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AN EPIDEMIC MODEL WITH CONVALESCENT CARRIER

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Abstract. In this paper, an epidemic model with convalescent carrier was established, without considering the birth rate and natural mortality rate. The stabilities of the disease-free equilibrium and endemic equilibrium were analyzed, and the results of the analysis were verified by numerical solutions.

Keywords: convalescent carrier; disease-free equilibrium; endemic equilibrium; stability; numerical solutions.

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1. Introduction

Pathogen carriers means that there is no clinical symptoms, but people can carry pathogens. Those who carry the bacteria in the body, who carry the virus in the body, who carry the parasites are collectively known as pathogen carriers. Pathogen carriers are often undetected, not isolated, because they are no symptoms and signs. Therefore, they are more important sources of infection. The so-called convalescent carrier refers to those who can still discharge pathogens within a certain period of time after the clinical symptoms disappeared, such as typhoid, cholera,

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diphtheria, hepatitis B and other diseases existing such carry status. Therefore, considering the convalescent carrier has more practical significance in the process of construction mathematical models of such diseases.

The article [1] established a kind of dynamic model of infectious diseases with latent period; a SEIQR model with latent period and quarantine was established in the article[2]; a SEIR epidemic model with immunization and latency was established in the article [3]; the article [4] established a hepatitis B epidemic model with hepatitis B virus carriers. On the basis of these articles, a new and different epidemic model was established. The stability of the disease-free equilibrium and endemic equilibrium were analyzed, and the results of the analysis were verified by numerical solutions.

2. Mathematical Model

We divide the host population into five epidemiological groups: susceptible S , latent Q (due to the late incubation period was infectious, therefore its infectious is not considered), infectious I , convalescent carriers C , recovered R (including cured and vaccinated, don't consider the loss rate of vaccination immunity). This model does not consider the birth rate and natural mortality rate. A represents immigration; β_1 denotes the rate at which the susceptible individuals become infected by one infectious person in unit time; β_2 is the proportion that the susceptible population is infected by one convalescent carrier per unit time; the vaccination rate is r ; ε denotes the incidence of the latent individuals in unit time; k_1 represents the rate at which the infections are cured per unit time; α is the rate moving from infectious to convalescent carrier; k_2 is the rate at which the convalescent carriers are cured in unit time; the disease-related death rate is d . We assume that all parameters are non-negative. According to the rule of constructing compartment epidemic model, a system with five ordinary differential equation is established:

$$\begin{cases} S'(t) = A - (\beta_1 S(t)I(t) + \beta_2 S(t)C(t)) - rS(t) \\ Q'(t) = \beta_1 S(t)I(t) + \beta_2 S(t)C(t) - \varepsilon Q(t) \\ I'(t) = \varepsilon Q(t) - \alpha I(t) - k_1 I(t) \\ C'(t) = \alpha I(t) - k_2 C(t) - dC(t) \\ R'(t) = rS(t) + k_1 I(t) + k_2 I(t) \end{cases} \quad (2.1)$$

3. Steady states analysis

Because the first four equations in the system (2.1) are irrelevant to the $R(t)$, we can discuss the following system of the four equations:

$$\begin{cases} S'(t) = A - (\beta_1 S(t)I(t) + \beta_2 S(t)C(t)) - rS(t) \\ Q'(t) = \beta_1 S(t)I(t) + \beta_2 S(t)C(t) - \varepsilon Q(t) \\ I'(t) = \varepsilon Q(t) - \alpha I(t) - k_1 I(t) \\ C'(t) = \alpha I(t) - k_2 C(t) - dC(t) \end{cases} \quad (3.1)$$

The system(3.1) has two equilibriums: disease-free equilibrium: $E_0(S^0, 0, 0, 0)$, where $S^0 = \frac{A}{r}$; and endemic equilibrium: $E_1(S^*, Q^*, I^*, C^*)$, where $S^* = \frac{(\alpha+k_1)(k_2+d)}{\beta_1(k_2+d)+\alpha\beta_2}$, $Q^* = \frac{\alpha+k_1}{\varepsilon}I^*$, $I^* = \frac{A-rS^*}{\alpha+k_1}$, $C^* = \frac{\alpha}{k_2+d}I^*$. The basic reproduction number of the system is $R_0 = \frac{S^0(\beta_1(k_2+d)+\alpha\beta_2)}{(\alpha+k_1)(k_2+d)}$.

Lemma 3.1. If $R_0 < 1$, E_0 is globally asymptotically stable; if $R_0 > 1$, E_1 is locally asymptotically stable.

We use the method in [4] to prove the theorem.

Proof.The Jacobian matrix at E_0 is

$$J(E_0) = \begin{bmatrix} -r & 0 & -\beta_1 S^0 & -\beta_2 S^0 \\ 0 & -\varepsilon & \beta_1 S^0 & \beta_2 S^0 \\ 0 & \varepsilon & -\alpha - k_1 & 0 \\ 0 & 0 & \alpha & -k_2 - d \end{bmatrix}.$$

The characteristic equation is

$$(\lambda + r)(\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0) = 0,$$

where

$$a_2 = \alpha + k_1 + \varepsilon + k_2 + d,$$

$$a_1 = \varepsilon(\alpha + k_1) + (\alpha + k_1 + \varepsilon)(k_2 + d) - \varepsilon\beta_1S^0,$$

$$a_0 = \varepsilon(\alpha + k_1)(k_2 + d) - \varepsilon\beta_1S^0(k_2 + d) - \varepsilon\alpha\beta_2S^0.$$

When $R_0 < 1$, we have $\varepsilon S^0[\beta_1(k_2 + d) + \alpha\beta_2] < \varepsilon(\alpha + k_1)(k_2 + d)$. So

$$a_0 = \varepsilon(\alpha + k_1)(k_2 + d) - \varepsilon\beta_1S^0(k_2 + d) - \varepsilon\alpha\beta_2S^0 > 0,$$

$$a_2 = \alpha + k_1 + \varepsilon + k_2 + d > 0,$$

$$a_1a_2 - a_0 = [\varepsilon(\alpha + k_1) + (\alpha + k_1 + \varepsilon)(k_2 + d)$$

$$- \varepsilon\beta_1S^0](\alpha + k_1 + \varepsilon + k_2 + d) - [\varepsilon(\alpha + k_1)(k_2 + d) - \varepsilon\beta_1S^0(k_2 + d) - \varepsilon\alpha\beta_2S^0]$$

$$= \varepsilon(\alpha + k_1)(\alpha + k_1 + \varepsilon + k_2 + d) + (\alpha + k_1 + \varepsilon)(k_2 + d)(\alpha + k_1 + \varepsilon + k_2 + d)$$

$$- \varepsilon\beta_1S^0(\alpha + k_1 + \varepsilon + k_2 + d) - \varepsilon(\alpha + k_1)(k_2 + d) + \varepsilon\beta_1S^0(k_2 + d) + \varepsilon\alpha\beta_2S^0$$

$$= \varepsilon(\alpha + k_1)(\alpha + k_1 + \varepsilon) - \varepsilon\beta_1S^0(\alpha + k_1 + \varepsilon) + (\alpha + k_1 + \varepsilon)(k_2 + d)(\alpha + k_1 + \varepsilon + k_2 + d)$$

$$+ \varepsilon\alpha\beta_2S^0$$

$$> 0.$$

Therefore, by Routh-Herwitz[5,6] criteria, all roots of the characteristic equation have negative real parts, thus E_0 is locally asymptotically stable.

We construct the Lyapunov function[7] $L = Q + I + C$ to prove the globally asymptotically stable of E_0 . Then

$$L' = \beta_1SI + \beta_2SC - \varepsilon Q + \varepsilon Q - \alpha I - k_1I + \alpha I - k_2C - dC$$

$$= \beta_1SI + \beta_2SC - k_1I - k_2C - dC$$

$$= \beta_1SI + \beta_2S\frac{\alpha}{k_2+d}I - k_1I - \alpha I$$

$$= [\beta_1S + \frac{\alpha\beta_2S}{k_2+d} - (\alpha + k_1)]I$$

$$\leq [\frac{S^0\beta_1(k_2+d) + \alpha\beta_2S^0}{k_2+d} - (\alpha + k_1)]I$$

$$\leq 0,$$

if and only if $I = 0$, $L' = 0$. Set $M = \{(S, Q, I, C) | L' = 0\}$, when $t \rightarrow +\infty$, we have $M \rightarrow \{E_0\}$,

therefore $\{E_0\}$ is the only largest invariant subset of M . By LaSalle's invariance principle[6],

E_0 is globally asymptotically stable.

The Jacobian matrix at E_1 is

$$J(E_1) = \begin{bmatrix} -\beta_1 I^* - \beta_2 C^* - r & 0 & -\beta_1 S^* & -\beta_2 S^* \\ \beta_1 I^* + \beta_2 C^* & -\varepsilon & \beta_1 S^* & \beta_2 S^* \\ 0 & \varepsilon & -\alpha - k_1 & 0 \\ 0 & 0 & \alpha & -k_2 - d \end{bmatrix}.$$

We make an elementary row-transformation for the $J(E_1)$ and obtain the following matrix:

$$J = \begin{bmatrix} -\beta_1 I^* - \beta_2 C^* - r & 0 & -\beta_1 S^* & -\beta_2 S^* \\ 0 & -\varepsilon & \frac{\beta_1 S^* r}{\beta_1 I^* + \beta_2 C^* + r} & \frac{\beta_2 S^* r}{\beta_1 I^* + \beta_2 C^* + r} \\ 0 & 0 & M_1 & \frac{\beta_2 S^* r}{\beta_1 I^* + \beta_2 C^* + r} \\ 0 & 0 & 0 & M_2 \end{bmatrix},$$

where

$$M_1 = -\alpha - k_1 + \frac{\beta_1 S^* r}{\beta_1 I^* + \beta_2 C^* + r},$$

$$M_2 = -k_2 - d - \frac{\alpha \beta_2 S^* r}{(-\alpha - k_1)(\beta_1 I^* + \beta_2 C^* + r) + \beta_1 S^* r}.$$

When $R_0 > 1$, the eigenvalues of J are

$$\lambda_1 = -\beta_1 I^* - \beta_2 C^* - r < 0,$$

$$\lambda_2 = -\varepsilon < 0,$$

$$\lambda_3 = M_1$$

$$\begin{aligned} &= -\alpha - k_1 + \frac{\beta_1 S^* r}{\beta_1 I^* + \beta_2 C^* + r} \\ &= -\alpha - k_1 + \frac{r\beta_1(\alpha + k_1)(k_2 + d)}{(\beta_1 I^* + \beta_2 C^* + r)[\beta_1(k_2 + d) + \alpha\beta_2]} \\ &= (\alpha + k_1) \frac{r\beta_1(k_2 + d) - (\beta_1 I^* + \beta_2 C^* + r)[\beta_1(k_2 + d) + \alpha\beta_2]}{(\beta_1 I^* + \beta_2 C^* + r)[\beta_1(k_2 + d) + \alpha\beta_2]} \\ &< 0, \end{aligned}$$

$$\lambda_4 = M_2$$

$$\begin{aligned} &= -k_2 - d - \frac{\alpha\beta_2 S^* r}{(-\alpha - k_1)(\beta_1 I^* + \beta_2 C^* + r) + \beta_1 S^* r} \\ &= -k_2 - d + \frac{\alpha\beta_2 S^* r}{(\alpha + k_1)(\beta_1 I^* + \beta_2 C^* + r) - \beta_1 S^* r} \\ &= \frac{\alpha\beta_2 S^* r - (k_2 + d)(k_1 + \alpha)(\beta_1 I^* + \beta_2 C^* + r) + (k_2 + d)r\beta_1 S^*}{(\alpha + k_1)(\beta_1 I^* + \beta_2 C^* + r) - \beta_1 S^* r} \\ &= \frac{rS^*[\beta_1(k_2 + d) + \alpha\beta_2] - (k_2 + d)(k_1 + \alpha)(\beta_1 I^* + \beta_2 C^* + r)}{(\alpha + k_1)(\beta_1 I^* + \beta_2 C^* + r) - \beta_1 S^* r} \\ &= \frac{r(k_2 + d)(k_1 + \alpha) - (k_2 + d)(k_1 + \alpha)(\beta_1 I^* + \beta_2 C^* + r)}{(\alpha + k_1)(\beta_1 I^* + \beta_2 C^* + r) - \beta_1 S^* r} \\ &< 0. \end{aligned}$$

Therefore, the eigenvalues of $J(E_1)$ have negative real parts, thus E_1 is locally asymptotically stable.

Next, we use the numerical solutions to verify the Lemma 3.1. In order to meet the two different conditions of Lemma 3.1, we set the value of each parameter in system(3.1) as follows:

(a) $R_0 < 1$: $A = 80$, $\beta_1 = 0.001$, $\beta_2 = 0.007$, $r = 0.8$, $\varepsilon = 0.75$, $\alpha = 0.4$, $k_1 = 0.6$, $k_2 = 0.3$, $d = 0.03$; (b) $R_0 > 1$: $A = 80$, $\beta_1 = 0.01$, $\beta_2 = 0.03$, $r = 0.8$, $\varepsilon = 0.75$, $\alpha = 0.4$, $k_1 = 0.6$, $k_2 = 0.3$, $d = 0.03$.

Set the initial value of system(3.1) as: $S(0) = 40$, $Q(0) = 30$, $I(0) = 20$, $C(0) = 10$.

Figure. 1 is the numerical solutions of system(3.1):

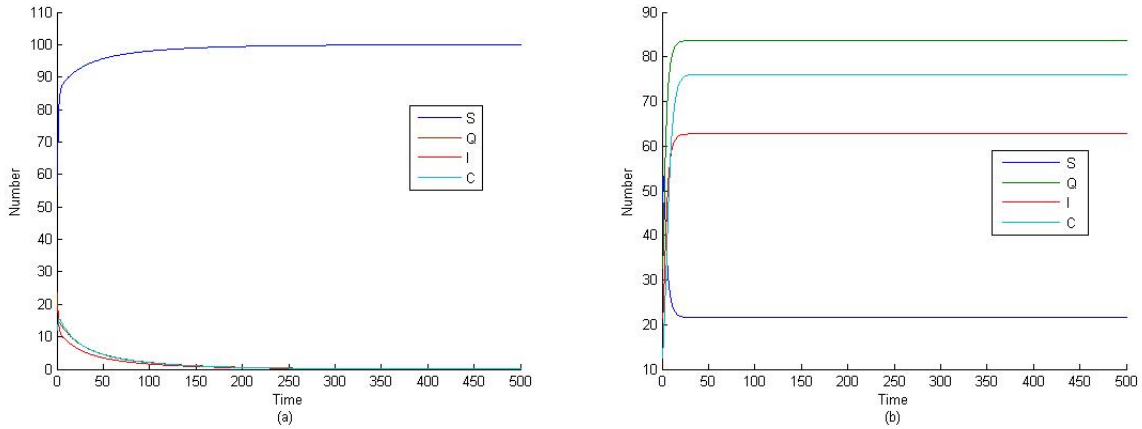


Figure. 1. Numerical solutions of system(3.1)

Figure. 1(a) shows that when $t \rightarrow \infty$, we have $S \rightarrow \frac{A}{r} = 100$, $Q \rightarrow 0$, $I \rightarrow 0$, $C \rightarrow 0$, disease eliminates; in figure. 1(b), when $t \rightarrow \infty$, we have $(S, Q, I, C) \rightarrow (S^*, Q^*, I^*, C^*)$, disease spreads and becomes endemic, and persists. In summary, the disease-eliminating or not is decided by the basic reproduction number R_0 . When $R_0 < 1$, with the time increase, the disease eventually eliminates; when $R_0 > 1$, with the increase of time, the disease eventually becomes stable.

4. Discussion

In this paper, we established an epidemic model with convalescent carrier, analyzed the stability of the disease-free equilibrium and endemic equilibrium, obtained the threshold of the

disease-eliminating: the basic reproduction number R_0 . When $R_0 < 1$, the disease-free equilibrium is globally asymptotically stable, and as the time increase, the disease eventually eliminates; when $R_0 > 1$, the endemic equilibrium is locally asymptotically stable, and with the increase of time, disease eventually becomes stable. The smaller R_0 , the more conducive to control and eliminate the disease. In practical problems, we can adjust the size of parameters in R_0 , for example, by isolating to reduce β_1 and β_2 , thereby reducing the R_0 , to control and eliminate the disease.

Conflict of Interests

The authors declare that there is no conflict of interests.

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