} \int_0^1 (1 - F(x))^2 dx = -\frac{1}{2} \left( 1 - 2\mathbb{E}[F(U)] + \mathbb{E}[F(U)^2] \right),$$

where  $U \sim \text{Uniform}(0, 1)$ . In particular,

$$-\frac{1}{2} < \Phi(X) < 0.$$

*Proof.* The identity follows by expanding the square and writing each integral over  $(0, 1)$  as an expectation with respect to  $U \sim \text{Uniform}(0, 1)$ . Since  $f(x) > 0$  on  $(0, 1)$  (note that  $1 - \mu \log x > 1$ ),  $F$  is strictly increasing there, with  $F(0^+) = 0$  and  $F(1) = 1$ ; hence  $0 < F(x) < 1$  for every  $x \in (0, 1)$ . Hence  $0 < (1 - F(x))^2 < 1$  on  $(0, 1)$ , and integrating gives  $0 < \int_0^1 (1 - F(x))^2 dx < 1$ . Multiplying by  $-\frac{1}{2}$  yields  $-\frac{1}{2} < \Phi(X) < 0$ .  $\square$

**2.4. Weighted cumulative residual extropy.** From (4) and (10), the WCuREX is

$$\Phi^w(X) = -\frac{1}{2} \int_0^1 x \left[ 1 - x^\lambda \left( 1 - \frac{\lambda\mu \log x}{\lambda+\mu} \right) \right]^2 dx, \quad \lambda > 0, \mu > 0.$$

Expanding the square, we obtain

$$\int_0^1 x \left[ 1 - x^\lambda \left( 1 - \frac{\lambda\mu \log x}{\lambda+\mu} \right) \right]^2 dx = I_1 + I_2 + I_3,$$

where

$$\begin{aligned} I_1 &= \int_0^1 x(1 - x^\lambda)^2 dx, \\ I_2 &= 2 \frac{\lambda\mu}{\lambda+\mu} \int_0^1 x(1 - x^\lambda)x^\lambda \log x dx, \\ I_3 &= \left( \frac{\lambda\mu}{\lambda+\mu} \right)^2 \int_0^1 x^{1+2\lambda} (\log x)^2 dx. \end{aligned}$$

Thus,

$$\begin{aligned} I_1 &= \int_0^1 x(1 - 2x^\lambda + x^{2\lambda}) dx = \int_0^1 x dx - 2 \int_0^1 x^{\lambda+1} dx + \int_0^1 x^{2\lambda+1} dx \\ &= \left[ \frac{x^2}{2} \right]_0^1 - 2 \left[ \frac{x^{\lambda+2}}{\lambda+2} \right]_0^1 + \left[ \frac{x^{2\lambda+2}}{2\lambda+2} \right]_0^1 = \frac{1}{2} - \frac{2}{\lambda+2} + \frac{1}{2(\lambda+1)}, \end{aligned}$$

$$\begin{aligned}
I_2 &= 2 \frac{\lambda\mu}{\lambda+\mu} \int_0^1 x(x^\lambda - x^{2\lambda}) \log x dx = 2 \frac{\lambda\mu}{\lambda+\mu} \left[ \int_0^1 x^{\lambda+1} \log x dx - \int_0^1 x^{2\lambda+1} \log x dx \right] \\
&= \frac{\lambda\mu}{\lambda+\mu} \left( \frac{1}{2(\lambda+1)^2} - \frac{2}{(\lambda+2)^2} \right),
\end{aligned}$$

and

$$I_3 = \left( \frac{\lambda\mu}{\lambda+\mu} \right)^2 \int_0^1 x^{1+2\lambda} (\log x)^2 dx = \left( \frac{\lambda\mu}{\lambda+\mu} \right)^2 \frac{1}{4(\lambda+1)^3}.$$

Thus

$$\begin{aligned}
(15) \quad \Phi^w(X) &= -\frac{1}{2} \left[ \frac{1}{2} - \frac{2}{\lambda+2} + \frac{1}{2(\lambda+1)} + \frac{\lambda\mu}{\lambda+\mu} \left( \frac{1}{2(\lambda+1)^2} - \frac{2}{(\lambda+2)^2} \right) \right. \\
&\quad \left. + \left( \frac{\lambda\mu}{\lambda+\mu} \right)^2 \frac{1}{4(\lambda+1)^3} \right].
\end{aligned}$$

**2.5. Negative cumulative extropy and its weighted version.** Based on (10) and (5) and the same technique as for the CuREX, we obtain

$$\begin{aligned}
(16) \quad \zeta(X) &= \frac{1}{2} \int_0^1 \left[ 1 - \left( \left( 1 - \frac{\mu\lambda \log x}{\lambda+\mu} \right) x^\lambda \right)^2 \right] dx \\
&= \frac{1}{2} \left[ 1 - \frac{1}{2\lambda+1} - \frac{2}{(2\lambda+1)^2} \frac{\mu\lambda}{\lambda+\mu} - \frac{2}{(2\lambda+1)^3} \left( \frac{\mu\lambda}{\lambda+\mu} \right)^2 \right].
\end{aligned}$$

Similarly, the WNegCEX is

$$\begin{aligned}
(17) \quad \zeta^w(X) &= \frac{1}{2} \int_0^1 x \left[ 1 - \left( \left( 1 - \frac{\mu\lambda \log x}{\lambda+\mu} \right) x^\lambda \right)^2 \right] dx \\
&= \frac{\lambda^2 (2\lambda^3 + 4\lambda^2(\mu+1) + 2\lambda(\mu(\mu+3)+1) + \mu(\mu+2))}{8(\lambda+1)^3(\lambda+\mu)^2}.
\end{aligned}$$

**Remark 1.** All closed-form expressions of this section were checked against direct numerical quadrature of the defining integrals (1)–(6), using (9) and (10), at all parameter values considered in Section 5. The absolute differences were below  $10^{-8}$  in every case.

### 3. FISHER INFORMATION STRUCTURE OF THE UEExpD

The FI is crucial in statistical inference, as it measures the amount of information that an observable RV carries about unknown parameters of the distribution. It provides the theoretical basis for estimation methods such as ML and serves as a benchmark for assessing the performance of the estimators. When considering the UEExpD, the analysis of the FIM is important, as it clarifies the interaction of parameters and their contribution to the overall precision of inference. This section focuses on obtaining the FIM of the UEExpD and analyzing its structural features. Based on (7) and (9), the log-likelihood contribution for one observation  $x$ :

$$\ell(\lambda, \mu; x) = \log f(x) = 2\log \lambda - \log(\lambda + \mu) + \log(1 - \mu \log x) + (\lambda - 1)\log x.$$

Then

$$\begin{aligned}\frac{\partial^2 \ell}{\partial \lambda^2} &= -\frac{2}{\lambda^2} + \frac{1}{(\lambda + \mu)^2}, \\ \frac{\partial^2 \ell}{\partial \lambda \partial \mu} &= \frac{1}{(\lambda + \mu)^2}, \\ \frac{\partial^2 \ell}{\partial \mu^2} &= \frac{1}{(\lambda + \mu)^2} - \frac{(\log x)^2}{(1 - \mu \log x)^2}.\end{aligned}$$

Using the second derivatives and taking expectations, we obtain the matrix entries as

$$I_{\lambda\lambda} = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \lambda^2}\right] = -\left(-\frac{2}{\lambda^2} + \frac{1}{(\lambda + \mu)^2}\right) = \frac{2}{\lambda^2} - \frac{1}{(\lambda + \mu)^2},$$

$$I_{\lambda\mu} = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \lambda \partial \mu}\right] = -\frac{1}{(\lambda + \mu)^2},$$

$$I_{\mu\mu} = -\mathbb{E}\left[\frac{\partial^2 \ell}{\partial \mu^2}\right] = -\frac{1}{(\lambda + \mu)^2} + \mathbb{E}\left[\frac{(\log X)^2}{(1 - \mu \log X)^2}\right].$$

We evaluate the expectation through the transformation  $T = -\log X$ , which takes values in  $(0, \infty)$  and, by (9), has density  $\frac{\lambda^2}{\lambda + \mu}(1 + \mu t)e^{-\lambda t}$ ,  $t > 0$ . Hence

$$\mathbb{E}\left[\frac{(\log X)^2}{(1 - \mu \log X)^2}\right] = \mathbb{E}\left[\frac{T^2}{(1 + \mu T)^2}\right] = \frac{\lambda^2}{\lambda + \mu} \int_0^\infty \frac{t^2}{1 + \mu t} e^{-\lambda t} dt.$$

To evaluate the last integral, use the decomposition

$$\frac{t^2}{1 + \mu t} = \frac{t}{\mu} - \frac{1}{\mu^2} + \frac{1}{\mu^2(1 + \mu t)},$$

together with  $\int_0^\infty t e^{-\lambda t} dt = \lambda^{-2}$ ,  $\int_0^\infty e^{-\lambda t} dt = \lambda^{-1}$  and

$$\int_0^\infty \frac{e^{-\lambda t}}{1 + \mu t} dt = \frac{1}{\mu} e^{\lambda/\mu} E_1\left(\frac{\lambda}{\mu}\right).$$

This gives

$$\int_0^\infty \frac{t^2}{1 + \mu t} e^{-\lambda t} dt = \frac{1}{\mu \lambda^2} - \frac{1}{\mu^2 \lambda} + \frac{1}{\mu^3} e^{\lambda/\mu} E_1\left(\frac{\lambda}{\mu}\right),$$

and therefore

$$I_{\mu\mu} = -\frac{1}{(\lambda + \mu)^2} + \frac{\lambda^2}{\lambda + \mu} \left[ \frac{1}{\mu \lambda^2} - \frac{1}{\mu^2 \lambda} + \frac{1}{\mu^3} e^{\lambda/\mu} E_1\left(\frac{\lambda}{\mu}\right) \right].$$

Simplifying, we obtain the closed form

$$I_{\mu\mu} = \frac{\lambda^2}{(\lambda + \mu)^2 \mu^2} \left( \frac{\lambda + \mu}{\mu} e^{\lambda/\mu} E_1\left(\frac{\lambda}{\mu}\right) - 1 \right),$$

where  $E_1(z) = \int_z^\infty \frac{e^{-t}}{t} dt$  is the exponential integral. The FIM for UEEExpD is

$$\mathbb{I}(X; \lambda, \mu) = \begin{pmatrix} \frac{2}{\lambda^2} - \frac{1}{(\lambda + \mu)^2} & -\frac{1}{(\lambda + \mu)^2} \\ -\frac{1}{(\lambda + \mu)^2} & \frac{\lambda^2}{(\lambda + \mu)^2 \mu^2} \left( \frac{\lambda + \mu}{\mu} e^{\lambda/\mu} E_1\left(\frac{\lambda}{\mu}\right) - 1 \right) \end{pmatrix}.$$

Since the support  $(0, 1)$  does not depend on  $(\lambda, \mu)$ , the usual regularity conditions hold and the two expressions in (8) coincide. For a random sample of size  $n$ , the information matrix is  $n\mathbb{I}(X; \lambda, \mu)$ .

#### 4. EXTROPY MEASURES AND FISHER INFORMATION OF ORDER STATISTICS

Order statistics provide a natural framework for investigating how information and uncertainty are distributed across different portions of a sample. In particular, the smallest observations, central observations, and largest observations may contribute differently to the estimation of model parameters and to the uncertainty associated with a random phenomenon. Motivated by this fact, this section extends the information-theoretic analysis of the proposed model from the parent RV to its order statistics. Let

$$X_1, X_2, \dots, X_n$$

be a random sample from the distribution under consideration, with distribution function  $F(x)$  and density function  $f(x)$ , and let

$$X_{1:n} < X_{2:n} < \cdots < X_{n:n}$$

denote the corresponding order statistics. For a continuous distribution, the density of the  $r$ th order statistic,  $1 \leq r \leq n$ , is

$$f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} [F(x)]^{r-1} [1-F(x)]^{n-r} f(x), \quad 0 < x < 1.$$

Based on the CDF (10) and PDF (9) of UEExpD, the density of the  $r$ th order statistic becomes

$$\begin{aligned} f_{r:n}(x; \lambda, \mu) &= \frac{n!}{(r-1)!(n-r)!} \left[ \left( 1 - \frac{\mu \lambda \log x}{\lambda + \mu} \right) x^\lambda \right]^{r-1} \\ &\times \left[ 1 - \left( 1 - \frac{\mu \lambda \log x}{\lambda + \mu} \right) x^\lambda \right]^{n-r} \\ &\times \frac{\lambda^2}{\lambda + \mu} (1 - \mu \log x) x^{\lambda-1}, \quad 0 < x < 1. \end{aligned} \quad (18)$$

Equation (18) provides the basis for studying the information structure of the ordered sample.

**4.1. Extropy measures of the  $r$ th Order Statistic.** The EX of  $X_{r:n}$  is defined as

$$\mathcal{E}_{r:n} = -\frac{1}{2} \int_0^1 f_{r:n}^2(x; \lambda, \mu) dx.$$

We obtain

$$\begin{aligned} \mathcal{E}_{r:n} &= -\frac{1}{2} \left[ \frac{n!}{(r-1)!(n-r)!} \right]^2 \left( \frac{\lambda^2}{\lambda + \mu} \right)^2 \\ &\times \int_0^1 \left[ \left( 1 - \frac{\mu \lambda \log x}{\lambda + \mu} \right) x^\lambda \right]^{2r-2} \\ &\times \left[ 1 - \left( 1 - \frac{\mu \lambda \log x}{\lambda + \mu} \right) x^\lambda \right]^{2n-2r} \\ &\times (1 - \mu \log x)^2 x^{2\lambda-2} dx. \end{aligned}$$

Although the integral does not generally reduce to a simple elementary expression for arbitrary  $(n, r)$ , it is a one-dimensional integral on a bounded interval and can be evaluated accurately using numerical quadrature. This representation is particularly useful for studying the variation of EX across the ordered sample. For  $r = 1$  and  $r = n$ , the corresponding quantities represent the EX of the minimum and maximum order statistics, respectively. For  $r = \lceil (n+1)/2 \rceil$ , the

measure corresponds to the order statistic closest to the sample median. Since  $F^{2r-2}(x)f^2(x)$  behaves like  $x^{2r\lambda-2}$ , up to logarithmic factors, as  $x \rightarrow 0^+$ ,  $\mathcal{E}_{r:n}$  is finite if and only if  $\lambda > 1/(2r)$ ; in particular,  $\mathcal{E}_{1:n}$  requires  $\lambda > \frac{1}{2}$ , as in (11).

The WEX of the  $r$ th order statistic is defined by

$$\mathcal{W}\mathcal{E}_{r:n} = -\frac{1}{2} \int_0^1 x f_{r:n}^2(x; \lambda, \mu) dx.$$

Then

$$\begin{aligned} \mathcal{W}\mathcal{E}_{r:n} &= -\frac{1}{2} \left[ \frac{n!}{(r-1)!(n-r)!} \right]^2 \\ &\quad \times \int_0^1 x F^{2r-2}(x) [1-F(x)]^{2n-2r} f^2(x) dx. \end{aligned}$$

The factor  $x$  assigns greater importance to larger observations. Therefore, comparison between  $\mathcal{E}_{r:n}$  and  $\mathcal{W}\mathcal{E}_{r:n}$  provides a useful way of assessing how the information structure changes when greater weight is assigned to observations near the upper boundary of the support. Let

$$\bar{F}_{r:n}(x) = 1 - F_{r:n}(x)$$

denote the survival function of  $X_{r:n}$ . The CuREX of  $X_{r:n}$  is defined by

$$\mathcal{C}\mathcal{R}\mathcal{E}_{r:n} = -\frac{1}{2} \int_0^1 \bar{F}_{r:n}^2(x) dx.$$

Since

$$F_{r:n}(x) = \sum_{j=r}^n \binom{n}{j} F^j(x) [1-F(x)]^{n-j},$$

we have

$$\bar{F}_{r:n}(x) = \sum_{j=0}^{r-1} \binom{n}{j} F^j(x) [1-F(x)]^{n-j}.$$

Hence,

$$\mathcal{C}\mathcal{R}\mathcal{E}_{r:n} = -\frac{1}{2} \int_0^1 \left\{ \sum_{j=0}^{r-1} \binom{n}{j} F^j(x) [1-F(x)]^{n-j} \right\}^2 dx.$$

This representation is especially useful for investigating the residual uncertainty associated with progressively larger order statistics.

The WCuREX of  $X_{r:n}$  is defined by

$$\mathcal{W}\mathcal{C}\mathcal{R}\mathcal{E}_{r:n} = -\frac{1}{2} \int_0^1 x \bar{F}_{r:n}^2(x) dx.$$

Thus

$$\mathcal{WCE}_{r:n} = -\frac{1}{2} \int_0^1 x \left\{ \sum_{j=0}^{r-1} \binom{n}{j} F^j(x) [1-F(x)]^{n-j} \right\}^2 dx.$$

The weighting factor  $x$  makes this measure particularly sensitive to the upper portion of the support. Consequently,  $\mathcal{WCE}_{r:n}$  can be used to investigate how residual uncertainty changes when greater emphasis is assigned to larger ordered observations.

The NegCEX associated with  $X_{r:n}$  is defined as

$$\mathcal{NCE}_{r:n} = \frac{1}{2} \int_0^1 [1 - F_{r:n}^2(x)] dx.$$

Consequently,

$$\mathcal{NCE}_{r:n} = \frac{1}{2} \int_0^1 \left\{ 1 - \left[ \sum_{j=r}^n \binom{n}{j} F^j(x) [1-F(x)]^{n-j} \right]^2 \right\} dx.$$

Unlike CuREX, which is based on the survival function, NegCEX is based on the distribution function itself. This complementary construction allows the lower and upper portions of the ordered sample to be examined from two different information perspectives.

The WNegCEX of  $X_{r:n}$  is defined as

$$\mathcal{WNECE}_{r:n} = \frac{1}{2} \int_0^1 x [1 - F_{r:n}^2(x)] dx.$$

Therefore,

$$(19) \quad \mathcal{WNECE}_{r:n} = \frac{1}{2} \int_0^1 x \left\{ 1 - \left[ \sum_{j=r}^n \binom{n}{j} F^j(x) [1-F(x)]^{n-j} \right]^2 \right\} dx.$$

The six quantities

$$\mathcal{E}_{r:n}, \quad \mathcal{WE}_{r:n}, \quad \mathcal{CE}_{r:n}, \quad \mathcal{WCE}_{r:n}, \quad \mathcal{NCE}_{r:n}, \quad \mathcal{WNECE}_{r:n}$$

together provide a complementary description of the information structure associated with the  $r$ th order statistic.

**4.2. Fisher information of order statistics.** The order-statistic framework can also be used to quantify the amount of information that each ordered observation provides about the unknown parameters  $\lambda$  and  $\mu$ .

For

$$\boldsymbol{\theta} = (\lambda, \mu)^\top,$$

the FIM associated with  $X_{r:n}$  is defined by

$$\mathcal{J}_{r:n}(\boldsymbol{\theta}) = -\mathbb{E} \left[ \frac{\partial^2}{\partial \boldsymbol{\theta} \partial \boldsymbol{\theta}^\top} \log f_{r:n}(X_{r:n}; \boldsymbol{\theta}) \right].$$

Equivalently, under the usual regularity conditions,

$$\mathcal{J}_{r:n}(\boldsymbol{\theta}) = \mathbb{E} \left[ \mathbf{U}_{r:n}(\boldsymbol{\theta}) \mathbf{U}_{r:n}^\top(\boldsymbol{\theta}) \right],$$

where

$$\mathbf{U}_{r:n}(\boldsymbol{\theta}) = \frac{\partial}{\partial \boldsymbol{\theta}} \log f_{r:n}(X_{r:n}; \boldsymbol{\theta})$$

is the score vector. Then

$$\begin{aligned} \log f_{r:n}(x) &= \log \left[ \frac{n!}{(r-1)!(n-r)!} \right] + (r-1) \log F(x) \\ &\quad + (n-r) \log[1 - F(x)] + \log f(x). \end{aligned}$$

Hence, for  $\boldsymbol{\theta}_j \in \{\lambda, \mu\}$ ,

$$(20) \quad \frac{\partial}{\partial \boldsymbol{\theta}_j} \log f_{r:n}(x) = \left[ \frac{r-1}{F(x)} - \frac{n-r}{1-F(x)} \right] \frac{\partial F(x)}{\partial \boldsymbol{\theta}_j} + \frac{\partial}{\partial \boldsymbol{\theta}_j} \log f(x).$$

Equation (20) is particularly important because it separates the information contribution of the order-statistic mechanism from the information contained in the parent density. For the UE-ExpD,

$$\log f(x) = 2 \log \lambda - \log(\lambda + \mu) + \log(1 - \mu \log x) + (\lambda - 1) \log x,$$

and therefore

$$\frac{\partial}{\partial \lambda} \log f(x) = \frac{2}{\lambda} - \frac{1}{\lambda + \mu} + \log x,$$

while

$$\frac{\partial}{\partial \mu} \log f(x) = -\frac{1}{\lambda + \mu} - \frac{\log x}{1 - \mu \log x}.$$

Furthermore,

$$F(x) = \left(1 - \frac{\mu\lambda \log x}{\lambda + \mu}\right) x^\lambda,$$

so that

$$\frac{\partial F(x)}{\partial \lambda} = F(x) \left[ \log x - \frac{\mu^2 \log x}{(\lambda + \mu)(\lambda + \mu - \mu\lambda \log x)} \right],$$

and

$$\frac{\partial F(x)}{\partial \mu} = -\frac{\lambda^2 \log x}{(\lambda + \mu)^2} x^\lambda.$$

We denote the diagonal elements by

$$\mathcal{J}_{r:n}^{(\lambda)} = [\mathcal{J}_{r:n}]_{11}, \quad \mathcal{J}_{r:n}^{(\mu)} = [\mathcal{J}_{r:n}]_{22}.$$

These quantities measure the parameter-specific information carried by the  $r$ th order statistic. The preceding formulation suggests a useful interpretation of order statistics in terms of parameter-specific information. By the score representation of  $\mathcal{J}_{r:n}(\theta)$ , these diagonal elements can be written as

$$\mathcal{J}_{r:n}^{(\lambda)} = \mathbb{E} \left[ \left( \frac{\partial}{\partial \lambda} \log f_{r:n}(X_{r:n}) \right)^2 \right],$$

and

$$\mathcal{J}_{r:n}^{(\mu)} = \mathbb{E} \left[ \left( \frac{\partial}{\partial \mu} \log f_{r:n}(X_{r:n}) \right)^2 \right].$$

## 5. SIMULATION STUDY

In the study of probability distributions, parameter estimation is crucial because accurate parameter estimates are necessary to draw reliable conclusions and forecasts. We analyze the performance of the suggested model estimators using the MLE technique. MLE is one of the most well-known techniques for parameter estimation in statistical analysis, having been first presented in [28]. Its favorable asymptotic characteristics, such as consistency, efficiency, and invariance under reparameterization, have led to its widespread adoption. Finding the parameter values that maximize the likelihood function and, consequently, the chance of observing the provided data is the fundamental idea behind MLE. The following is the expression for the equivalent log-likelihood function for the suggested model:

$$\log L = n \log \left( \frac{\lambda^2}{\lambda + \mu} \right) + (\lambda - 1) \sum_{i=1}^n \log(x_i) + \sum_{i=1}^n \log(1 - \mu \log(x_i)).$$

**5.1. Simulation for Fisher information measures.** This subsection details the computation of the estimated FI measures for proposed model parameters  $\lambda$  and  $\mu$ . To obtain these values, random datasets were generated using the inverse cumulative distribution function method applied to the proposed model. The resulting numerical estimates are presented in Tables 1 and 2. A comparative analysis was subsequently conducted to evaluate these information measures across different sample sizes ( $n$ ). The evaluation employs standard statistical metrics, including the bias (BS), mean square error (MSqEr), and mean relative error (MReEr), the latter of which is calculated as follows:

- (1) BS is given as  $|BS(\hat{T})| = (1/W) \sum_{\omega=1}^W |\hat{T}_{\omega} - T|$ , where  $T$  is the considered FI measure of samples with size  $W$ .
- (2) MSqEr indicated as  $MSqEr = (1/W) \sum_{\omega=1}^W (\hat{T}_{\omega} - T)^2$ ,
- (3) MReEr defined as  $MReEr = (1/W) \sum_{\omega=1}^W |\hat{T}_{\omega} - T|/T$ .

This subsection outlines the computational procedure for estimating two FI measures: FI with respect to parameter  $\lambda$  ( $FI(\lambda)$ ) and Fisher information with respect to parameter  $\mu$  ( $FI(\mu)$ ). The numerical results were derived according to the following protocol:

- Initialization: The known parameters of the proposed model ( $\lambda_0, \mu_0$ ) were fixed and used to generate multiple random datasets via the inverse cumulative distribution function method.
- Baseline calculation: For each generated dataset, MLEs are determined, then the theoretical values of the information measures  $FI(\lambda)$  and  $FI(\mu)$  are determined.
- For each FI, the BS, MSqEr, and MReEr were computed. This entire process was iterated a sufficient number of times to ensure the convergence and stability of the reported numerical values.

The numerical results in Table 1 demonstrate a consistent pattern across all parameter configurations. As the sample size ( $n$ ) increases, the estimators for both  $FI(\lambda)$  and  $FI(\mu)$  show marked improvement. This is evidenced by the systematic decrease in BS, MSqEr, and MReEr with larger  $n$ . Notably, the estimation of  $FI(\mu)$  presents a greater challenge, as indicated by its significantly higher initial MReEr values; however, these errors also exhibit a strong convergent trend towards zero as sample size increases. This pattern confirms the asymptotic consistency

and efficiency of the MLEs used in this simulation. The results in Table 2 confirm the expected asymptotic properties of the estimators. Across all parameter sets, a clear trend emerges: as the sample size ( $n$ ) increases, the BS, MSqEr, and MReEr for both  $FI(\lambda)$  and  $FI(\mu)$  consistently decrease. This pattern strongly indicates that the estimators are consistent, converging toward the initial values. Notably, the estimation of  $FI(\mu)$  proves more challenging, particularly for smaller sample sizes, as evidenced by its substantially higher initial MReEr values. However, these errors also demonstrate a robust convergent behavior, improving significantly as  $n$  grows.

**5.2. Numerical simulation for different extropy measures.** This study evaluates the diversity of estimated EX measures derived from our proposed model. To conduct this assessment, datasets were stochastically generated using the model's quantile function. The resulting numerical estimates are compiled in Tables 3 through 7. A comparative analysis of these extropy values is then performed across different sample sizes ( $W$ ) using several statistical measures. These include the bias (BS), mean square error (MSqEr), and mean relative error (MReEr), the latter of which is calculated as follows:

- (1) BS is given as  $|BS(\hat{H})| = (1/W) \sum_{\omega=1}^W |\hat{H}_{\omega} - H|$ , where  $H$  is the considered extropy of samples with size  $W$ .
- (2) MSqEr indicated as  $MSqEr = (1/W) \sum_{\omega=1}^W (\hat{H}_{\omega} - H)^2$ ,
- (3) MReEr defined as  $MReEr = (1/W) \sum_{\omega=1}^W |\hat{H}_{\omega} - H|/H$ .

The numerical values for the investigated EX measures, presented in Tables 3 through 7, were calculated according to the following procedure. Each measure is denoted by a specific acronym for clarity: EX, WEX, CuREX, WCuREX, NegCEX, and WNegCEX). The numerical simulation was conducted according to the following procedure. First, the parameters of the proposed model ( $\lambda_0, \mu_0$ ) were initialized and used to generate a series of random datasets. Subsequently, MLEs were derived from each generated dataset. These MLEs were then used to calculate the estimated EX measures, denoted as  $\hat{H}$ . For each EX, the BS, MSqEr, and MReEr were computed. This entire process was iterated a sufficient number of times to ensure the convergence of the resulting numerical values.

Tables 3 and 4 report the numerical values of the different EX measures for two distinct parameter settings, namely  $(\lambda = 0.9, \mu = 1.5)$  and  $(\lambda = 2.0, \mu = 0.5)$ . For each case, the BS,

TABLE 1. Numerical values for Fisher information for different values of parameters.

| $n$     | Measure | $\lambda = 0.9, \mu = 1.5$ |           | $\lambda = 1.5, \mu = 0.7$ |           | $\lambda = 1.25, \mu = 2.5$ |           |
|---------|---------|----------------------------|-----------|----------------------------|-----------|-----------------------------|-----------|
|         |         | $FI(\lambda)$              | $FI(\mu)$ | $FI(\lambda)$              | $FI(\mu)$ | $FI(\lambda)$               | $FI(\mu)$ |
| $\Xi_0$ |         | 2.29552                    | 0.02029   | 0.68227                    | 0.07278   | 1.20888                     | 0.00683   |
| 30      | Est.    | 2.335890                   | 0.168071  | 0.713848                   | 0.174735  | 1.215840                    | 0.069298  |
|         | BS      | 0.535678                   | 0.154514  | 0.175010                   | 0.125811  | 0.267200                    | 0.064845  |
|         | MSqEr   | 0.522253                   | 0.166996  | 0.050017                   | 0.046260  | 0.111243                    | 0.035651  |
|         | MReEr   | 0.233357                   | 7.614022  | 0.256509                   | 1.728627  | 0.221029                    | 9.489736  |
| 90      | Est.    | 2.305855                   | 0.079097  | 0.680489                   | 0.140308  | 1.202778                    | 0.025425  |
|         | BS      | 0.341066                   | 0.066595  | 0.100747                   | 0.094407  | 0.174163                    | 0.021430  |
|         | MSqEr   | 0.182423                   | 0.046546  | 0.016879                   | 0.026748  | 0.049940                    | 0.004248  |
|         | MReEr   | 0.148579                   | 3.281615  | 0.147663                   | 1.297146  | 0.144069                    | 3.136135  |
| 130     | Est.    | 2.288887                   | 0.055432  | 0.680725                   | 0.134610  | 1.202116                    | 0.020136  |
|         | BS      | 0.294734                   | 0.043738  | 0.090582                   | 0.088227  | 0.154568                    | 0.016263  |
|         | MSqEr   | 0.134155                   | 0.015967  | 0.013632                   | 0.023708  | 0.039680                    | 0.002373  |
|         | MReEr   | 0.128395                   | 2.155292  | 0.132764                   | 1.212234  | 0.127859                    | 2.380007  |
| 170     | Est.    | 2.278488                   | 0.044158  | 0.683353                   | 0.128524  | 1.194039                    | 0.017736  |
|         | BS      | 0.248084                   | 0.032036  | 0.082075                   | 0.082015  | 0.141284                    | 0.013811  |
|         | MSqEr   | 0.092825                   | 0.005913  | 0.011112                   | 0.020112  | 0.032333                    | 0.001660  |
|         | MReEr   | 0.108073                   | 1.578657  | 0.120295                   | 1.126875  | 0.116871                    | 2.021140  |
| 210     | Est.    | 2.261568                   | 0.039209  | 0.685386                   | 0.125566  | 1.205481                    | 0.014969  |
|         | BS      | 0.223942                   | 0.027269  | 0.077291                   | 0.077585  | 0.128181                    | 0.011052  |
|         | MSqEr   | 0.077539                   | 0.003183  | 0.009688                   | 0.018460  | 0.025502                    | 0.000629  |
|         | MReEr   | 0.097556                   | 1.343723  | 0.113284                   | 1.066006  | 0.106032                    | 1.617390  |
| 250     | Est.    | 2.253720                   | 0.034215  | 0.682556                   | 0.118203  | 1.199035                    | 0.012476  |
|         | BS      | 0.210427                   | 0.022033  | 0.070981                   | 0.071016  | 0.121866                    | 0.008643  |
|         | MSqEr   | 0.068931                   | 0.001537  | 0.008040                   | 0.015227  | 0.023115                    | 0.000305  |
|         | MReEr   | 0.091668                   | 1.085726  | 0.104036                   | 0.975748  | 0.100808                    | 1.264818  |
| 290     | Est.    | 2.256105                   | 0.031817  | 0.681969                   | 0.113016  | 1.196820                    | 0.011355  |
|         | BS      | 0.185435                   | 0.019994  | 0.068719                   | 0.067007  | 0.108246                    | 0.007453  |
|         | MSqEr   | 0.055687                   | 0.001512  | 0.007487                   | 0.013092  | 0.018057                    | 0.000252  |
|         | MReEr   | 0.080781                   | 0.985235  | 0.100721                   | 0.920664  | 0.089542                    | 1.090767  |
| 330     | Est.    | 2.240236                   | 0.029646  | 0.681963                   | 0.113281  | 1.191077                    | 0.010126  |
|         | BS      | 0.179392                   | 0.017592  | 0.063952                   | 0.065623  | 0.100291                    | 0.006396  |
|         | MSqEr   | 0.050346                   | 0.000900  | 0.006249                   | 0.013024  | 0.015911                    | 0.000153  |
|         | MReEr   | 0.078149                   | 0.866895  | 0.093733                   | 0.901651  | 0.082961                    | 0.935980  |
| 370     | Est.    | 2.232635                   | 0.028026  | 0.683987                   | 0.112530  | 1.200405                    | 0.009893  |
|         | BS      | 0.178892                   | 0.016364  | 0.061803                   | 0.064084  | 0.092341                    | 0.005835  |
|         | MSqEr   | 0.048851                   | 0.000767  | 0.005935                   | 0.012423  | 0.013152                    | 0.000113  |
|         | MReEr   | 0.077931                   | 0.806375  | 0.090583                   | 0.880502  | 0.076385                    | 0.853850  |
| 410     | Est.    | 2.239391                   | 0.026684  | 0.677209                   | 0.107939  | 1.187216                    | 0.009000  |
|         | BS      | 0.165811                   | 0.015093  | 0.058467                   | 0.060422  | 0.091012                    | 0.005041  |
|         | MSqEr   | 0.043693                   | 0.000708  | 0.005177                   | 0.011008  | 0.012373                    | 0.000080  |
|         | MReEr   | 0.072232                   | 0.743740  | 0.085694                   | 0.830186  | 0.075286                    | 0.737684  |

TABLE 2. Numerical values for Fisher information for different values of parameters.

| $n$     | Measure | $\lambda = 2.5, \mu = 0.9$ |           | $\lambda = 1.0, \mu = 3.0$ |           | $\lambda = 3.5, \mu = 1.0$ |           |
|---------|---------|----------------------------|-----------|----------------------------|-----------|----------------------------|-----------|
|         |         | $FI(\lambda)$              | $FI(\mu)$ | $FI(\lambda)$              | $FI(\mu)$ | $FI(\lambda)$              | $FI(\mu)$ |
| $\Xi_0$ |         | 0.23349                    | 0.03596   | 1.9375                     | 0.00376   | 0.11388                    | 0.0234    |
| 30      | Est.    | 0.247715                   | 0.067252  | 1.859861                   | 0.089508  | 0.128027                   | 0.043902  |
|         | BS      | 0.058167                   | 0.043219  | 0.398148                   | 0.087060  | 0.033378                   | 0.027435  |
|         | MSqEr   | 0.006299                   | 0.004977  | 0.266521                   | 0.098637  | 0.001730                   | 0.001964  |
|         | MReEr   | 0.249113                   | 1.201737  | 0.205496                   | 23.112868 | 0.293089                   | 1.172254  |
| 90      | Est.    | 0.254535                   | 0.061999  | 1.876959                   | 0.015576  | 0.119100                   | 0.036240  |
|         | BS      | 0.042975                   | 0.039359  | 0.259974                   | 0.013454  | 0.016689                   | 0.021243  |
|         | MSqEr   | 0.003248                   | 0.004697  | 0.111045                   | 0.002794  | 0.000491                   | 0.001104  |
|         | MReEr   | 0.184053                   | 1.094410  | 0.134180                   | 3.571823  | 0.146543                   | 0.907675  |
| 130     | Est.    | 0.247870                   | 0.051831  | 1.882612                   | 0.012431  | 0.119153                   | 0.033176  |
|         | BS      | 0.035318                   | 0.030787  | 0.223323                   | 0.010375  | 0.015063                   | 0.019013  |
|         | MSqEr   | 0.001955                   | 0.002761  | 0.078470                   | 0.000891  | 0.000437                   | 0.000984  |
|         | MReEr   | 0.151259                   | 0.856075  | 0.115263                   | 2.754511  | 0.132269                   | 0.812398  |
| 170     | Est.    | 0.248688                   | 0.051078  | 1.901929                   | 0.009132  | 0.119152                   | 0.031761  |
|         | BS      | 0.031516                   | 0.029885  | 0.204677                   | 0.007060  | 0.014010                   | 0.018331  |
|         | MSqEr   | 0.001548                   | 0.002607  | 0.063340                   | 0.000271  | 0.000369                   | 0.000904  |
|         | MReEr   | 0.134976                   | 0.830986  | 0.105640                   | 1.874271  | 0.123019                   | 0.783251  |
| 210     | Est.    | 0.246392                   | 0.049836  | 1.880039                   | 0.008103  | 0.118905                   | 0.030440  |
|         | BS      | 0.027669                   | 0.028960  | 0.190207                   | 0.006063  | 0.012343                   | 0.016861  |
|         | MSqEr   | 0.001159                   | 0.002361  | 0.054792                   | 0.000167  | 0.000294                   | 0.000798  |
|         | MReEr   | 0.118501                   | 0.805272  | 0.098172                   | 1.609540  | 0.108387                   | 0.720439  |
| 250     | Est.    | 0.244456                   | 0.046287  | 1.893600                   | 0.007382  | 0.118403                   | 0.027900  |
|         | BS      | 0.024172                   | 0.025931  | 0.164359                   | 0.005323  | 0.011666                   | 0.015150  |
|         | MSqEr   | 0.000899                   | 0.001942  | 0.041205                   | 0.000125  | 0.000257                   | 0.000621  |
|         | MReEr   | 0.103521                   | 0.721045  | 0.084830                   | 1.413124  | 0.102437                   | 0.647327  |
| 290     | Est.    | 0.244016                   | 0.045127  | 1.881859                   | 0.006780  | 0.119040                   | 0.027752  |
|         | BS      | 0.023455                   | 0.025213  | 0.157674                   | 0.004718  | 0.011603                   | 0.014813  |
|         | MSqEr   | 0.000865                   | 0.001787  | 0.036476                   | 0.000086  | 0.000242                   | 0.000571  |
|         | MReEr   | 0.100454                   | 0.701078  | 0.081380                   | 1.252495  | 0.101883                   | 0.632941  |
| 330     | Est.    | 0.242447                   | 0.042872  | 1.887141                   | 0.006461  | 0.118941                   | 0.027029  |
|         | BS      | 0.021540                   | 0.022745  | 0.149753                   | 0.004361  | 0.010962                   | 0.014632  |
|         | MSqEr   | 0.000725                   | 0.001434  | 0.033324                   | 0.000060  | 0.000203                   | 0.000561  |
|         | MReEr   | 0.092250                   | 0.632456  | 0.077292                   | 1.157676  | 0.096260                   | 0.625233  |
| 370     | Est.    | 0.241839                   | 0.041245  | 1.883455                   | 0.005898  | 0.119248                   | 0.025085  |
|         | BS      | 0.019454                   | 0.021505  | 0.138791                   | 0.003916  | 0.010952                   | 0.013165  |
|         | MSqEr   | 0.000619                   | 0.001227  | 0.028867                   | 0.000047  | 0.000194                   | 0.000419  |
|         | MReEr   | 0.083316                   | 0.597976  | 0.071634                   | 1.039555  | 0.096173                   | 0.562528  |
| 410     | Est.    | 0.241400                   | 0.038824  | 1.880839                   | 0.005539  | 0.119117                   | 0.023988  |
|         | BS      | 0.019105                   | 0.020328  | 0.132633                   | 0.003512  | 0.010834                   | 0.012243  |
|         | MSqEr   | 0.000605                   | 0.001025  | 0.025990                   | 0.000035  | 0.000182                   | 0.000364  |
|         | MReEr   | 0.081823                   | 0.565242  | 0.068456                   | 0.932278  | 0.095131                   | 0.523113  |

MSqEr, and MReEr are provided across increasing sample sizes ( $n$ ). From Table 3, it is evident that the estimated values of all measures (EX, WEX, CuREX, WCuREX, NegCEX, and WNegCEX) converge steadily toward their respective initial (true) values as  $n$  increases. For instance, the estimate of EX improves from  $-0.8931$  at  $n = 30$  to  $-0.8189$  at  $n = 410$ , approaching the theoretical value of  $-0.8387$ . The BS exhibits a clear decreasing trend across all measures, reflecting improved estimator accuracy with larger samples. Similarly, MSqEr declines significantly with increasing  $n$ , indicating enhanced precision. The MReEr follows a consistent downward trend, with particularly noticeable reductions for WCuREX, which decreases from  $0.2372$  at  $n = 30$  to  $0.0738$  at  $n = 410$ . In Table 4, a similar convergence pattern is observed. The estimates align closely with the true values even for relatively small  $n$ , with further refinement as the sample size grows. The BS values decrease systematically, with EX bias reducing from  $0.0394$  at  $n = 30$  to  $0.0121$  at  $n = 410$ . MSqEr again demonstrates strong convergence properties, with EX decreasing from  $0.0026$  at  $n = 30$  to  $0.0003$  at  $n = 410$ . The MReEr values reveal a marked improvement in relative estimation accuracy with increasing  $n$ .

For Table 5, which reports the EX measures under  $\lambda = 1.5$  and  $\mu = 2.5$ , the Est. of all measures converge closely to their initial values as  $n$  increases. The BS and MSqEr consistently decrease with sample size, highlighting the efficiency of the estimators. Notably, the MReEr shows higher values for WCuREX, while the most stable performance is observed for EX and NegCEX. Overall, this indicates that a larger  $n$  substantially improves the reliability of all EX estimators.

For Table 6, corresponding to  $\lambda = 3.5$  and  $\mu = 0.2$ , the estimates also approach their true values with increasing  $n$ . While the initial estimation at  $n = 30$  shows noticeable deviations, especially for EX and WEX, the BS and MSqEr decline rapidly as  $n$  grows. The MReEr decreases steadily, with the best performance obtained for NegCEX and WNegCEX, whereas WEX exhibits comparatively higher MReEr across all sample sizes. This behavior reflects that the weighted versions of extropy require larger  $n$  to stabilize, while the negative extropy-based measures demonstrate robustness even at smaller samples.

Finally, Table 7, for  $\lambda = 1.0$  and  $\mu = 4.0$ , demonstrates a similar trend of convergence. The BS and MSqEr values decrease monotonically as  $n$  increases, confirming consistency. Among

the measures, EX and CuREX display relatively higher MReEr, especially at smaller  $n$ , whereas WEX and WCuREX stabilize more quickly. Both NegCEX and WNegCEX consistently provide the most accurate estimates with minimal error, confirming their robustness in this parameter setting.

## 6. REAL DATA ANALYSIS

To demonstrate the practical efficacy of the proposed framework, its performance is evaluated using two datasets. The first dataset obtained by Quesenberry and Hales [29] consists of 30 measurements of tensile strength of polyester fibers. The first dataset's values are: 0.023, 0.032, 0.054, 0.069, 0.081, 0.094, 0.105, 0.127, 0.148, 0.169, 0.188, 0.216, 0.255, 0.277, 0.311, 0.361, 0.376, 0.395, 0.432, 0.463, 0.481, 0.519, 0.529, 0.567, 0.642, 0.674, 0.752, 0.823, 0.887, 0.926. The second one studied by Jodrá [30], which represents the annual proportions of antimicrobial resistant isolates in Portugal in 2012. Its values are: 0.01, 0.01, 0.03, 0.05, 0.08, 0.12, 0.14, 0.15, 0.15, 0.16, 0.19, 0.2, 0.2, 0.23, 0.26, 0.3, 0.32, 0.36, 0.39, 0.43, 0.54, 0.58, 0.59, 0.94.

We evaluated the model's fit to the analyzed data via the Kolmogorov-Smirnov test. The high p-values (0.999973 and 0.956219) for both datasets fail to reject the null hypothesis at the  $\alpha = 0.05$  level, supporting the model's flexibility in accommodating the observed data. Supplementary visual assessments, provided in Figures 1 and 2, comprise estimated PDFs with histograms, P-P plots, and estimated CDFs and survival functions.

Table 8 presents estimated EX, WEX, CuREX, WCuREX, NegCEX, and WNegCEX for the two real datasets. From this table, we have

- A key observation is the consistently negative values for the EX, WEX, CuREX, and WCuREX measures. For Data I, the standard entropy is calculated as  $EX = -0.603864$ , while for Data II, it is  $EX = -0.808213$ .
- Data I shows a higher NegCEX (NegCEX = 0.257802) than Data II (NegCEX = 0.209818), which is consistent with the interpretation from the standard entropy measures.
- The weighted variants (WEX, WCuREX, WNegCEX) consistently return values closer to zero than their non-weighted counterparts.

TABLE 3. Numerical values of the different extropy measures for  $\lambda = 0.9$  and  $\mu = 1.5$ .

| $n$     | Measure | EX        | WEX       | CuREX     | WCuREX    | NegCEX   | WNegCEX  |
|---------|---------|-----------|-----------|-----------|-----------|----------|----------|
| Initial |         | -0.83872  | -0.12832  | -0.0825   | -0.01458  | 0.23526  | 0.0737   |
| 30      | Est.    | -0.893144 | -0.136150 | -0.089347 | -0.016795 | 0.245934 | 0.079303 |
|         | BS      | 0.222338  | 0.014358  | 0.012301  | 0.003461  | 0.018626 | 0.009151 |
|         | MSqEr   | 0.223521  | 0.000340  | 0.000241  | 0.000020  | 0.000530 | 0.000130 |
|         | MReEr   | 0.265090  | 0.111893  | 0.148920  | 0.237241  | 0.079171 | 0.124171 |
| 90      | Est.    | -0.851424 | -0.133603 | -0.087104 | -0.015983 | 0.242104 | 0.077187 |
|         | BS      | 0.138603  | 0.010506  | 0.009071  | 0.002496  | 0.013816 | 0.006775 |
|         | MSqEr   | 0.037951  | 0.000178  | 0.000130  | 0.000010  | 0.000295 | 0.000072 |
|         | MReEr   | 0.165254  | 0.081876  | 0.109814  | 0.171082  | 0.058724 | 0.091931 |
| 130     | Est.    | -0.836350 | -0.132558 | -0.086162 | -0.015661 | 0.240600 | 0.076364 |
|         | BS      | 0.104549  | 0.008292  | 0.007197  | 0.001996  | 0.011272 | 0.005602 |
|         | MSqEr   | 0.017605  | 0.000109  | 0.000082  | 0.000007  | 0.000203 | 0.000051 |
|         | MReEr   | 0.124652  | 0.064617  | 0.087126  | 0.136864  | 0.047913 | 0.076015 |
| 170     | Est.    | -0.828683 | -0.132032 | -0.085667 | -0.015461 | 0.239618 | 0.075801 |
|         | BS      | 0.087228  | 0.006904  | 0.006006  | 0.001667  | 0.009598 | 0.004824 |
|         | MSqEr   | 0.012156  | 0.000077  | 0.000058  | 0.000005  | 0.000142 | 0.000036 |
|         | MReEr   | 0.104001  | 0.053803  | 0.072715  | 0.114297  | 0.040797 | 0.065458 |
| 210     | Est.    | -0.822680 | -0.131900 | -0.085556 | -0.015411 | 0.239435 | 0.075685 |
|         | BS      | 0.074280  | 0.006014  | 0.005224  | 0.001475  | 0.008553 | 0.004346 |
|         | MSqEr   | 0.009170  | 0.000060  | 0.000045  | 0.000004  | 0.000117 | 0.000030 |
|         | MReEr   | 0.088563  | 0.046864  | 0.063247  | 0.101088  | 0.036355 | 0.058965 |
| 250     | Est.    | -0.821275 | -0.131652 | -0.085299 | -0.015321 | 0.238919 | 0.075408 |
|         | BS      | 0.068848  | 0.005678  | 0.004975  | 0.001405  | 0.008204 | 0.004180 |
|         | MSqEr   | 0.007992  | 0.000053  | 0.000040  | 0.000003  | 0.000106 | 0.000027 |
|         | MReEr   | 0.082087  | 0.044246  | 0.060235  | 0.096332  | 0.034870 | 0.056718 |
| 290     | Est.    | -0.817245 | -0.131690 | -0.085345 | -0.015327 | 0.238980 | 0.075430 |
|         | BS      | 0.065687  | 0.005495  | 0.004851  | 0.001372  | 0.008020 | 0.004049 |
|         | MSqEr   | 0.006718  | 0.000048  | 0.000038  | 0.000003  | 0.000102 | 0.000026 |
|         | MReEr   | 0.078317  | 0.042825  | 0.058726  | 0.094082  | 0.034090 | 0.054939 |
| 330     | Est.    | -0.822324 | -0.131053 | -0.084868 | -0.015209 | 0.238449 | 0.075203 |
|         | BS      | 0.059470  | 0.004840  | 0.004294  | 0.001203  | 0.007082 | 0.003613 |
|         | MSqEr   | 0.005712  | 0.000038  | 0.000030  | 0.000002  | 0.000080 | 0.000021 |
|         | MReEr   | 0.070905  | 0.037721  | 0.051980  | 0.082501  | 0.030102 | 0.049021 |
| 370     | Est.    | -0.821220 | -0.131013 | -0.084772 | -0.015157 | 0.238120 | 0.075003 |
|         | BS      | 0.058911  | 0.004682  | 0.004054  | 0.001123  | 0.006634 | 0.003379 |
|         | MSqEr   | 0.005688  | 0.000035  | 0.000027  | 0.000002  | 0.000070 | 0.000018 |
|         | MReEr   | 0.070239  | 0.036486  | 0.049075  | 0.076965  | 0.028199 | 0.045848 |
| 410     | Est.    | -0.818907 | -0.130909 | -0.084624 | -0.015098 | 0.237757 | 0.074799 |
|         | BS      | 0.053187  | 0.004374  | 0.003823  | 0.001076  | 0.006429 | 0.003306 |
|         | MSqEr   | 0.004413  | 0.000030  | 0.000024  | 0.000002  | 0.000066 | 0.000017 |
|         | MReEr   | 0.063414  | 0.034083  | 0.046285  | 0.073760  | 0.027327 | 0.044856 |

TABLE 4. Numerical values of the different extropy measures for  $\lambda = 2.0$  and  $\mu = 0.5$ .

| $n$     | Measure | EX        | WEX       | CuREX     | WCuREX    | NegCEX   | WNegCEX  |
|---------|---------|-----------|-----------|-----------|-----------|----------|----------|
| Initial |         | -0.59259  | -0.41     | -0.2395   | -0.07018  | 0.38272  | 0.15481  |
| 30      | Est.    | -0.604358 | -0.423643 | -0.241757 | -0.071777 | 0.384472 | 0.156252 |
|         | BS      | 0.039390  | 0.048816  | 0.016640  | 0.007562  | 0.010363 | 0.006873 |
|         | MSqEr   | 0.002627  | 0.003927  | 0.000434  | 0.000091  | 0.000168 | 0.000074 |
|         | MReEr   | 0.066471  | 0.119062  | 0.069476  | 0.107743  | 0.027077 | 0.044394 |
| 90      | Est.    | -0.602094 | -0.421737 | -0.242006 | -0.071695 | 0.384651 | 0.156267 |
|         | BS      | 0.029802  | 0.037036  | 0.012553  | 0.005746  | 0.007905 | 0.005271 |
|         | MSqEr   | 0.001462  | 0.002195  | 0.000242  | 0.000051  | 0.000094 | 0.000042 |
|         | MReEr   | 0.050291  | 0.090331  | 0.052413  | 0.081864  | 0.020654 | 0.034045 |
| 130     | Est.    | -0.600022 | -0.419684 | -0.241719 | -0.071478 | 0.384517 | 0.156134 |
|         | BS      | 0.022696  | 0.028328  | 0.009602  | 0.004403  | 0.006102 | 0.004081 |
|         | MSqEr   | 0.000858  | 0.001307  | 0.000145  | 0.000031  | 0.000058 | 0.000026 |
|         | MReEr   | 0.038299  | 0.069092  | 0.040091  | 0.062739  | 0.015945 | 0.026362 |
| 170     | Est.    | -0.597379 | -0.416454 | -0.240836 | -0.071023 | 0.383924 | 0.155714 |
|         | BS      | 0.019061  | 0.023832  | 0.008172  | 0.003725  | 0.005192 | 0.003468 |
|         | MSqEr   | 0.000585  | 0.000908  | 0.000103  | 0.000022  | 0.000042 | 0.000019 |
|         | MReEr   | 0.032165  | 0.058126  | 0.034119  | 0.053077  | 0.013565 | 0.022403 |
| 210     | Est.    | -0.600067 | -0.419961 | -0.242093 | -0.071589 | 0.384748 | 0.156257 |
|         | BS      | 0.018188  | 0.022818  | 0.007696  | 0.003543  | 0.004943 | 0.003315 |
|         | MSqEr   | 0.000520  | 0.000812  | 0.000090  | 0.000019  | 0.000037 | 0.000017 |
|         | MReEr   | 0.030692  | 0.055654  | 0.032132  | 0.050481  | 0.012916 | 0.021413 |
| 250     | Est.    | -0.598314 | -0.417738 | -0.241441 | -0.071263 | 0.384298 | 0.155944 |
|         | BS      | 0.015595  | 0.019732  | 0.006644  | 0.003067  | 0.004330 | 0.002915 |
|         | MSqEr   | 0.000410  | 0.000648  | 0.000072  | 0.000015  | 0.000030 | 0.000014 |
|         | MReEr   | 0.026317  | 0.048127  | 0.027742  | 0.043697  | 0.011313 | 0.018831 |
| 290     | Est.    | -0.598828 | -0.418158 | -0.241723 | -0.071353 | 0.384376 | 0.155979 |
|         | BS      | 0.015072  | 0.019033  | 0.006409  | 0.002959  | 0.004168 | 0.002807 |
|         | MSqEr   | 0.000380  | 0.000599  | 0.000066  | 0.000014  | 0.000028 | 0.000013 |
|         | MReEr   | 0.025434  | 0.046423  | 0.026758  | 0.042155  | 0.010890 | 0.018130 |
| 330     | Est.    | -0.598049 | -0.417166 | -0.241458 | -0.071213 | 0.384184 | 0.155843 |
|         | BS      | 0.013385  | 0.017046  | 0.005729  | 0.002651  | 0.003776 | 0.002546 |
|         | MSqEr   | 0.000303  | 0.000491  | 0.000055  | 0.000012  | 0.000024 | 0.000011 |
|         | MReEr   | 0.022587  | 0.041577  | 0.023921  | 0.037774  | 0.009866 | 0.016447 |
| 370     | Est.    | -0.597672 | -0.416816 | -0.241298 | -0.071153 | 0.384129 | 0.155811 |
|         | BS      | 0.013336  | 0.016860  | 0.005713  | 0.002631  | 0.003721 | 0.002507 |
|         | MSqEr   | 0.000296  | 0.000475  | 0.000053  | 0.000011  | 0.000023 | 0.000010 |
|         | MReEr   | 0.022504  | 0.041121  | 0.023854  | 0.037480  | 0.009722 | 0.016196 |
| 410     | Est.    | -0.597219 | -0.416092 | -0.241159 | -0.071059 | 0.383967 | 0.155690 |
|         | BS      | 0.012088  | 0.015331  | 0.005190  | 0.002390  | 0.003401 | 0.002297 |
|         | MSqEr   | 0.000257  | 0.000411  | 0.000046  | 0.000010  | 0.000020 | 0.000009 |
|         | MReEr   | 0.020398  | 0.037393  | 0.021671  | 0.034055  | 0.008886 | 0.014836 |

TABLE 5. Numerical values of the different extropy measures for  $\lambda = 1.5$  and  $\mu = 2.5$ .

| $n$     | Measure | EX        | WEX       | CuREX     | WCuREX    | NegCEX   | WNegCEX  |
|---------|---------|-----------|-----------|-----------|-----------|----------|----------|
| Initial |         | -0.52404  | -0.21386  | -0.14732  | -0.03228  | 0.30267  | 0.10546  |
| 30      | Est.    | -0.521963 | -0.227609 | -0.154963 | -0.035582 | 0.312110 | 0.111336 |
|         | BS      | 0.012054  | 0.022472  | 0.014012  | 0.005166  | 0.015386 | 0.009124 |
|         | MSqEr   | 0.000238  | 0.000842  | 0.000309  | 0.000044  | 0.000365 | 0.000130 |
|         | MReEr   | 0.023002  | 0.105075  | 0.095108  | 0.160017  | 0.050834 | 0.086507 |
| 90      | Est.    | -0.521367 | -0.223667 | -0.152992 | -0.034647 | 0.309628 | 0.109715 |
|         | BS      | 0.010629  | 0.015899  | 0.010124  | 0.003757  | 0.011592 | 0.006904 |
|         | MSqEr   | 0.000170  | 0.000417  | 0.000163  | 0.000022  | 0.000204 | 0.000073 |
|         | MReEr   | 0.020283  | 0.074339  | 0.068721  | 0.116376  | 0.038297 | 0.065458 |
| 130     | Est.    | -0.521512 | -0.221409 | -0.151778 | -0.034145 | 0.308263 | 0.108891 |
|         | BS      | 0.009467  | 0.013239  | 0.008520  | 0.003094  | 0.009627 | 0.005714 |
|         | MSqEr   | 0.000139  | 0.000296  | 0.000120  | 0.000016  | 0.000148 | 0.000053 |
|         | MReEr   | 0.018065  | 0.061903  | 0.057828  | 0.095825  | 0.031806 | 0.054173 |
| 170     | Est.    | -0.522203 | -0.220815 | -0.151432 | -0.033909 | 0.307386 | 0.108277 |
|         | BS      | 0.008894  | 0.011576  | 0.007458  | 0.002701  | 0.008536 | 0.005128 |
|         | MSqEr   | 0.000120  | 0.000230  | 0.000094  | 0.000013  | 0.000119 | 0.000043 |
|         | MReEr   | 0.016972  | 0.054126  | 0.050619  | 0.083661  | 0.028203 | 0.048624 |
| 210     | Est.    | -0.522751 | -0.218971 | -0.150311 | -0.033477 | 0.306136 | 0.107540 |
|         | BS      | 0.008460  | 0.009791  | 0.006426  | 0.002351  | 0.007598 | 0.004567 |
|         | MSqEr   | 0.000109  | 0.000162  | 0.000069  | 0.000009  | 0.000092 | 0.000033 |
|         | MReEr   | 0.016144  | 0.045782  | 0.043617  | 0.072826  | 0.025102 | 0.043300 |
| 250     | Est.    | -0.523421 | -0.218420 | -0.149979 | -0.033308 | 0.305525 | 0.107144 |
|         | BS      | 0.007849  | 0.009373  | 0.006162  | 0.002192  | 0.006995 | 0.004171 |
|         | MSqEr   | 0.000094  | 0.000150  | 0.000065  | 0.000008  | 0.000080 | 0.000028 |
|         | MReEr   | 0.014979  | 0.043825  | 0.041823  | 0.067900  | 0.023110 | 0.039545 |
| 290     | Est.    | -0.523275 | -0.217954 | -0.149712 | -0.033224 | 0.305348 | 0.107056 |
|         | BS      | 0.007675  | 0.008966  | 0.005917  | 0.002100  | 0.006739 | 0.004030 |
|         | MSqEr   | 0.000091  | 0.000131  | 0.000057  | 0.000007  | 0.000073 | 0.000026 |
|         | MReEr   | 0.014646  | 0.041922  | 0.040161  | 0.065052  | 0.022266 | 0.038212 |
| 330     | Est.    | -0.523794 | -0.217557 | -0.149457 | -0.033086 | 0.304840 | 0.106720 |
|         | BS      | 0.007299  | 0.008373  | 0.005576  | 0.001980  | 0.006349 | 0.003769 |
|         | MSqEr   | 0.000082  | 0.000115  | 0.000051  | 0.000006  | 0.000063 | 0.000022 |
|         | MReEr   | 0.013928  | 0.039153  | 0.037850  | 0.061311  | 0.020975 | 0.035737 |
| 370     | Est.    | -0.524281 | -0.217048 | -0.149125 | -0.032933 | 0.304311 | 0.106383 |
|         | BS      | 0.006819  | 0.007750  | 0.005124  | 0.001788  | 0.005737 | 0.003425 |
|         | MSqEr   | 0.000073  | 0.000098  | 0.000043  | 0.000005  | 0.000053 | 0.000019 |
|         | MReEr   | 0.013013  | 0.036239  | 0.034777  | 0.055381  | 0.018955 | 0.032475 |
| 410     | Est.    | -0.524415 | -0.216204 | -0.148596 | -0.032770 | 0.303904 | 0.106181 |
|         | BS      | 0.006524  | 0.007669  | 0.005101  | 0.001749  | 0.005553 | 0.003301 |
|         | MSqEr   | 0.000067  | 0.000092  | 0.000041  | 0.000005  | 0.000049 | 0.000017 |
|         | MReEr   | 0.012450  | 0.035859  | 0.034622  | 0.054179  | 0.018347 | 0.031295 |

TABLE 6. Numerical values of the different extropy measures for  $\lambda = 3.5$  and  $\mu = 0.2$ .

| $n$     | Measure | EX        | WEX       | CuREX     | WCuREX    | NegCEX   | WNegCEX  |
|---------|---------|-----------|-----------|-----------|-----------|----------|----------|
| Initial |         | -0.97638  | -0.82898  | -0.33396  | -0.11986  | 0.43447  | 0.19205  |
| 30      | Est.    | -0.998055 | -0.851021 | -0.335612 | -0.121100 | 0.435184 | 0.192690 |
|         | BS      | 0.085629  | 0.088326  | 0.012436  | 0.007528  | 0.005838 | 0.004575 |
|         | MSqEr   | 0.011525  | 0.012227  | 0.000230  | 0.000085  | 0.000051 | 0.000031 |
|         | MReEr   | 0.087700  | 0.106547  | 0.037238  | 0.062800  | 0.013437 | 0.023820 |
| 90      | Est.    | -0.982881 | -0.835621 | -0.334123 | -0.120093 | 0.434553 | 0.192160 |
|         | BS      | 0.058108  | 0.059959  | 0.008586  | 0.005183  | 0.004031 | 0.003155 |
|         | MSqEr   | 0.005367  | 0.005700  | 0.000110  | 0.000041  | 0.000024 | 0.000015 |
|         | MReEr   | 0.059514  | 0.072328  | 0.025711  | 0.043238  | 0.009277 | 0.016429 |
| 130     | Est.    | -0.985553 | -0.838472 | -0.334795 | -0.120459 | 0.434892 | 0.192413 |
|         | BS      | 0.045321  | 0.046759  | 0.006650  | 0.004024  | 0.003121 | 0.002447 |
|         | MSqEr   | 0.003400  | 0.003610  | 0.000068  | 0.000025  | 0.000015 | 0.000009 |
|         | MReEr   | 0.046417  | 0.056406  | 0.019913  | 0.033570  | 0.007183 | 0.012738 |
| 170     | Est.    | -0.988437 | -0.841477 | -0.335355 | -0.120778 | 0.435161 | 0.192617 |
|         | BS      | 0.039877  | 0.041140  | 0.005838  | 0.003536  | 0.002739 | 0.002148 |
|         | MSqEr   | 0.002480  | 0.002638  | 0.000051  | 0.000019  | 0.000011 | 0.000007 |
|         | MReEr   | 0.040841  | 0.049627  | 0.017482  | 0.029502  | 0.006303 | 0.011185 |
| 210     | Est.    | -0.993971 | -0.847192 | -0.336208 | -0.121287 | 0.435562 | 0.192930 |
|         | BS      | 0.037739  | 0.038939  | 0.005450  | 0.003314  | 0.002560 | 0.002012 |
|         | MSqEr   | 0.002409  | 0.002559  | 0.000048  | 0.000018  | 0.000010 | 0.000006 |
|         | MReEr   | 0.038652  | 0.046972  | 0.016319  | 0.027648  | 0.005893 | 0.010478 |
| 250     | Est.    | -0.995917 | -0.849211 | -0.336531 | -0.121478 | 0.435716 | 0.193049 |
|         | BS      | 0.035975  | 0.037129  | 0.005183  | 0.003155  | 0.002439 | 0.001918 |
|         | MSqEr   | 0.002180  | 0.002318  | 0.000043  | 0.000016  | 0.000010 | 0.000006 |
|         | MReEr   | 0.036845  | 0.044789  | 0.015521  | 0.026319  | 0.005614 | 0.009985 |
| 290     | Est.    | -0.995417 | -0.848702 | -0.336488 | -0.121447 | 0.435698 | 0.193033 |
|         | BS      | 0.033350  | 0.034430  | 0.004795  | 0.002921  | 0.002260 | 0.001778 |
|         | MSqEr   | 0.001942  | 0.002066  | 0.000038  | 0.000014  | 0.000008 | 0.000005 |
|         | MReEr   | 0.034157  | 0.041533  | 0.014359  | 0.024369  | 0.005202 | 0.009256 |
| 330     | Est.    | -0.992777 | -0.845995 | -0.336140 | -0.121232 | 0.435539 | 0.192907 |
|         | BS      | 0.030045  | 0.031029  | 0.004361  | 0.002650  | 0.002058 | 0.001616 |
|         | MSqEr   | 0.001532  | 0.001632  | 0.000031  | 0.000012  | 0.000007 | 0.000004 |
|         | MReEr   | 0.030772  | 0.037430  | 0.013058  | 0.022110  | 0.004736 | 0.008416 |
| 370     | Est.    | -0.990229 | -0.843376 | -0.335801 | -0.121021 | 0.435382 | 0.192782 |
|         | BS      | 0.026891  | 0.027776  | 0.003927  | 0.002383  | 0.001854 | 0.001456 |
|         | MSqEr   | 0.001198  | 0.001278  | 0.000025  | 0.000009  | 0.000006 | 0.000003 |
|         | MReEr   | 0.027541  | 0.033506  | 0.011760  | 0.019884  | 0.004268 | 0.007578 |
| 410     | Est.    | -0.991195 | -0.844377 | -0.335961 | -0.121115 | 0.435458 | 0.192841 |
|         | BS      | 0.025279  | 0.026128  | 0.003676  | 0.002234  | 0.001742 | 0.001368 |
|         | MSqEr   | 0.001099  | 0.001173  | 0.000023  | 0.000008  | 0.000005 | 0.000003 |
|         | MReEr   | 0.025890  | 0.031519  | 0.011006  | 0.018639  | 0.004009 | 0.007124 |

TABLE 7. Numerical values of the different extropy measures for  $\lambda = 1.0$  and  $\mu = 4.0$ .

| $n$     | Measure | EX        | WEX       | CuREX     | WCuREX    | NegCEX   | WNegCEX  |
|---------|---------|-----------|-----------|-----------|-----------|----------|----------|
| Initial |         | -0.82     | -0.13     | -0.07925  | -0.01277  | 0.22074  | 0.065    |
| 30      | Est.    | -0.767846 | -0.139185 | -0.087930 | -0.015172 | 0.234162 | 0.071506 |
|         | BS      | 0.095099  | 0.011101  | 0.010091  | 0.002711  | 0.015800 | 0.007698 |
|         | MSqEr   | 0.019927  | 0.000233  | 0.000182  | 0.000014  | 0.000428 | 0.000106 |
|         | MReEr   | 0.115975  | 0.085390  | 0.127314  | 0.212174  | 0.071576 | 0.118437 |
| 90      | Est.    | -0.766238 | -0.137525 | -0.086753 | -0.014850 | 0.232854 | 0.070911 |
|         | BS      | 0.086506  | 0.009263  | 0.008857  | 0.002382  | 0.014373 | 0.007021 |
|         | MSqEr   | 0.013739  | 0.000144  | 0.000124  | 0.000009  | 0.000322 | 0.000080 |
|         | MReEr   | 0.105495  | 0.071256  | 0.111744  | 0.186404  | 0.065114 | 0.108011 |
| 130     | Est.    | -0.769935 | -0.136027 | -0.085252 | -0.014399 | 0.230412 | 0.069657 |
|         | BS      | 0.066713  | 0.007170  | 0.006773  | 0.001810  | 0.011186 | 0.005511 |
|         | MSqEr   | 0.007211  | 0.000085  | 0.000075  | 0.000006  | 0.000204 | 0.000051 |
|         | MReEr   | 0.081357  | 0.055153  | 0.085448  | 0.141615  | 0.050677 | 0.084791 |
| 170     | Est.    | -0.789090 | -0.133905 | -0.083128 | -0.013817 | 0.226948 | 0.067978 |
|         | BS      | 0.057228  | 0.005714  | 0.005252  | 0.001385  | 0.008877 | 0.004399 |
|         | MSqEr   | 0.005612  | 0.000054  | 0.000046  | 0.000003  | 0.000125 | 0.000031 |
|         | MReEr   | 0.069790  | 0.043953  | 0.066267  | 0.108409  | 0.040214 | 0.067675 |
| 210     | Est.    | -0.788415 | -0.133715 | -0.082900 | -0.013745 | 0.226542 | 0.067760 |
|         | BS      | 0.050446  | 0.005073  | 0.004741  | 0.001263  | 0.008188 | 0.004113 |
|         | MSqEr   | 0.004179  | 0.000042  | 0.000036  | 0.000003  | 0.000106 | 0.000027 |
|         | MReEr   | 0.061520  | 0.039021  | 0.059819  | 0.098864  | 0.037091 | 0.063273 |
| 250     | Est.    | -0.789388 | -0.133571 | -0.082546 | -0.013611 | 0.225617 | 0.067241 |
|         | BS      | 0.047410  | 0.004854  | 0.004388  | 0.001133  | 0.007311 | 0.003629 |
|         | MSqEr   | 0.003520  | 0.000037  | 0.000031  | 0.000002  | 0.000084 | 0.000021 |
|         | MReEr   | 0.057818  | 0.037335  | 0.055358  | 0.088696  | 0.033119 | 0.055823 |
| 290     | Est.    | -0.790272 | -0.133438 | -0.082469 | -0.013597 | 0.225595 | 0.067244 |
|         | BS      | 0.046254  | 0.004713  | 0.004203  | 0.001076  | 0.006923 | 0.003452 |
|         | MSqEr   | 0.003222  | 0.000035  | 0.000028  | 0.000002  | 0.000076 | 0.000019 |
|         | MReEr   | 0.056407  | 0.036257  | 0.053032  | 0.084210  | 0.031361 | 0.053106 |
| 330     | Est.    | -0.789244 | -0.133541 | -0.082431 | -0.013566 | 0.225321 | 0.067068 |
|         | BS      | 0.044395  | 0.004621  | 0.004080  | 0.001047  | 0.006792 | 0.003416 |
|         | MSqEr   | 0.002937  | 0.000033  | 0.000026  | 0.000002  | 0.000072 | 0.000018 |
|         | MReEr   | 0.054140  | 0.035547  | 0.051472  | 0.081910  | 0.030771 | 0.052556 |
| 370     | Est.    | -0.790626 | -0.133280 | -0.082336 | -0.013563 | 0.225430 | 0.067168 |
|         | BS      | 0.041494  | 0.004283  | 0.003861  | 0.001005  | 0.006536 | 0.003277 |
|         | MSqEr   | 0.002595  | 0.000030  | 0.000023  | 0.000002  | 0.000067 | 0.000017 |
|         | MReEr   | 0.050602  | 0.032946  | 0.048712  | 0.078675  | 0.029608 | 0.050417 |
| 410     | Est.    | -0.790401 | -0.133386 | -0.082181 | -0.013482 | 0.224770 | 0.066770 |
|         | BS      | 0.040288  | 0.004281  | 0.003647  | 0.000930  | 0.006039 | 0.003060 |
|         | MSqEr   | 0.002448  | 0.000028  | 0.000021  | 0.000001  | 0.000058 | 0.000015 |
|         | MReEr   | 0.049131  | 0.032933  | 0.046010  | 0.072792  | 0.027360 | 0.047075 |

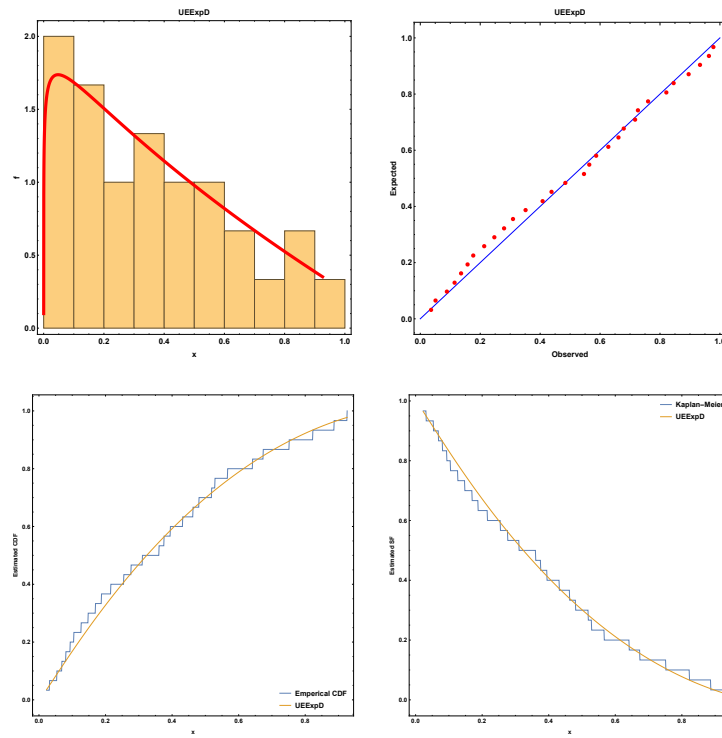


FIGURE 1. Graphical representation for the first data set.

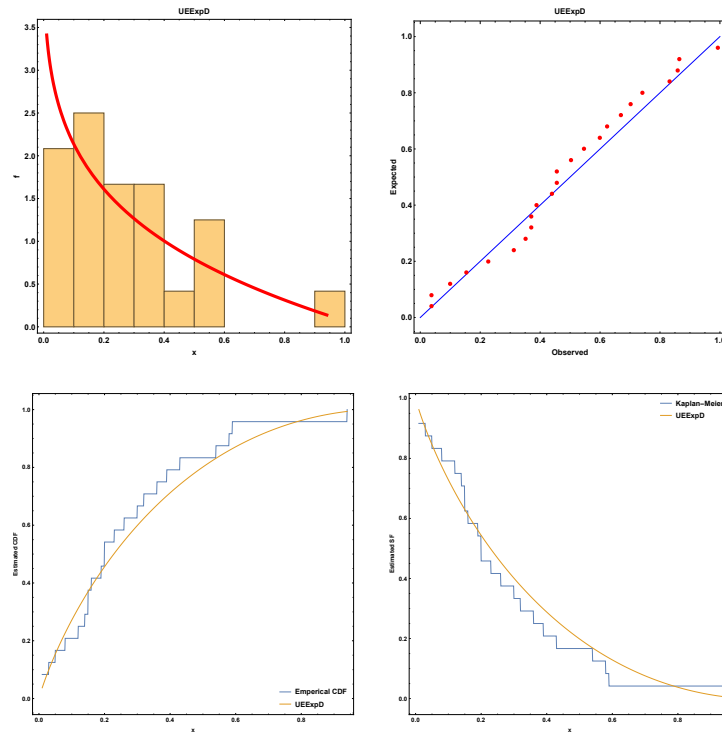


FIGURE 2. Graphical representation for the second data set.

TABLE 8. Estimates for different extropy values for the two real datasets.

| Data    | $\hat{\lambda}$ | $\hat{\mu}$ | EX        | WEX       | CuREX      | WCuREX     | NegCEX   | WNegCEX   |
|---------|-----------------|-------------|-----------|-----------|------------|------------|----------|-----------|
| Data I  | 1.31034         | 5.45248     | -0.603864 | -0.169941 | -0.111437  | -0.0205133 | 0.257802 | 0.0809966 |
| Data II | 1.0918          | 15.8679     | -0.808213 | -0.137041 | -0.0786694 | -0.0118312 | 0.209818 | 0.0578711 |

The estimated  $FI(\lambda)$  and  $FI(\mu)$  for the two real datasets are presented in Table 9.

TABLE 9. Estimates of the Fisher information measures for the two real datasets.

| Data    | $\hat{\lambda}$ | $\hat{\mu}$ | $FI(\lambda)$ | $FI(\mu)$ |
|---------|-----------------|-------------|---------------|-----------|
| Data I  | 1.31034         | 5.45248     | 1.14297       | 0.000878  |
| Data II | 1.0918          | 15.8679     | 1.67434       | 0.000024  |

**Analysis of ordered real data.** The sample order statistics from a continuous population with CDF  $F(x)$  and PDF  $f(x)$  of UEExpD are denoted by  $X_{1:n}, \dots, X_{n:n}$ . In such case, the density function of  $X_{r:n}$  is  $f_{r:n}(x) = \frac{n!}{(r-1)!(n-r)!} F(x)^{r-1} (1-F(x))^{n-r} f(x)$ . To gain a deeper understanding of the uncertainty structure in the data, we computed and presented the EX and FI measures as functions of the order statistic  $r$ ; for details, refer to Figures 3, 4, 5 and 6. These graphs visually evaluate the variation in uncertainty across different points in the sequence of the two genuine datasets, shedding light on the overall information richness and system unpredictability.

Figures 3 and 4 provide the graphical representations of six extropy measures: EX, WEX, CuREX, WCuREX, NegCEX, and WNegCEX, for the first and second ordered real datasets. Here we can see various interesting patterns: Both are negative, EX falls more at lower  $r$ -values and then levels off, while WEX falls more evenly. Both residual measures cumulatively decrease monotonically with increasing  $r$ . Unlike the prior measurements, these NegCEX measures escalate with  $r$ . Figure 5 displays the FI values for the parameters  $\lambda$  and  $\mu$  of the UEExpD, derived from the first real dataset. The left panel illustrates the FI for  $\lambda$ , which initially ascends with  $r$ , attains a maximum at the mid-range, and thereafter diminishes as  $r$  continues to increase. This bell-shaped distribution means that the central part of the ordered data essentially determines the information about  $\lambda$ . The influence of both the microscopic and the broad observations is comparatively reduced. The right panel shows the FI for  $\mu$  which monotonically increases with  $r$  and maximizes at the top end of the sorting data. This monotonicity means that larger observations bear more information about  $\mu$ , and thus this parameter is particularly

sensitive to the tail properties of the data set. The Fisher information for the parameters  $\lambda$  (left panel) and  $\mu$  (right panel) of the UEExpD obtained from the second real data set is shown in Figure 6. The Fisher information curve for  $\lambda$  is bell-shaped: it rises strongly for lower  $r$ , peaks in the middle of the ordered dataset, and then progressively falls as  $r$  increases. The influence of the extremes on inference about  $\lambda$  is less than that of the central observations under this pattern. The Fisher information in  $\mu$  increases almost monotonically with  $r$  and is maximized at the top end of the ordered data. This indicates that the estimation of  $\mu$  is sensitive to the tail behavior in the dataset and is highly dependent on the larger observations.

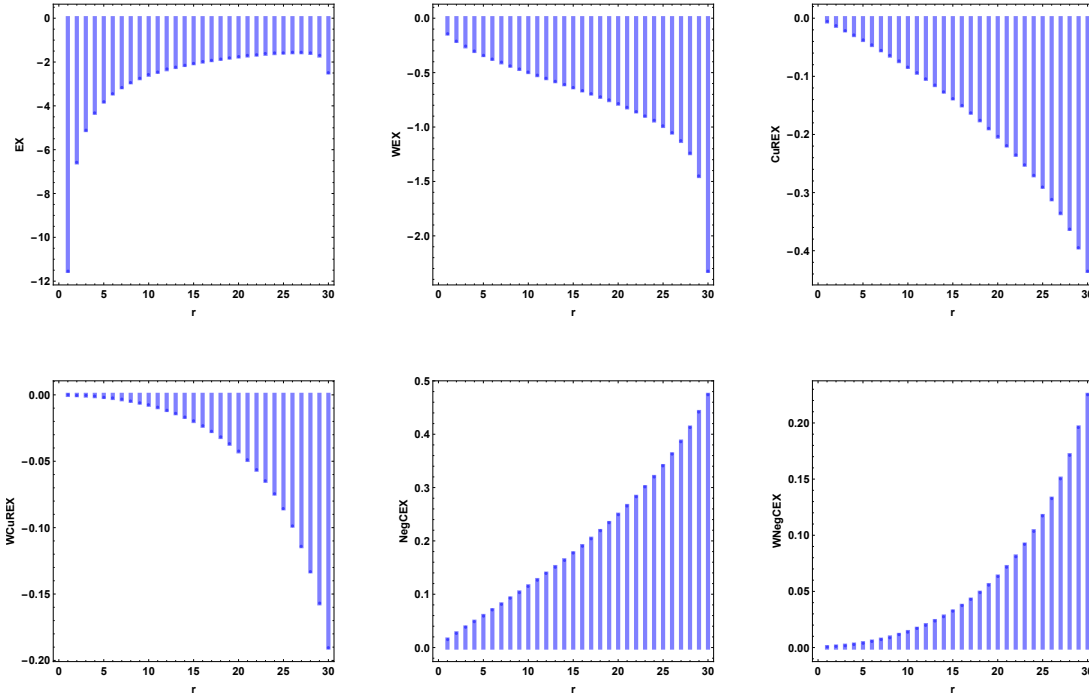


FIGURE 3. Graphical representation of EX, WEX, CuREX, WCuREX, NegCEX, and WNegCEX for the first dataset.

## 7. CONCLUSION AND FUTURE PERSPECTIVES

This paper developed an information-theoretic treatment of the UEExpD, a flexible two-parameter model for bounded data on  $(0, 1)$ . We derived closed-form expressions for the EX, WEX, CuREX, WCuREX, NegCEX and WNegCEX of the UEExpD, and confirmed all of them by direct numerical integration. The FIM was obtained in closed form. For order statistics, the

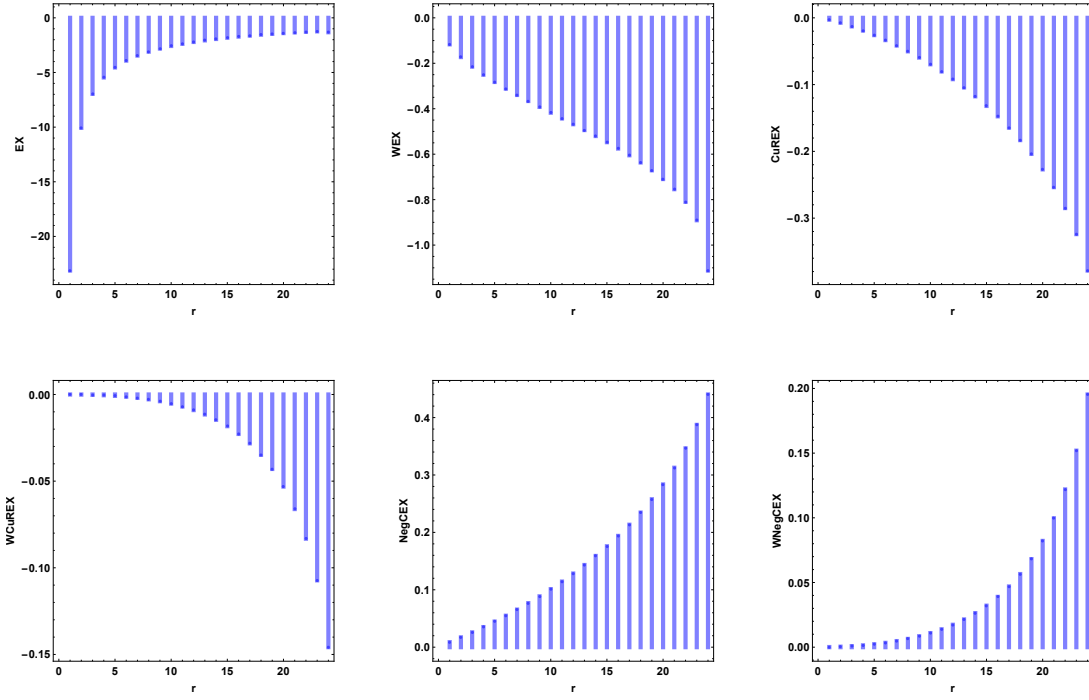


FIGURE 4. Graphical representation of EX, WEX, CuREX, WCuREX, NegCEX, and WNegCEX for the second dataset.

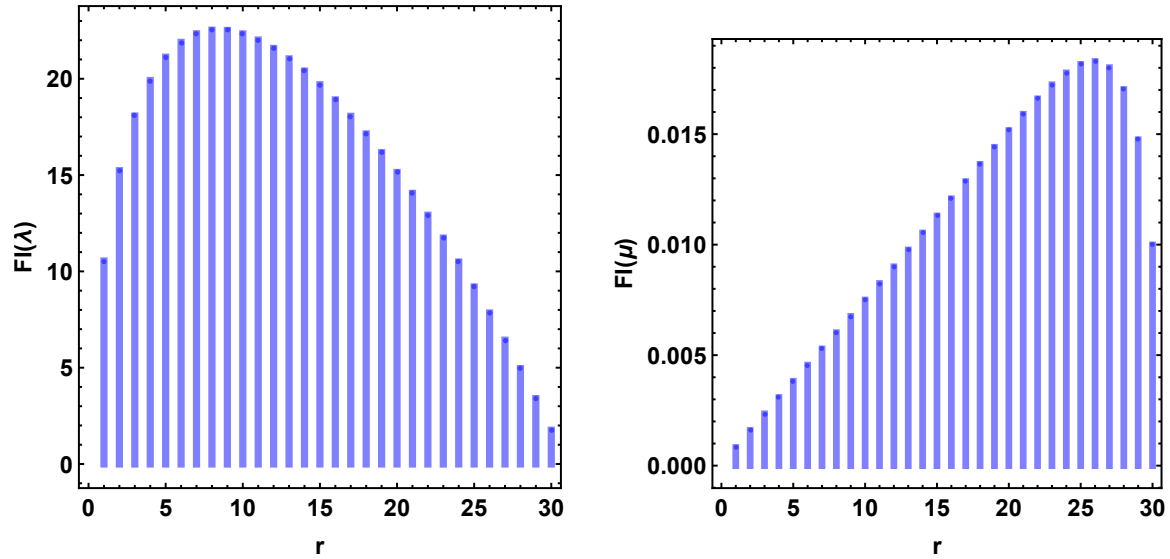


FIGURE 5. Graphical representation of  $FI(\lambda)$  and  $FI(\mu)$  for the first dataset.

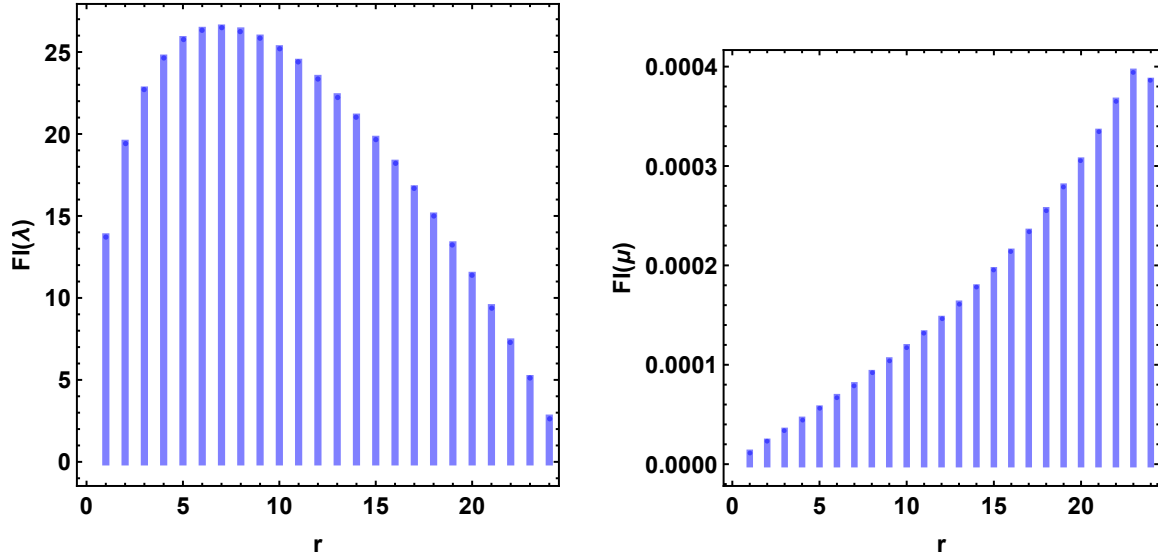


FIGURE 6. Graphical representation of  $FI(\lambda)$  and  $FI(\mu)$  for the second dataset.

extropy measures and the FIM were represented as one-dimensional integrals over  $(0, 1)$  that are evaluated accurately by numerical quadrature. The simulation study showed that the bias, mean square error and mean relative error of the maximum likelihood estimates of all measures decrease as the sample size increases. The UEExpD provided an adequate fit to both real datasets. The order-statistic analysis showed that information about  $\lambda$  is concentrated in the lower-to-middle part of the ordered sample. By contrast, information about  $\mu$  increases with  $r$  over most of the ordered sample and is carried mainly by the larger observations. This distinction can guide the choice of which ordered observations to record in inspection and censoring schemes.

Several directions remain open. A systematic comparison of the UEExpD with competing unit-interval models, such as the beta, Kumaraswamy and unit-Weibull distributions, based on information criteria and goodness-of-fit statistics, would further clarify its practical scope. Given the weak information about  $\mu$ , Bayesian and interval estimation deserve particular attention. The extropy and Fisher information results can be extended to censored samples, record values and progressive censoring schemes, where  $\mathcal{J}_{r:n}$  may be used to design optimal inspection plans. Further directions include goodness-of-fit tests based on nonparametric extropy estimators, and multivariate and regression versions of the UEExpD.

## **AUTHORS CONTRIBUTIONS**

All authors have contributed equally to writing and reviewing the manuscript.

## **DATA AVAILABILITY STATEMENT**

The data that support the findings of this study are available within the article.

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## **CONFLICT OF INTERESTS**

The author(s) declare that there is no conflict of interests.

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