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GLOBAL STABILITY OF A COMMENSAL SYMBIOSIS MODEL WITH FEEDBACK CONTROLS

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Abstract. In this paper, a commensal symbiosis model with feedback controls is investigated. By constructing a suitable Lyapunov function, we show that the positive equilibrium of the system is globally stable, which means that feedback control variables only change the position of the positive equilibrium and retain its global stability property.

Keywords: Commensal symbiosis model; Feedback controls; Global stability.

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1. Introduction

Mutualism model could be clarified as: commensalism, facultative and obligate. The Lotka-Volterra mutualism model has been studied extensively. Permanence and global attractivity are important concepts to describe the dynamic behaviors of the system. In the past few years, many excellent results concerned with the permanence, existence of positive periodic solution, and global stability of the mutualism system were obtained, see [1-10] and the references cited

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therein. For example, Chen *et al.* [4] focused on the permanence of the discrete mutualism model with time delays. Chen [5] analyzed the permanence for the discrete mutualism with time delays. In [7-8], the authors proposed and studied a delayed two-species model of facultative mutualism, they investigated the existence and globally asymptotic stability of positive periodic of the system. Chen *et al.* [9] studied an obligate Lotka-Volterra mutualism model, by constructing a suitable Lyapunov function, sufficient conditions could ensure the global asymptotical of the nonnegative equilibria was obtained. While there are few works on commensalism model.

In [1], the authors proposed and studied the following two species commensalism mutualism model:

$$\begin{aligned} \frac{dx}{dt} &= r_1x\Big(\frac{k_1-x+\alpha y}{k_1}\Big), \\ \frac{dy}{dt} &= r_2y\Big(\frac{k_2-y}{k_2}\Big), \end{aligned} \tag{1.1}$$

where x and y are the densities of population 1 and population 2 at time t, respectively. r_i, $i=1,2$ are the intrinsic growth rate of two populations. k_i, $i=1,2$ are the environmental carrying capacity of the population 1 and population 2, respectively. α reflects the efficiency of every single in population 2 can contribute to population 1. Here, we can see x is harmless to specie y and x will survival without y. System (1.1) admits four equilibria $E_1(0,0), E_2(k_1,0), E_3(0,k_2)$ and $E_4(k_1+\alpha k_2, k_2)$. Concerned with the local stability property of the system (1.1), by analyzing the characteristic root of the characteristic equation, Sun and Wei [1] obtained the following result.

Theorem A. $E_1(0,0), E_2(k_1,0), E_3(0,k_2)$ *are all unstable,* $E_4(k_1+\alpha k_2, k_2)$ *is locally stable.*

It is natural for us to study the global property of the system. On the other hand, Gopalsamy and Weng [11] first time proposed and studied a single specie model with feedback control, since then, many scholars attach great attention to ecological model with feedback controls, see [12-18] and the references cited therein. However, to the best of the author' knowledge, to this day, still no scholars propose and study the commensal symbiosis model with feedback controls. This motivates us to propose and study the following system:

$$
\begin{aligned}
\dot{x} &= x\big(b_1 - a_{11}x + a_{12}y - \alpha_1 u_1\big),\\
\dot{y} &= y\big(b_2 - a_{22}y - \alpha_2 u_2\big),\\
\dot{u}_1 &= -\eta_1 u_1 + a_1 x,\\
\dot{u}_2 &= -\eta_2 u_2 + a_2 y.
\end{aligned} \tag{1.2}
$$

The rest of the paper is organized as follows. We will state and prove the global stability property in the next section. In Section 3, numerical simulations are presented to illustrate our results. We end this work by a brief discussion.

2. Global stability

First we consider the commensalism Lotka-Volterra mutualism model as follow:

$$
\begin{aligned}
\frac{dx}{dt} &= x\big(b_1 - a_{11}x + a_{12}y\big),\\
\frac{dy}{dt} &= y\big(b_2 - a_{22}y\big).
\end{aligned} \tag{2.1}
$$

System (2.1) admits a positive equilibria $P_0(x_0, y_0)$, where

$$
x_0 = \frac{b_1 a_{22} + b_2 a_{12}}{a_{11}a_{22}},\ y_0 = \frac{b_2}{a_{22}}.
$$

Theorem 2.1. *The positive equilibrium $P_0(x_0, y_0)$ of system (2.1) is globally stable.*

Proof. Now we construct a Lyapunov function

$$
V_1 = \eta_1\Big(x - x_0 - x_0 \ln\frac{x}{x_0}\Big) + \eta_2\Big(y - y_0 - y_0 \ln\frac{y}{y_0}\Big), \tag{2.2}
$$

where η_1, η_2 are positive constants be determined later.

Calculating the derivative along the solution of system (2.1), we have

$$\begin{aligned}
\frac{dV_1}{dt} &= \eta_1(x-x_0)[-a_{11}(x-x_0)+a_{12}(y-y_0)] \\
&\quad +\eta_2(y-y_0)[-a_{22}(y-y_0)] \\
&= -a_{11}\eta_1(x-x_0)^2+a_{12}\eta_1(x-x_0)(y-y_0) \\
&\quad -a_{22}\eta_2(y-y_0)^2 \\
&= -(x-x_0,y-y_0)\begin{pmatrix} a_{11}\eta_1 & -\frac{a_{12}\eta_1}{2} \\ -\frac{a_{12}\eta_1}{2} & a_{22}\eta_2 \end{pmatrix}\begin{pmatrix} x-x_0 \\ y-y_0 \end{pmatrix}.
\end{aligned} \tag{2.3}$$

Now we show that one could choose suitable constants η_i, $i=1,2$ such that the matrix

$$\begin{pmatrix} a_{11}\eta_1 & -\frac{a_{12}\eta_1}{2} \\ -\frac{a_{12}\eta_1}{2} & a_{22}\eta_2 \end{pmatrix}$$

is positive definite. Indeed, one could choose $\eta_1 = \dfrac{a_{11}a_{22}}{a_{12}^2}$, $\eta_2 = 1$, then

$$\begin{aligned}
&\begin{vmatrix} a_{11}\eta_1 & -\frac{a_{12}\eta_1}{2} \\ -\frac{a_{12}\eta_1}{2} & a_{22}\eta_2 \end{vmatrix} = a_{11}a_{22}\eta_1\eta_2 - \frac{a_{12}^2\eta_1^2}{4} \\
&= \frac{a_{11}^2a_{22}^2}{a_{12}^2} - \frac{a_{11}^2a_{22}^2}{4a_{12}^2} \\
&= \frac{3a_{11}^2a_{22}^2}{4a_{12}^2} > 0.
\end{aligned} \tag{2.4}$$

So $\frac{dV_1}{dt} < 0$ strictly for all $x > 0, y > 0$ except the positive equilibrium $P_0(x_0,y_0)$, where $\frac{dV_1}{dt} = 0$. Thus, $V_1(t)$ satisfies Lyapunov's asymptotic stability theorem, and the positive equilibrium $P_0(x_0,y_0)$ of system (2.1) is globally stable. This ends the proof of Theorem 2.1.

Now let's consider the dynamic behaviors of the system (1.2). Obviously, $P(x^*, y^*, u_1^*, u_2^*)$ satisfies the equations

$$\begin{cases} b_1 - a_{11}x^* + a_{12}y^* - \alpha_1 u_1^* = 0, \\ b_2 - a_{22}y^* - \alpha_2 u_2^* = 0, \\ -\eta_1 u_1^* + a_1 x^* = 0, \\ -\eta_2 u_2^* + a_2 y^* = 0. \end{cases}$$

From above equations, one could easily see that system (1.2) admits a unique positive equilibrium $P(x^*, y^*, u_1^*, u_2^*)$, where

$$x^* = \frac{(b_1 a_{22} + b_2 a_{12})\eta_1\eta_2 + b_1 a_2 \alpha_2 \eta_1}{(a_{11}\eta_1 + a_1\alpha_1)(a_{22}\eta_2 + a_2\alpha_2)}, \ y^* = \frac{b_2\eta_2}{a_{22}\eta_2 + a_2\alpha_2},$$
$$u_1^* = \frac{a_1 x^*}{\eta_1}, \ u_2^* = \frac{a_2 y^*}{\eta_2},$$

Concerned with the stability property of the positive equilibrium $P(x^*, y^*, u_1^*, u_2^*)$, we have

Theorem 2.2. *The positive equilibrium $P(x^*, y^*, u_1^*, u_2^*)$ of system (1.2) is globally stable.*

Proof. Now let's construct a Lyapunov function

$$V_2 = \delta_1\left(x - x^* - x^* \ln\frac{x}{x^*}\right) + \delta_2\left(y - y^* - y^* \ln\frac{y}{y^*}\right) + \delta_3(u_1 - u_1^*)^2 + \delta_4(u_2 - u_2^*)^2, \qquad (2.5)$$

where $\delta_1 = \dfrac{a_{11}a_{22}}{a_{12}^2}$, $\delta_2 = 1$, $\delta_3 = \dfrac{\delta_1\alpha_1}{2a_1}$, $\delta_4 = \dfrac{\delta_2\alpha_2}{2a_2}$.

Calculating the derivative along the solution of system (1.2), we have

$$\begin{aligned}
\frac{dV_2}{dt} &= \delta_1(x-x^*)[-a_{11}(x-x^*)+a_{12}(y-y^*)-\alpha_1(u_1-u_1^*)] \\
&\quad +\delta_2(y-y^*)[-a_{22}(y-y^*)-\alpha_2(u_2-u_2^*)] \\
&\quad +2\delta_3(u_1-u_1^*)[-\eta_1(u_1-u_1^*)+a_1(x-x^*)] \\
&\quad +2\delta_4(u_2-u_2^*)[-\eta_2(u_2-u_2^*)+a_2(y-y^*)] \\
&= -a_{11}\delta_1(x-x^*)^2-a_{22}\delta_2(y-y^*)^2-2\delta_3\eta_1(u_1-u_1^*)^2 \\
&\quad -2\delta_4\eta_2(u_2-u_2^*)^2+a_{12}\delta_1(x-x^*)(y-y^*) \\
&\quad +(2\delta_3a_1-\delta_1\alpha_1)(x-x^*)(u_1-u_1^*) \\
&\quad +(2\delta_4a_2-\delta_2\alpha_2)(y-y^*)(u_2-u_2^*) \\
&= -(x-x^*,y-y^*)\begin{pmatrix} a_{11}\delta_1 & -\frac{a_{12}\delta_1}{2} \\ -\frac{a_{12}\delta_1}{2} & a_{22}\delta_2 \end{pmatrix}\begin{pmatrix} x-x^* \\ y-y^* \end{pmatrix} \\
&\quad -2\delta_3\eta_1(u_1-u_1^*)^2-2\delta_4\eta_2(u_2-u_2^*)^2.
\end{aligned} \tag{2.6}$$

Through a simple algebraic computations, we know that the matrix

$$\begin{pmatrix} a_{11}\delta_1 & -\frac{a_{12}\delta_1}{2} \\ -\frac{a_{12}\delta_1}{2} & a_{22}\delta_2 \end{pmatrix}$$

is positive definite. Thus $\frac{dV_2}{dt} < 0$ strictly for all $x > 0, y > 0, u_i > 0, i = 1,2$ except the positive equilibrium $P(x^*,y^*,u_1^*,u_2^*)$, where $\frac{dV_2}{dt} = 0$. Thus, $V_2(t)$ satisfies Lyapunov's asymptotic stability theorem, and the positive equilibrium $P(x^*,y^*,u_1^*,u_2^*)$ of system (1.2) is globally stable. This ends the proof of Theorem 2.2.

Remark 1 After comparison, we found that $x^* < x_0$, $y^* < y_0$, so Theorem 2.2 shows that the feedback controls only change the position of the positive equilibrium and reduce the density of the population.

3. Examples

The following two examples show the feasibility of the main results.

Example 3.1 Consider the following equations

$$\begin{aligned} \dot{x} &= x(1-x+2y), \\ \dot{y} &= y(1-2y). \end{aligned} \tag{3.1}$$

Here, corresponding to system (2.1), we take $b_1 = b_2 = a_{11} = 1$, $a_{12} = a_{22} = 2$. It follows from Theorem 2.1 that the positive equilibrium $P_0(2, 0.5)$ is globally stable.

Numeric simulation (Fig 1) also indicates that P_0 is globally stable.

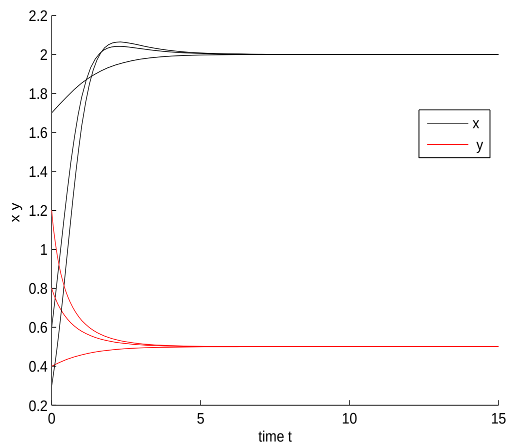

FIGURE 1. Dynamics behaviors of the solution $(x(t), y(t))$ of system (3.1) with the initial conditions $(x(0), y(0))$=$(1.7, 0.4)$, $(0.6, 0.8)$ and $(0.3, 1.2)$, respectively.

Example 3.2 Now let's further incorporate the feedback control variables to the system (3.1) and consider the following system.

$$\begin{aligned} \dot{x} &= x(1-x+2y-0.5u_1), \\ \dot{y} &= y(1-2y-0.25u_2), \\ \dot{u}_1 &= -2u_1 + x, \\ \dot{u}_2 &= -u_2 + 2y. \end{aligned} \tag{3.2}$$

Here, corresponding to system (1.2), we take $b_1 = b_2 = a_{11} = 1$, $a_{12} = a_{22} = 2$, $\alpha_1 = 0.5, \alpha_2 = 0.25$, $\eta_1 = a_2 = 2$, $\eta_2 = a_1 = 1$. From Theorem 2.2 the positive equilibrium $P(1.44, 0.4, 0.72, 0.8)$ is globally stable.

Numeric simulation (Fig 2) also indicates that P is globally stable.

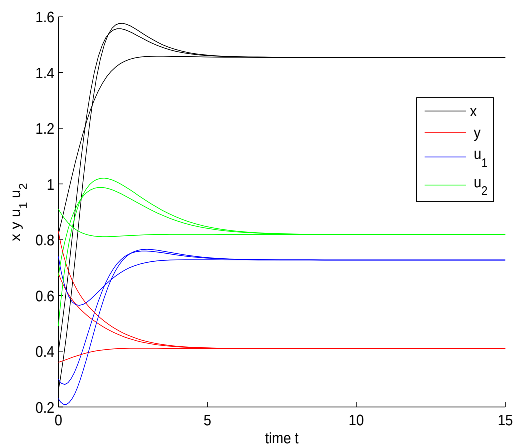

FIGURE 2. Dynamics behaviors of the solution (x, y, u_1, u_2) of system (3.2) with the initial conditions $(x(0), y(0), u_1(0), u_2(0)) =$ $0.4, 0.68, 0.30, 0.67)$, $(0.82, 0.36, 0.74, 0.91)$ *and* $(0.26, 0.83, 0.23, 0.49)$, respectively.

4. Discussion

By constructing some suitable Lyapunov functions, we show that the positive equilibrium of Lotka-Volterra commensalism system is global stability, Theorem 2.1 complement and supplement the main results of [1]. Furthermore, we incorporate feedback control variables to the system (2.1). Our study shows that feedback controls only change the position of the interior equilibrium, reduce the density of the population and retain the global stability of the unique interior equilibrium. This indicates that, in the realistic environment, the epiphytes can keep a balance with their host plants and the feedback controls have no influence on the persistent property of the system. In other word, the epiphytes and their host plants would finally coexist in a stable state.

Remark 2 There are still many interesting and challenging problems that need to research for system (2.1). For example, a more suitable system should consider some of the past state of the species, and this leads to the system with delays, for such kind of system, whether feedback control variables still have no influence on the stability property of the system or not is still unknown, we leave this for future investigation.

Conflict of Interests

The authors declare that there is no conflict of interests.

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