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EXTINCTION OF A DELAY DIFFERENTIAL EQUATION MODEL OF PLANKTON ALLELOPATHY

LIN LI*, RUNXIN WU

Department of Mathematics and Physics, Fujian University of Technology, Fuzhou, Fujian 350108, China

Abstract. In this paper, a delay differential equation model of the growth of two-species competitive plankton with one toxin producing phytoplankton is studied in this paper. Under some suitable assumption, we prove that one of the components will be driven to extinction while the other one will stabilize at a certain solution of a Logistic equation. Our results supplement one of the main results in [Dynamic behaviors of a delay differential equation model of plankton allelopathy, J. Comput. Appl. Math. 206 (2007), 733-754].

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1. Introduction

During the last two decades, competitive system with the effect of toxic substances become one of the important research topic, many excellent results have been obtained, see [1]-[21] and the references cited therein.

*Corresponding author

E-mail address: dalaotu@163.com

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Smith [1] proposed the following two species competitive system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[K_1-\alpha_1 x_1(t)-\beta_{12}x_2(t)-\gamma_1 x_1(t)x_2(t)], \\ \dot{x}_2(t) &= x_2(t)[K_2-\alpha_2 x_2(t)-\beta_{21}x_2(t)-\gamma_2 x_1(t)x_2(t)], \end{aligned} \tag{1.1}$$

where $x_1(t)$ and $x_2(t)$ denote the population density of two competing species at time t for a common pool of resources. The terms $\gamma_1 x_1(t)x_2(t)$ and $\gamma_2 x_1(t)x_2(t)$ denote the effect of toxic substance, here the author made the assumption that each species produces a substance toxic to the other, but only when the other is present. By constructing a suitable Lyapunov function, Chattopadhyay [2] obtained a set of sufficient conditions which guarantee the global attractivity of the positive equilibrium of above system.

Mukhopadhyay *et al.* [3] argued that the production of the toxic substance allelopathic to the competing species will not be instantaneous, but delayed by different discrete time lags required for the maturity of both species, and they modified system (1.2) to the following system:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)[K_1-\alpha_1 x_1(t)-\beta_{12}x_2(t)-\gamma_1 x_1(t)x_2(t-\tau_1)], \\ \dot{x}_2(t) &= x_2(t)[K_2-\alpha_2 x_2(t)-\beta_{21}x_2(t)-\gamma_2 x_1(t-\tau_2)x_2(t)], \end{aligned} \tag{1.3}$$

where $\tau_i>0, i=1,2$ are the time required for the maturity of the first species and second species, respectively. By using an iterative method, Li *et al.* [4] investigated the global stability of the interior equilibrium point of the system, they showed that toxic substances are harmless for the stability of the interior equilibrium point.

Recently, Jin and Ma [24] argued that the environmental fluctuation is important in an ecosystem, and more realistic models require the inclusion of the effect of environmental changing, especially environmental parameters which are time-dependent periodic changing (e.g., seasonal changes, food supplies, etc.). They proposed the following two-species competition model:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\Big[r_1(t)-\sum_{j=1}^{2}a_{1j}(t)\int_{-T_{1j}}^{0}K_{1j}(s)x_j(t+s)ds \\ &\qquad -b_1(t)x_1(t)\int_{-\tau_2}^{0}f_2(s)x_2(t+s)ds\Big], \\ \dot{x}_2(t) &= x_2(t)\Big[r_2(t)-\sum_{j=1}^{2}a_{2j}(t)\int_{-T_{2j}}^{0}K_{1j}(s)x_j(t+s)ds \\ &\qquad -b_2(t)x_2(t)\int_{-\tau_1}^{0}f_1(s)x_1(t+s)ds\Big]. \end{aligned} \tag{1.4}$$

By using the coincidence degree theory, sufficient conditions which guarantee the existence of positive periodic solutions of system (1.4) are obtained.

Stimulated by the works of [5], Chen *et al.* [6] studied the extinction property of the following two dimensional system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\Big[r_1(t)-a_{11}(t)x_1(t)-a_{12}(t)\int_{-T_{12}}^{0}K_{12}(s)x_2(t+s)ds \\ &\quad -b_{12}(t)x_1(t)\int_{-\tau_{12}}^{0}f_{12}(s)x_2(t+s)ds\Big], \\ \dot{x}_2(t) &= x_2(t)\Big[r_2(t)-a_{21}(t)\int_{-T_{21}}^{0}K_{21}(s)x_1(t+s)ds-a_{22}(t)x_2(t) \\ &\quad -b_{21}(t)x_2(t)\int_{-\tau_{21}}^{0}f_{21}(s)x_1(t+s)ds\Big],\end{aligned} \tag{1.5}$$

where $r_i(t), a_{ij}(t), b_{ij}(t) (i \neq j), i, j = 1,2$ are continuous and bounded above and below by positive constants on $[0,+\infty)$; T_{ij}, τ_{ij} are positive constants, $K_{ij} \in C([-T_{ij},0],(0,+\infty))$ and $\int_{-T_{ij}}^{0} K_{ij}(s)ds = 1, f_{ij} \in C([-\tau_{ij},0],(0,+\infty))$ and $\int_{-\tau_{ij}}^{0} f_{ij}(s)ds = 1 (i, j = 1,2, i \neq j)$. the authors of [6] showed that if

$$r_1^l a_{21}^l > a_{11}^u r_2^u,\ r_1^l a_{22}^l \geq r_2^u a_{12}^u \text{ and } r_1^l b_{21}^l \geq r_2^u b_{12}^u, \tag{1.6}$$

holds, the second species will be driven to extinction while the first one will stabilize at a certain solution of a logistic equation.

Recently, Solé *et al.* [14] considered a Lotka-Volterra type of model for two interacting phytoplankton species, where one species could produce toxic, while the other one is non-toxic produce. The model takes the form:

$$\begin{aligned}\frac{dx_1}{dt} &= x_1\Big(b_1-a_{11}x_1-\gamma x_1x_2^2\Big), \\ \frac{dx_2}{dt} &= x_2\Big(b_2-a_{22}x_2\Big).\end{aligned} \tag{1.7}$$

Noting that in system (1.7), the solutions of the system could be expressed in a explicit form, and the dynamic behaviors of the system could be discussed thoroughly. Bandyopadhyay [15] argued that it maybe better to incorporate the inter-species competition, and he proposed the following two species competition model:

$$\begin{aligned} \frac{dx_1}{dt} &= x_1\Big(b_1 - a_{11}x_1 - a_{12}x_2 - \gamma x_1 x_2^2\Big), \\ \frac{dx_2}{dt} &= x_2\Big(b_2 - a_{21}x_1 - a_{22}x_2\Big). \end{aligned} \tag{1.8}$$

His study implies that the toxic substance may change the local stability property of the positive equilibrium. Lin *et al.* [21] further studied the non-autonomous case of system (1.8), sufficient conditions which ensure the permanence and global attractivity of the system were obtained. Stimulated by the works of [6, 14, 15, 21], we propose the following two species competitive plankton with one toxin producing phytoplankton:

$$\begin{aligned} \dot{x}_1(t) &= x_1(t)\Big[r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)\int_{-T_{12}}^{0} K_{12}(s)x_2(t+s)ds \\ &\quad -b_{12}(t)x_1(t)\int_{-\tau_{12}}^{0} f_{12}(s)x_2(t+s)ds\Big], \\ \dot{x}_2(t) &= x_2(t)\Big[r_2(t) - a_{21}(t)\int_{-T_{21}}^{0} K_{21}(s)x_1(t+s)ds - a_{22}(t)x_2(t)\Big]. \end{aligned} \tag{1.9}$$

Throughout this paper, it is assumed that:

(H_1) $r_i(t), a_{ij}(t), (i \neq j), i, j = 1, 2, b_{12}(t)$ are continuous and bounded above and below by positive constants on $[0, +\infty)$;

(H_2) T_{ij}, τ_{12} are positive constants, $K_{ij} \in C([-T_{ij}, 0], (0, +\infty))$,

$$\int_{-T_{ij}}^{0} K_{ij}(s)ds = 1, f_{12} \in C([-\tau_{12}, 0], (0, +\infty))$$

and $\int_{-\tau_{12}}^{0} f_{12}(s)ds = 1$.

We consider (1.9) together with the initial conditions

$$x_i(\theta) = \phi_i(\theta) \geq 0,\ \theta \in [-\tau, 0]; \phi_i(0) > 0, \tag{1.10}$$

where $\tau = \max_{i,j}\{T_{ij}, \tau_{12}\}$, ϕ_i are continuous on $[-\tau, 0]$. It is not difficult to see that solutions of (1.9)-(1.10) are well defined for all $t \geq 0$ and satisfy

$$x_i(t) > 0 \text{ for } t \geq 0, i = 1, 2, ..., n.$$

Throughout this paper, we shall use the following notations:

$$g^l = \min_{t\geq 0} g(t),\ \ g^u = \max_{t\geq 0} g(t),$$

where g is a continuous bounded function defined on $[0,+\infty)$;

Comparing system (1.9) with (1.5), one could easily see that system (1.9) is the special case of system (1.5)(with $b_{21}(t)\equiv 0$). Hence, it is nature for One to conjecture that the results of [6] could be applied directly to system (1.9). Indeed, as far as the permanence and stability property of the system is concerned, the results of [6] could be applied to system (1.9) directly. However, since $b_{21}(t)\equiv 0, b_{12}^u > 0$, the inequality

$$r_1^l b_{21}^l \geq r_2^u b_{12}^u \tag{1.11}$$

in (1.6) no longer holds. This means that the extinction result of [6] can no longer be applied to system (1.9). To investigate the extinction property of the system (1.9), one needs to develop some new analysis technique.

The aim of this paper is, by developing the analysis technique of Chen et al.[6] and Montes De Oca and Vivas[22], to investigate the extinction property of the system (1.9). The organization of this paper is as follows. We state and prove the main results in the next section and we end this paper by a briefly discussion.

2. Main results

Lemma 2.1. [6] *If $a>0, b>0$ and $\dot{x}(t)\leq (\geq) x(t)(b-ax(t)),\, x(t_0)>0$, we have*

$$\limsup_{t\to+\infty} x(t) \leq \frac{b}{a}\ (\liminf_{t\to+\infty} x(t) \geq \frac{b}{a}).$$

Lemma 2.2. *Let $col(x_1(t), x_2(t))$ be any solution of system (1.9) with initial conditions (1.10), then*

$$\limsup_{t\to\infty} x_i(t) \leq r_i^u / a_{ii}^l \stackrel{\text{def}}{=} M_i,\ i=1,2.$$

Proof. From system (1.9) one has

$$\dot{x}_i(t) \leq x_i(t)\Big[r_i(t) - a_{ii}(t)x_i(t)\Big],\ i=1,2.$$

By applying Lemma 2.1, it immediately follows

$$\limsup_{t\to+\infty} x_i(t) \le M_i, i = 1,2.$$

This ends the proof of Lemma 2.2.

Lemma 2.3. $(Fluctuation\ lemma\ )$([22, Lemma 4]) *Let $x(t)$ be a bounded differentiable function on (α,∞), then there exist sequences $\tau_n \to \infty, \sigma_n \to \infty$ such that*

$$(a)\ \dot{x}(\tau_n) \to 0 \text{ and } x(\tau_n) \to \limsup_{t\to\infty} x(t) = \bar{x} \text{ as } n \to \infty,$$

$$(b)\ \dot{x}(\sigma_n) \to 0 \text{ and } x(\sigma_n) \to \liminf_{t\to\infty} x(t) = \underline{x} \text{ as } n \to \infty.$$

Lemma 2.4 ([22, Lemma 7]). *There exists a unique solution $x_1^*(t)$ of the logistic equation*

$$\dot{x}_1(t) = x_1(t)\Big[r_1(t) - a_{11}(t)x_1(t)\Big] \tag{2.1}$$

such that $\delta \le x_1^*(t) \le \Delta$ on $(-\infty,\infty)$, where Δ and δ are any numbers satisfying the inequalities $0 < \delta < r_1^l/a_{11}^u$ and $r_1^u/a_{11}^l < \Delta$.

Lemma 2.5. *Let $col(x_1(t),x_2(t))$ be any solution of system (1.9) with initial conditions (1.10), assume that*

$$r_1^l a_{21}^l > a_{11}^u r_2^u,\ r_1^l a_{22}^l > r_2^u a_{12}^u \tag{2.2}$$

hold, then there exists $\alpha > 0$ such that $x_1(t) \ge \alpha$ for all $t \ge 0$.

Proof. The proof of the Lemma 2.5 is similarly to the proof of Lemma 6 of Montes De Oca and Vivas[22] and we omit the detail here.

Our main results are the following Theorem 2.1-2.6.

Theorem 2.1. *In addition to (2.2), further assume that the following inequality*

$$b_{12}^u < \min\left\{ \frac{r_1^l - \frac{a_{11}^u}{a_{21}^l} r_2^u}{\frac{r_1^u}{a_{11}^l}\frac{r_2^u}{a_{22}^l}}, \frac{r_1^l - \frac{a_{12}^u}{a_{22}^l} r_2^u}{\frac{r_1^u}{a_{11}^l}\frac{r_2^u}{a_{22}^l}} \right\} \tag{2.3}$$

holds, then the species x_2 will be driven to extinction, that is, for any positive solution $col(x_1(t),x_2(t))$ of system (1.9), $x_2(t) \to 0$ as $t \to +\infty$.

Proof. It follows from (2.3) that one could choose enough small positive constant $\varepsilon_1 > 0$ such that

$$b_{12}^u < \min\left\{ \frac{r_1^l - \frac{a_{11}^u}{a_{21}^l} r_2^u}{\left(\frac{r_1^u}{a_{11}^l} + \varepsilon_1\right)\left(\frac{r_2^u}{a_{22}^l} + \varepsilon_1\right)}, \frac{r_1^l - \frac{a_{12}^u}{a_{22}^l} r_2^u}{\left(\frac{r_1^u}{a_{11}^l} + \varepsilon_1\right)\left(\frac{r_2^u}{a_{22}^l} + \varepsilon_1\right)} \right\}. \tag{2.4}$$

(2.4) is equivalent to

$$\frac{\triangle}{r_2^u} a_{21}^l - a_{11}^u > 0, \;\; a_{12}^u - \frac{\triangle}{r_2^u} a_{22}^l < 0. \tag{2.5}$$

where $\triangle \overset{\text{def}}{=} r_1^l - b_{12}^u\left(\frac{r_1^u}{a_{11}^l} + \varepsilon_1\right)\left(\frac{r_2^u}{a_{22}^l} + \varepsilon_1\right)$. Let $\underline{x}_1 = \liminf_{t\to\infty} x_1(t)$ and $\bar{x}_2 = \limsup_{t\to\infty} x_2(t)$, from Lemma 2.5 we know that $\underline{x}_1 \geq \alpha > 0$, obviously $\bar{x}_2 \geq 0$. It follows from Lemma 2.2 that $x_i(t), i = 1,2$ satisfies

$$\underline{x}_1 < \frac{r_1^u}{a_{11}^l} + \varepsilon_1, \bar{x}_2 < \frac{r_2^u}{a_{22}^l} + \varepsilon_1. \tag{2.6}$$

To end the proof of Theorem 2.1, it suffices to show that $\bar{x}_2 = 0$. In order to get a contradiction, suppose that $\bar{x}_2 > 0$. According to Fluctuation lemma, there exists sequences $\tau_n \to \infty, \sigma_n \to \infty$ such that $\dot{x}_1(\tau_n) \to 0, \dot{x}_2(\sigma_n) \to 0, x_1(\tau_n) \to \underline{x}_1$ and $x_2(\sigma_n) \to \bar{x}_2$. Since the functions $\int_{-T_{12}}^0 K_{12}(s)x_2(t+s)ds, \int_{-\tau_{12}}^0 f_{12}(s)x_2(t+s)ds$ and $\int_{-T_{21}}^0 K_{21}(s)x_1(t+s)ds$ are bounded, we can assume that

$$\int_{-T_{12}}^0 K_{12}(s)x_2(\tau_n+s)ds \to \alpha_1,$$
$$\int_{-\tau_{12}}^0 f_{12}(s)x_2(\tau_n+s)ds \to \alpha_2,$$
$$\int_{-T_{21}}^0 K_{21}(s)x_1(\sigma_n+s)ds \to \beta_1.$$

It's clear that $\alpha_i \leq \bar{x}_2, \beta_1 \geq \underline{x}_1,\ i = 1,2$. Therefore, it follows from (1.9) that

$$\begin{aligned} 0 &\geq \underline{x}_1[r_1^l - a_{11}^u\underline{x}_1 - a_{12}^u\bar{x}_2 - b_{12}^u\underline{x}_1\bar{x}_2], \\ 0 &\leq \bar{x}_2[r_2^u - a_{21}^l\underline{x}_1 - a_{22}^l\bar{x}_2], \end{aligned}$$

Since $\underline{x}_1 \geq \alpha > 0$ and $\bar{x}_2 > 0$, it follows

$$r_1^l \leq a_{11}^u\underline{x}_1 + a_{12}^u\bar{x}_2 + b_{12}^u\underline{x}_1\bar{x}_2, \tag{2.7}$$

$$r_2^u \geq a_{21}^l\underline{x}_1 + a_{22}^l\bar{x}_2. \tag{2.8}$$

Now, by applying inequalities (2.6) to (2.7), we get

$$r_1^l - b_{12}^u\big(\frac{r_1^u}{a_{11}^l}+\varepsilon_1\big)\big(\frac{r_2^u}{a_{22}^l}+\varepsilon_1\big) \le a_{11}^u\underline{x}_1 + a_{12}^u\bar{x}_2,$$

or

$$\triangle \le a_{11}^u\underline{x}_1 + a_{12}^u\bar{x}_2. \tag{2.9}$$

Multiplying (2.8) by $-\triangle/r_2^u$ leads to

$$-\triangle \quad \le \quad -\frac{\triangle}{r_2^u}a_{21}^l\underline{x}_1 - \frac{\triangle}{r_2^u}a_{22}^l\bar{x}_2. \tag{2.10}$$

Adding (2.10) to (2.9), it follows

$$0 \le \left(a_{11}^u - \frac{\triangle}{r_2^u}a_{21}^l\right)\underline{x}_1 + \left(a_{12}^u - \frac{\triangle}{r_2^u}a_{22}^l\right)\bar{x}_2,$$

that is

$$\left(a_{12}^u - \frac{\triangle}{r_2^u}a_{22}^l\right)\bar{x}_2 \ge \left(\frac{\triangle}{r_2^u}a_{21}^l - a_{11}^u\right)\underline{x}_1. \tag{2.11}$$

From the first inequality of (2.5) and $\underline{x}_1 \ge \alpha > 0$, we get

$$\left(\frac{\triangle}{r_2^u}a_{21}^l - a_{11}^u\right)\underline{x}_1 > 0,$$

therefore, (2.11) implies that

$$\left(a_{12}^u - \frac{\triangle}{r_2^u}a_{22}^l\right)\bar{x}_2 > 0. \tag{2.12}$$

(2.12) together with (2.5) leads to $\bar{x}_2 < 0$, which is a contradiction. This completes the proof of Theorem 2.1.

Theorem 2.2. *In addition to (2.2), further assume that the following inequality*

$$b_{12}^u < \left(\frac{r_1^l}{r_2^u} - \frac{a_{11}^u}{a_{21}^l}\right)\frac{a_{21}^l a_{22}^l}{r_2^u} \tag{2.13}$$

holds, then the species x_2 will be driven to extinction, that is, for any positive solution $col(x_1(t), x_2(t))$ of system (1.9), $x_2(t) \to 0$ as $t \to +\infty$.

Proof. It follows from (2.13) that one could choose enough small positive constant $\varepsilon_2 > 0$ such that

$$b_{12}^u\Big(\frac{r_2^u}{a_{22}^l}+\varepsilon_2\Big) < \frac{r_1^l a_{21}^l}{r_2^u} - a_{11}^u. \tag{2.14}$$

Let $\underline{x}_1 = \liminf_{t\to\infty} x_1(t)$ and $\bar{x}_2 = \limsup_{t\to\infty} x_2(t)$, similarly to the analysis of Theorem 2.1, we can show that (2.6)-(2.8) hold. Now, by applying the second inequality in (2.6) to (2.7), we get

$$r_1^l \le \Big(a_{11}^u + b_{12}^u(\frac{r_2^u}{a_{22}^l}+\varepsilon_2)\Big)\underline{x}_1 + a_{12}^u \bar{x}_2, \tag{2.15}$$

Multiplying (2.8) by $-r_1^l/r_2^u$ leads to

$$-r_1^l \quad \le \quad -\frac{r_1^l a_{21}^l}{r_2^u}\underline{x}_1 - \frac{r_1^l a_{22}^l}{r_2^u}\bar{x}_2. \tag{2.16}$$

Adding (2.15) to (2.16), it follows

$$0 \le \left(a_{11}^u + b_{12}^u(\frac{r_2^u}{a_{22}^l}+\varepsilon_2) - \frac{r_1^l a_{21}^l}{r_2^u}\right)\underline{x}_1 + \left(a_{12}^u - \frac{r_1^l a_{22}^l}{r_2^u}\right)\bar{x}_2,$$

that is

$$\left(a_{12}^u - \frac{r_1^l a_{22}^l}{r_2^u}\right)\bar{x}_2 \ge \left(\frac{r_1^l a_{21}^l}{r_2^u} - a_{11}^u - b_{12}^u(\frac{r_2^u}{a_{22}^l}+\varepsilon_2)\right)\underline{x}_1 \tag{2.17}$$

From (2.14) and $\underline{x}_1 \ge \alpha > 0$, we get

$$\left(\frac{r_1^l a_{21}^l}{r_2^u} - a_{11}^u - b_{12}^u(\frac{r_2^u}{a_{22}^l}+\varepsilon_2)\right)\underline{x}_1 > 0,$$

therefore, (2.17) implies that

$$\left(a_{12}^u - \frac{r_1^l a_{22}^l}{r_2^u}\right)\bar{x}_2 > 0. \tag{2.18}$$

(2.18) together with the second inequality of (2.2) leads to $\bar{x}_2 < 0$, which is a contradiction. This completes the proof of Theorem 2.2.

Theorem 2.3. *In addition to (2.2), further assume that the following inequality*

$$b_{12}^u < \left(\frac{r_1^l}{r_2^u} - \frac{a_{12}^u}{a_{22}^l}\right)\frac{a_{22}^l a_{11}^l}{r_1^u} \tag{2.19}$$

holds, then the species x_2 will be driven to extinction, that is, for any positive solution $col(x_1(t), x_2(t))$ of system (1.9), $x_2(t) \to 0$ as $t \to +\infty$.

Proof. It follows from (2.19) that one could choose enough small positive constant $\varepsilon_3 > 0$ such that

$$b_{12}^u\Big(\frac{r_1^u}{a_{11}^l}+\varepsilon_3\Big) < \frac{r_1^l a_{22}^l}{r_2^u} - a_{12}^u. \tag{2.20}$$

Let $\underline{x}_1 = \liminf_{t\to\infty} x_1(t)$ and $\overline{x}_2 = \limsup_{t\to\infty} x_2(t)$, similarly to the analysis of Theorem 2.1, we can show that (2.6)-(2.8) hold. Now, by applying the first inequality in (2.6) to (2.7), we get

$$r_1^l \le a_{11}^u \underline{x}_1 + \Big(a_{12}^u + b_{12}^u(\frac{r_1^u}{a_{11}^l}+\varepsilon_3)\Big)\overline{x}_2, \tag{2.21}$$

Multiplying (2.8) by $-r_1^l/r_2^u$ leads to

$$-r_1^l \quad \le \quad -\frac{r_1^l a_{21}^l}{r_2^u}\underline{x}_1 - \frac{r_1^l a_{22}^l}{r_2^u}\overline{x}_2. \tag{2.22}$$

Adding (2.21) to (2.22), it follows

$$0 \le \left(a_{11}^u - \frac{r_1^l a_{21}^l}{r_2^u}\right)\underline{x}_1 + \left(a_{12}^u + b_{12}^u(\frac{r_1^u}{a_{11}^l}+\varepsilon_3) - \frac{r_1^l a_{22}^l}{r_2^u}\right)\overline{x}_2,$$

that is

$$\left(a_{12}^u + b_{12}^u(\frac{r_1^u}{a_{11}^l}+\varepsilon_3) - \frac{r_1^l a_{22}^l}{r_2^u}\right)\overline{x}_2 \ge \left(\frac{r_1^l a_{21}^l}{r_2^u} - a_{11}^u\right)\underline{x}_1 \tag{2.23}$$

From the first inequality of (2.2) and $\underline{x}_1 \ge \alpha > 0$, we get

$$\left(\frac{r_1^l a_{21}^l}{r_2^u} - a_{11}^u\right)\underline{x}_1 > 0,$$

therefore, (2.23) implies that

$$\left(a_{12}^u + b_{12}^u(\frac{r_1^u}{a_{11}^l}+\varepsilon_3) - \frac{r_1^l a_{22}^l}{r_2^u}\right)\overline{x}_2 > 0. \tag{2.24}$$

(2.18) together with the second inequality of (2.2) leads to $\overline{x}_2 < 0$, which is a contradiction. This completes the proof of Theorem 2.3.

Theorem 2.4. *Assume that the conditions of Theorem 2.1 or 2.2 or 2.3 hold, let $x(t) = col(x_1(t), x_2(t))$ be any positive solution of system (1.9), then the species x_2 will be driven to extinction, that is, $x_2(t) \to 0$ as $t \to +\infty$, and $x_1(t) \to x_1^*(t)$ as $t \to +\infty$, where $x_1^*(t)$ is defined by Lemma 2.4.*

Proof. By applying Lemma 2.3-2.5, the proof of Theorem 2.4 is similar to that of the proof of Theorem 4.1 in [6]. We omit the detail here.

Now let's consider the following system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\Big[r_1(t)-a_{11}(t)x_1(t)-a_{12}(t)\textstyle\int_{-\infty}^{0}K_{12}(s)x_2(t+s)ds\\ &\qquad -b_{12}(t)x_1(t)\textstyle\int_{-\infty}^{0}f_{12}(s)x_2(t+s)ds\Big],\\ \dot{x}_2(t) &= x_2(t)\Big[r_2(t)-a_{21}(t)\textstyle\int_{-\infty}^{0}K_{21}(s)x_1(t+s)ds-a_{22}(t)x_2(t)\Big]\end{aligned}\tag{2.25}$$

together with the initial conditions

$$x_i(\theta)=\phi_i(\theta)\geq 0,\ \theta\in(-\infty,0];\phi_i(0)>0,i=1,2,\tag{2.26}$$

where ϕ_i are continuous on $(-\infty,0]$. We introduce a condition

$(H_2^{'})$ $K_{ij}\in C((-\infty,0],(0,+\infty))$ and $\int\limits_{-\infty}^{0}K_{ij}(s)ds=1,i\neq j,i,j=1,2;f_{12}\in C((-\infty,0],(0,+\infty))$ and $\int\limits_{-\infty}^{0}f_{12}(s)ds=1$.

Theorem 2.5. *In addition to* (H_1) *and* $(H_2^{'})$*, assume that the conditions of Theorem 2.1 or 2.2 or 2.3 hold, let* $col(x_1(t),x_2(t))$ *be any solution of system (2.25) with initial conditions (2.26), then* $\lim_{t\to\infty}x_2(t)=0$ *and* $\lim_{t\to\infty}\left[x_1(t)-x_1^*(t)\right]=0$*, where* $x_1^*(t)$ *is defined by Lemma 2.4.*

Now let's consider following system:

$$\begin{aligned}\dot{x}_1(t) &= x_1(t)\Big[r_1(t)-a_{11}(t)x_1(t)-a_{12}(t)x_2(t-\tau_{12}(t))\\ &\qquad -b_{12}(t)x_1(t)x_2(t-\eta_{12}(t))\Big],\\ \dot{x}_2(t) &= x_2(t)\Big[r_2(t)-a_{21}(t)x_1(t-\tau_{21}(t))-a_{22}(t)x_2(t)\Big]\end{aligned}\tag{2.27}$$

together with the initial conditions

$$x_i(\theta)=\phi_i(\theta)\geq 0,\ \theta\in[-\tau,0];\phi_i(0)>0,i=1,2,\tag{2.28}$$

where $\tau_{ij}(t),\eta_{ij}(t),i,j=1,2$ are nonnegative continuous bounded functions, $\tau=\max\limits_{t}\{\tau_{ij}(t),\eta_{ij}(t),i,j=1,2\}$, ϕ_i are continuous on $[-\tau,0]$.

Theorem 2.6. *In addition to* (H_1)*, assume that the conditions of Theorem 2.1 or 2.2 or 2.3 hold, let* $col(x_1(t),x_2(t))$ *be any solution of system (2.27) with initial condition (2.28), then* $\lim_{t\to\infty}x_2(t)=0$ *and* $\lim_{t\to\infty}\left[x_1(t)-x_1^*(t)\right]=0$*, where* $x_1^*(t)$ *is defined by Lemma 2.4.*

The proof of Theorem 2.5 and 2.6 are similarly to that of the proof of Theorem 2.4, we omit the detail here.

3. Conclusion

Chen et al.[6] proposed a delay differential equation model of plankton allelopathy, which is described by system (1.5). Using a fluctuation theorem, sufficient conditions which guarantee one of the components will be driven to extinction while the other will stabilize at a certain solution of a logistic equation is obtained. In this paper, we assume that one species is non-toxic phytoplankton while the other one is toxic liberating phytoplankton, this results in model (1.9). Though system (1.9) is the special case of system (1.5), the results obtained in [6] which concerned with the extinction of the system could not be applied to the system (1.9) directly. By developing some new analysis technique, we are able to establish some sufficient conditions which ensure the extinction of one species and the global attractivity of the other species. Our results indicate that if the second species in the system without toxic substance is extinction (this is ensured by condition (2.2)), then although the second species could emit the toxic to improve their chance of living, if the toxic rate is enough low such that inequality (2.2) or (2.13) or (2.19) holds, then second species is still be driven to extinction.

Conflict of Interests

The authors declare that there is no conflict of interests.

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