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POSITIVE PERIODIC SOLUTION OF A DISCRETE OBLIGATE LOTKA-VOLTERRA MODEL

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Abstract. In this paper, sufficient conditions are obtained for the existence of positive periodic solution of the following discrete obligate Lotka-Volterra model

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\big\{-a_1(k)-b_1(k)x_1(k)+c_1(k)x_2(k)\big\},\\ x_2(k+1) &= x_2(k)\exp\big\{a_2(k)-b_2(k)x_2(k)\big\}, \end{aligned}$$

where $\{a_i(k)\}, \{b_i(k)\}, i=1,2$ and $\{c_1(k)\}$ are all positive ω-periodic sequences, ω is a fixed positive integer.

Keywords: Obligate Lotka-Volterra model; Positive periodic solution.

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1. Introduction

In [1], Sun and Wei proposed the following intraspecific commensal model:

$$\begin{aligned} \frac{dx}{dt} &= r_1 x\Big(\frac{k_1-x+ay}{k_1}\Big),\\ \frac{dy}{dt} &= r_2 y\Big(\frac{k_2-y}{k_2}\Big), \end{aligned} \tag{1.1}$$

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where $r_i, k_i, i = 1,2$ and a are all positive constants. In this model, the second species is favorable to the first one while the first species has no influence on the second one. The authors investigated the local stability of all equilibrium points. They showed that there is only one local stable equilibrium point in the system.

Stimulated by the works of Sun and Wei, Zhu, Li and Xu [2] proposed the following obligate Lotka-Volterra model:

$$\begin{aligned} \frac{dx}{dt} &= x\Big(a_1 + b_1 x + c_1 y\Big), \\ \frac{dy}{dt} &= y\Big(a_2 + c_2 y\Big), \end{aligned} \tag{1.2}$$

where $x \geq 0, y \geq 0, a_1 < 0, a_2 > 0, b_1 < 0, c_2 < 0, c_1 > 0$ are all positive constants. The authors gave a detail qualitative analysis on the model, they also presented the thresholds of persistency and extinction for above system in a polluted environment.

As was pointed out by Fan and Wang [4], the discrete time models governed by difference equations are more appropriate than the continuous ones when the populations have non-overlapping generations. Recently, Xie, Miao and Xue[3] proposed the following discrete commensal symbiosis model

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\big\{a_1(k) - b_1(k)x_1(k) + c_1(k)x_2(k)\big\}, \\ x_2(k+1) &= x_2(k)\exp\big\{a_2(k) - b_2(k)x_2(k)\big\}, \end{aligned} \tag{1.3}$$

where $\{b_i(k)\}, i = 1,2, \{c_1(k)\}$ are all positive ω-periodic sequences, ω is a fixed positive integer, $\{a_i(k)\}$ are ω-periodic sequences, which satisfies $\overline{a}_i = \frac{1}{\omega}\sum\limits_{k=0}^{\omega-1} a_i(k) > 0, i = 1,2$. By using the coincidence degree theory, they showed that the system (1.3) admits at least one positive ω-periodic solutions.

The success of Xie, Miao and Xue [3] motivated us to propose a discrete analogue of system (1.2), i.e., the following discrete obligate Lotka-Volterra model

$$\begin{aligned} x_1(k+1) &= x_1(k)\exp\big\{-a_1(k) - b_1(k)x_1(k) + c_1(k)x_2(k)\big\}, \\ x_2(k+1) &= x_2(k)\exp\big\{a_2(k) - b_2(k)x_2(k)\big\}, \end{aligned} \tag{1.4}$$

where $\{a_i(k)\}, \{b_i(k)\}, i = 1,2$ and $\{c_1(k)\}$ are all positive ω-periodic sequences, ω is a fixed positive integer. Here we also focus our attention to the study of the positive periodic solution of the system (1.4). For more works on mutualism system, one could refer to [1]-[16] and the references cited therein.

2. Main results

In order to obtain the existence of positive periodic solutions of (1.4), for the reader's convenience, we shall summarize in the following a few concepts and results from [5] that will be basic for this paper.

Let X, Z be normed vector spaces, $L : DomL \subset X \to Z$ be a linear mapping, $N : X \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $dimKerL = CodimImL < +\infty$ and ImL is closed in Z. If L is a Fredholm mapping of index zero there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $ImP = KerL, ImL = KerQ = Im(I-Q)$. It follows that $L|DomL \cap KerP : (I-P)X \to ImL$ is invertible. We denote the inverse of that map by K_P. If Ω be an open bounded subset of X, the mapping N will be called L-compact on $\bar{\Omega}$ if $QN(\bar{\Omega})$ is bounded and $K_P(I-Q)N : \bar{\Omega} \to X$ is compact. Since ImQ is isomorphic to $KerL$, there exists an isomorphisms $J : ImQ \to KerL$.

In the proof of our existence theorem below, we will use the continuation theorem of Gaines and Mawhin ([5, p40]).

Lemma 2.1 (Continuation Theorem) *Let L be a Fredholm mapping of index zero and let N be L-compact on $\bar{\Omega}$. Suppose*

(a). For each $\lambda \in (0,1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$;

(b). $QNx \neq 0$ for each $x \in \partial\Omega \cap KerL$ and

$$deg\{JQN, \Omega \cap KerL, 0\} \neq 0.$$

Then the equation $Lx = Nx$ has at least one solution lying in $DomL \cap \bar{\Omega}$.

Let Z, Z^+, R and R^+ denote the sets of all integers, nonnegative integers, real unumbers, and nonnegative real numbers, respectively. For convenience, in the following discussion, we will use the notation below throughout this paper:

$$I_\omega = \{0, 1, ..., \omega - 1\},\ \overline{g} = \frac{1}{\omega}\sum_{k=0}^{\omega-1} g(k),\ \ g^u = \max_{k\in I_\omega} g(k),\ \ g^l = \min_{k\in I_\omega} g(k),$$

where $\{g(k)\}$ is an ω-periodic sequence of real numbers defined for $k \in Z$.

Lemma 2.2 [4] *Let $g : Z \to R$ be ω-periodic, i. e., $g(k+\omega) = g(k)$. Then for any fixed $k_1, k_2 \in I_\omega$, and any $k \in Z$, one has*

$$g(k) \le g(k_1) + \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|,$$

$$g(k) \ge g(k_2) - \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|.$$

We now reach the position to establish our main result.

Theorem 2.1 *Assume that $\bar{c}_1\bar{a}_2 > \bar{a}_1\bar{b}_2 \exp\{(\bar{A}_2 + \bar{a}_2)\omega\}$ holds. Then system (1.4) admits at least one positive ω-periodic solution.*

Proof. Let

$$x_i(k) = \exp\{u_i(k)\}, \;\; i = 1, 2,$$

so that system (1.4) becomes

$$\begin{aligned} u_1(k+1) - u_1(k) = &\; -a_1(k) - b_1(k)\exp\{u_1(k)\} + c_1(k)\exp\{u_2(k)\}, \\ u_2(k+1) - u_2(k) = &\; a_2(k) - b_2(k)\exp\{u_2(k)\}. \end{aligned} \tag{2.1}$$

Define

$$l_2 = \left\{ y = \{y(k)\}, y(k) = (y_1(k), y_2(k))^T \in R^2 \right\}.$$

For $a = (a_1, a_2)^T \in R^2$, define $|a| = \max\{|a_1|, |a_2|\}$. Let $l^\omega \subset l_2$ denote the subspace of all ω sequences equipped with the usual normal form $\|y\| = \max_{k \in I_\omega} |y(k)|$. It is not difficult to show that l^ω is a finite-dimensional Banach space. Let

$$l_0^\omega = \{y = \{y(k)\} \in l^\omega : \sum_{k=0}^{\omega-1} y(k) = 0\},\; l_c^\omega = \{y = \{y(k)\} \in l^\omega : y(k) = h \in R^2, k \in Z\},$$

then l_0^ω and l_c^ω are both closed linear subspace of l^ω, and

$$l^\omega = l_0^\omega \oplus l_c^\omega, \;\; dim l_c^\omega = 2.$$

Now let us define $X = Y = l^\omega$, $(Ly)(k) = y(k+1) - y(k)$, and

$$N(u_1, u_2)^T = (N_1, N_2)^T := N(u, k),$$

where

$$\begin{cases} N_1 &= -a_1(k) - b_1(k)\exp\{u_1(k)\} + c_1(k)\exp\{u_2(k)\}, \\ N_2 &= a_2(k) - b_2(k)\exp\{u_2(k)\}. \end{cases}$$

$$Px = \frac{1}{\omega}\sum_{s=0}^{\omega-1} x(s), x \in X, \ \ Qy = \frac{1}{\omega}\sum_{s=0}^{\omega-1} y(s), y \in Y.$$

Then similarly to the analysis of [3], one could see that L is a Fredholm mapping of index zero. N is L-compact on any open bounded set $\Omega \subset X$.

Now we are at the point to search for an appropriate open, bounded subset Ω in X for the application of the continuation theorem. Corresponding to the operator equation $Lx = \lambda Nx, \lambda \in (0,1)$, we have

$$\begin{aligned} u_1(k+1) - u_1(k) = &\ \lambda[-a_1(k) - b_1(k)\exp\{u_1(k)\} + c_1(k)\exp\{u_2(k)\}], \\ u_2(k+1) - u_2(k) = &\ \lambda[a_2(k) - b_2(k)\exp\{u_2(k)\}]. \end{aligned} \tag{2.2}$$

Suppose that $y = (y_1(k), y_2(k))^T \in X$ is an arbitrary solution of system (2.2) for a certain $\lambda \in (0,1)$. Summing on both sides of (2.2) from 0 to $\omega - 1$ with respect to k, we reach

$$\begin{aligned} &\sum_{k=0}^{\omega-1}[-a_1(k) - b_1(k)\exp\{u_1(k)\} + c_1(k)\exp\{u_2(k)\}] = 0, \\ &\sum_{k=0}^{\omega-1}[a_2(k) - b_2(k)\exp\{u_2(k)\}] = 0. \end{aligned}$$

That is,

$$\sum_{k=0}^{\omega-1} b_1(k)\exp\{u_1(k)\} + \bar{a}_1\omega = \sum_{k=0}^{\omega-1} c_1(k)\exp\{u_2(k)\}, \tag{2.3}$$

$$\sum_{k=0}^{\omega-1} b_2(k)\exp\{u_2(k)\} = \bar{a}_2\omega. \tag{2.4}$$

From (2.3) and (2.4), we have

$$\begin{aligned} & \sum_{k=0}^{\omega-1}|u_1(k+1) - u_1(k)| \\ = \ & \lambda\sum_{k=0}^{\omega-1}|-a_1(k) - b_1(k)\exp\{u_1(k)\} + c_1(k)\exp\{u_2(k)\}| \\ \le \ & \sum_{k=0}^{\omega-1}|a_1(k)| - \sum_{k=0}^{\omega-1}a_1(k) \\ & + \sum_{k=0}^{\omega-1}\Big(a_1(k) + b_1(k)\exp\{u_1(k)\} + c_1(k)\exp\{u_2(k)\}\Big) \\ = \ & \sum_{k=0}^{\omega-1}|a_1(k)| - \bar{a}_1\omega + 2\sum_{k=0}^{\omega-1}c_1(k)\exp\{u_2(k)\} \end{aligned}$$

$$\begin{aligned}
&= (\bar{A}_1-\bar{a}_1)\omega+2\sum_{k=0}^{\omega-1}c_1(k)\exp\{u_2(k)\},\\
&\sum_{k=0}^{\omega-1}|u_2(k+1)-u_2(k)|\\
&= \lambda\sum_{k=0}^{\omega-1}|a_2(k)-b_2(k)\exp\{u_2(k)\}|\\
&\le (\bar{A}_2+\bar{a}_2)\omega,
\end{aligned}\tag{2.5}$$

where $\bar{A}_1=\frac{1}{\omega}\sum_{k=0}^{\omega-1}|a_1(k)|,\ \bar{A}_2=\frac{1}{\omega}\sum_{k=0}^{\omega-1}|a_2(k)|$. Since $\{u(k)\}=\{(u_1(k),u_2(k))^T\}\in X$, there exist $\eta_i,\delta_i, i=1,2$ such that

$$u_i(\eta_i)=\min_{k\in I_\omega}u_i(k),\ u_i(\delta_i)=\max_{k\in I_\omega}u_i(k).$$

With the same analysis as that of (2.8)-(2.11) in [3], one could see that

$$\ln\frac{\bar{a}_2}{\bar{b}_2}-(\bar{A}_2+\bar{a}_2)\omega\le u_2(k)\le\ln\frac{\bar{a}_2}{\bar{b}_2}+(\bar{A}_2+\bar{a}_2)\omega,\tag{2.6}$$

and so,

$$|u_2(k)|\le\max\left\{|\ln\frac{\bar{a}_2}{\bar{b}_2}+(\bar{A}_2+\bar{a}_2)\omega|,|\ln\frac{\bar{a}_2}{\bar{b}_2}-(\bar{A}_2+\bar{a}_2)\omega|\right\}\stackrel{\text{def}}{=}H_2.\tag{2.7}$$

It follows from (2.3) and (2.6) that

$$\sum_{k=0}^{\omega-1}b_1(k)\exp\{u_1(\eta_1)\}\quad\le\quad-\bar{a}_1\omega+\sum_{k=0}^{\omega-1}c_1(k)\exp\{(\bar{A}_2+\bar{a}_2)\omega+\ln\tfrac{\bar{a}_2}{\bar{b}_2}\}.\tag{2.8}$$

Noting that from the conditions of Theorem one could easily see that

$$\frac{\bar{c}_1\bar{a}_2}{\bar{b}_2}\exp\{(\bar{A}_2+\bar{a}_2)\omega\}>\frac{\bar{c}_1\bar{a}_2}{\bar{b}_2}>\bar{a}_1,$$

and so, it follows from (2.8) that

$$u_1(\eta_1)\le\ln\frac{\Delta_1}{\bar{b}_1},\tag{2.9}$$

where

$$\Delta_1=-\bar{a}_1+\frac{\bar{c}_1\bar{a}_2}{\bar{b}_2}\exp\{(\bar{A}_2+\bar{a}_2)\omega\}.$$

It follows from Lemma 2.2, (2.5) and (2.9) that

$$\begin{aligned}
u_1(k)&\le u_1(\eta_1)+\sum_{k=0}^{\omega-1}|u_1(k+1)-u_1(k)|\\
&\le (\bar{A}_1-\bar{a}_1)\omega+\ln\tfrac{\Delta_1}{\bar{b}_1}+2\frac{\bar{c}_1\bar{a}_2\omega}{\bar{b}_2}\exp\{(\bar{A}_2+\bar{a}_2)\omega\}\stackrel{\text{def}}{=}M_1.
\end{aligned}\tag{2.10}$$

It follows from (2.3), (2.6) and the condition of the theorem that

$$\sum_{k=0}^{\omega-1} b_1(k)\exp\{u_1(\delta_1)\} \geq -\bar{a}_1\omega + \sum_{k=0}^{\omega-1} c_1(k)\exp\{\ln\tfrac{\bar{a}_2}{\bar{b}_2} - (\bar{A}_2+\bar{a}_2)\omega\} > 0,$$

and

$$u_1(\delta_1) \geq \ln\frac{\Delta_2}{\bar{b}_1}, \tag{2.11}$$

where

$$\Delta_2 = -\bar{a}_1 + \frac{\bar{c}_1\bar{a}_2}{\bar{b}_2}\exp\{-(\bar{A}_2+\bar{a}_2)\omega\}.$$

It follows from Lemma 2.2, (2.5) and (2.11) that

$$\begin{aligned} u_1(k) &\geq u_1(\delta_1) - \sum_{k=0}^{\omega-1}|u_1(k+1)-u_1(k)| \\ &\geq \ln\tfrac{\Delta_2}{\bar{b}_1} - (\bar{A}_1+\bar{a}_1)\omega - 2\frac{\bar{c}_1\bar{a}_2\omega}{\bar{b}_2}\exp\{-(\bar{A}_2+\bar{a}_2)\omega\} \overset{\text{def}}{=} M_2. \end{aligned} \tag{2.12}$$

Combine with (2.10) and (2.12) leads to

$$|u_1(k)| \leq \max\Big\{|M_1|, |M_2|\Big\} \overset{\text{def}}{=} H_1. \tag{2.13}$$

Clearly, H_1 and H_2 are independent on the choice of λ. From the condition of the Theorem, one could easily see that $\bar{c}_1\bar{a}_2 > \bar{a}_1\bar{b}_2$, and so, the system of algebraic equations

$$-\bar{a}_1 - \bar{b}_1x_1 + \bar{c}_1x_2 = 0,\ \ \bar{a}_2 - \bar{b}_2x_2 = 0 \tag{2.14}$$

admits a unique positive solution $(x_1^*, x_2^*) \in R_2^+$, where

$$x_1^* = \frac{-\bar{a}_1 + \bar{c}_1x_2^*}{\bar{b}_1},\ x_2^* = \frac{\bar{a}_2}{\bar{b}_2}.$$

Let $H = H_1+H_2+H_3$, where $H_3 > 0$ is taken sufficiently enough large such that $||(\ln\{x_1^*\},\ \ln\{x_2^*\})^T|| = |\ln\{x_1^*\}| + |\ln\{x_2^*\}| < H_3$.

Let $H = H_1+H_2+H_3$, and define

$$\Omega = \left\{u(t) = (u_1(k), u_2(k))^T \in X : \|u\| < H\right\}.$$

It is clear that Ω verifies requirement (a) in Lemma 2.1. When $u \in \partial\Omega \cap KerL = \partial\Omega \cap R^2$, u is constant vector in R^2 with $||u|| = B$. Then

$$QNu = \begin{pmatrix} -\bar{a}_1 - \bar{b}_1\exp\{u_1\} + \bar{c}_1\exp\{u_2\} \\ \bar{a}_2 - \bar{b}_2\exp\{u_2\} \end{pmatrix} \neq 0.$$

Moreover, direct calculation shows that

$$deg\{JQN, \Omega \cap KerL, 0\} = \operatorname{sgn}\Big(\bar{b}_1 \bar{b}_2 \exp\{x_1^*\} \exp\{x_2^*\}\Big) = 1 \neq 0.$$

where $deg(.)$ is the Brouwer degree and the J is the identity mapping since $ImQ = KerL$.

By now we have proved that Ω verifies all the requirements in Lemma 2.1. Hence (2.1) has at least one solution $(u_1^*(k), u_2^*(k))^T$, in $DomL \cap \bar{\Omega}$. And so, system (1.4) admits a positive periodic solution $(x_1^*(k), x_2^*(k))^T$, where $x_i^*(k) = \exp\{u_i^*(k)\}, i = 1, 2$, This completes the proof of the claim.

Remark 2.1 The condition of the Theorem 2.1 could be rewritten as

$$\bar{c}_1 > \frac{\bar{a}_1 \bar{b}_2}{\bar{a}_2} \exp\{(\bar{A}_2 + \bar{a}_2)\omega\}.$$

Noting that $c_1(k)$ reflects the contribution rate of the second species to the first species, hence our theorem implies that if the contribution rate is enough large, then, despite the negative growth rate of the first species, two species could be coexistent in a periodic fluctuation form.

Conflict of Interests

The authors declare that there is no conflict of interests.

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