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EXISTENCE OF MULTIPLE POSITIVE ALMOST PERIODIC SOLUTIONS TO AN IMPULSIVE NON-AUTONOMOUS LOTKA-VOLTERRA PREDATOR-PREY SYSTEM WITH HARVESTING TERMS

CHUNHUA ZHANG, YUYING ZHANG, XIAOXING CHEN*

College of Mathematics and Computer Science, Fuzhou University, Fuzhou, Fujian, 350116, China

Abstract. This paper is concerned with an almost periodic predator-prey system with impulsive non-autonomous Lotka-Volterra functional response and harvesting terms. By using Mawhins continuation theorem of coincidence degree theory and some analytical approaches, we establish the existence of eight positive almost periodic solutions for the system. Furthermore, our results improve the main results of paper [1]. An example is given to illustrate the effectiveness of our results.

Keywords: Eight positive almost periodic solutions; Lotka-Volterra predator-prey system; Coincidence degree; Harvesting term.

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1. Introduction

In recent years, the existence of positive periodic solutions for biological models with harvesting terms has been widely investigated by many researchers (see[1-4]). In[1], the authors proposed a non-autonomous three species Lotka-Volterra predator-prey system with harvesting

*Corresponding author

E-mail address: cxxing79@163.com

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terms model:

$$
\begin{aligned}
x_1'(t) &= x_1(t)\Big(r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t)\Big) - h_1(t),\\
x_2'(t) &= x_2(t)\Big(r_2(t) + a_{21}(t)x_1(t) - a_{22}(t)x_2(t) - a_{23}(t)x_3(t)\Big) - h_2(t),\\
x_3'(t) &= x_3(t)\Big(r_3(t) + a_{32}(t)x_2(t) - a_{33}(t)x_3(t)\Big) - h_3(t).
\end{aligned}
\tag{1.1}
$$

Under the assumptions of periodicity of the parameters of (1.1), using Mawhin's continuation theorem of coincidence degree theory, the authors of [1] established the eight positive periodic solutions to (1.1). In fact, it is more realistic and reasonable to study almost periodic system than periodic system. Recently, there are two main approaches to obtain sufficient conditions for the existence and stability of the almost periodic solutions of biological models: one is using the fixed point theorem, Lyapunov functional method, and differential inequality techniques (see [5-11]); the other is using functional hull theory and Lyapunov functional method (see [12-17]). However, to the best of our knowledge, there are very few published letters considering the almost periodic solutions for impulsive nonautonomous Lotka-Volterra predator-prey system with time delay and harvesting terms by applying the method of coincidence degree theory. Motivated by above, in this paper, we are concerned with the following impulsive nonautonomous three species Lotka-Volterra predator-prey system with time delay and harvesting terms model:

$$
\begin{aligned}
&x_1'(t) = x_1(t)\Big(r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t-\tau_2(t))\Big) - h_1(t), t \neq t_k,\\
&x_2'(t) = x_2(t)\Big(r_2(t) + a_{21}(t)x_1(t-\tau_1(t)) - a_{22}(t)x_2(t) - a_{23}(t)x_3(t-\tau_3(t))\Big) - h_2(t), t \neq t_k,\\
&x_3'(t) = x_3(t)\Big(r_3(t) + a_{32}(t)x_2(t-\tau_2(t)) - a_{33}(t)x_3(t)\Big) - h_3(t), t \neq t_k,\\
&x_1(t_k^+) = (1+\Gamma_{1k})x_1(t_k), t = t_k,\\
&x_2(t_k^+) = (1+\Gamma_{2k})x_2(t_k), t = t_k,\\
&x_3(t_k^+) = (1+\Gamma_{3k})x_3(t_k), t = t_k,
\end{aligned}
\tag{1.2}
$$

where $x_i(t)$ denotes the densities of the ith species respectively; $r_i(t)$ represents the ith species intrinsic growth rates; $a_{ii}(t)$ denotes the intra-specific competition rates of the ith species; $a_{12}(t), a_{23}(t)$ are the predation rates for the second species to first the species, the third species to the second species, respectively; $a_{21}(t), a_{32}(t)$ are the nutrition conversion rates for the first species to the second species, the second species to the third species, respectively; $h_i(t)$ is the harvesting term for the ith species. Moreover, $r_i(t)$,$a_{ii}(t)$, $a_{12}(t)$,$a_{23}(t)$, $a_{21}(t)$,$a_{32}(t)$, $h_i(t)$, are all bounded and positive continuous almost periodic functions defined on $[0,\infty)(i=1,2,3)$; the time delay $\tau_i(t)$ (i= 1,2,3) are all nonnegative continuous almost periodic functions; $\Gamma_{ik} > -1(i=1,2,3)$ are constants and $0=t_0<t_1<t_2<...t_k<t_{k+1}<...$, are impulse points with $\lim\limits_{k\to+\infty} t_k=+\infty$.

The organization of this paper is as follows. In Section2, we state some definitions lemmas which are useful in later sections and make some preparations. In Section3, using Mawhins continuation theorem of coincidence degree theory and some analytical approaches, we establish sufficient conditions for the existence of eight positive almost periodic solutions to system (1.2). In Section 4, an example is given to illustrate our results obtained.

2. Preliminaries

In this section, we give a short introduction to some referred definitions and lemmas that will come into play later on.

$AP(R)=\{f(t): f(t)$ is a continuous, real valued, almost periodic function on $R\}$. Suppose that $f(t,\phi)$ is almost periodic in t, uniformly with respect to $\phi\in C([-\sigma,0],R)$. $T(f,\varepsilon,S)$ will denote the set of $\varepsilon-$ almost periods with respect to $S\subset C([-\sigma,0],R)$, $l(\varepsilon,S)$ the inclusion interval, $\Lambda(f)$ the set of Fourier exponents, mod(f) the module of f, and $m(f)$ the mean value.

Let $PC(R,R^n)$=$\{\varphi: R\to R^n,\varphi$ is a piecewise continuous function with points of discontinuity of the first kind at $t_k, k=1,2,...$, at which $\varphi(t_k^-)$ and $\varphi(t_k^+)$ exist and $\varphi(t_k^-)=\varphi(t_k)\}$.

Definition 2.1 [21] The family of sequences $\{t_k^j=t_{k+j}-t_k, k,j\in Z\}$ is said to be equipotentially almost periodic if for arbitrarye $\varepsilon>0$, there exists a relatively dense set $\varepsilon-$ almost periods, that are common for any sequences.

Definition 2.2 [21] The function $\varphi\in PC(R,R)$ is said to be almost periodic, if the following conditions hold:

(1) the set of sequences $\{t_k^j=t_{k+j}-t_k, k,j\in Z\}$ is equipotentially almost periodic;

(2) for any $\varepsilon > 0$ there exists a real number $\delta = \delta(\varepsilon) > 0$ such that if the points t_1 and t_2 belong to the same interval of continuity of $\varphi(t)$ and $|t_1 - t_2| < \delta$, then $|\varphi(t_1) - \varphi(t_2)| < \varepsilon$;

(3) for any $\varepsilon > 0$ there exists a relatively dense set T of $\varepsilon-$ almost periodic such that if $\tau \in T$, then $|\varphi(t+\tau) - \varphi(t)| < \varepsilon$ for all $t \in R$ which satisfy the condition $|t - t_k| > \varepsilon, k \in Z$.

Lemma 2.1 [18] If $f(t) \in AP(R)$, then there exists $t_0 \in R$ such that $f(t_0) = m(f)$.

lemma 2.2 [22] *Assume that $f(t) \in AP(R)$, then $f(t)$ is bounded on R.*

Lemma 2.3 [18] *Assume that $x(t) \in AP(R) \cap C^1(R,R)$, then there exist two point sequences $\{\xi_k\}_{k=1}^{\infty}$, $\{\eta_k\}_{k=1}^{\infty}$, such that $x'(\xi_k) = x'(\eta_k) = 0$, $\lim\limits_{k\to+\infty} \xi_k = +\infty$ and $\lim\limits_{k\to+\infty} \eta_k = -\infty$.*

Lemma 2.4 [18] *Assume that $x(t) \in AP(R) \cap C^1(R,R)$, then $x(t)$ falls into one of the following four cases:*

(i) there are $\xi, \eta \in R$ such that $x(\xi) = \sup\limits_{t\in R} x(t)$ and $x(\eta) = \inf\limits_{t\in R} x(t)$. In this case, $x'(\xi) = x'(\eta) = 0$.

(ii) there are no $\xi, \eta \in R$ such that $x(\xi) = \sup\limits_{t\in R} x(t)$ and $x(\eta) = \inf\limits_{t\in R} x(t)$. In this case, for any $\varepsilon > 0$, there exist two points $\xi, \eta \in R$ such that $x'(\xi) = x'(\eta) = 0$, $x(\xi) > \sup\limits_{t\in R} x(t) - \varepsilon$ and $x(\eta) < \inf\limits_{t\in R} x(t) + \varepsilon$.

(iii) there is a $\xi \in R$ such that $x(\xi) = \sup\limits_{t\in R} x(t)$ and there is no $\eta \in R$ such that $x(\eta) = \inf\limits_{t\in R} x(t)$. In this case, $x'(\xi) = 0$ and for any $\varepsilon > 0$, there exist an η such that $x'(\eta) = 0$ and $x(\eta) < \inf\limits_{t\in R} x(t) + \varepsilon$.

(iv) there is an $\eta \in R$ such that $x(\eta) = \inf\limits_{t\in R} x(t)$ and there is no $\xi \in R$ such that $x(\xi) = \sup\limits_{t\in R} x(t)$. In this case, $x'(\eta) = 0$ and for any $\varepsilon > 0$, there exist a ξ such that $x'(\xi) = 0$ and $x(\xi) > \sup\limits_{t\in R} x(t) - \varepsilon$.

Consider the following system

$$
\begin{aligned}
&x_1'(t) = x_1(t)\Big(r_1(t) - \overline{a}_{11}(t)x_1(t) - \overline{a}_{12}(t)x_2(t-\tau_2(t))\Big) - \overline{h}_1(t),\\
&x_2'(t) = x_2(t)\Big(r_2(t) + \overline{a}_{21}(t)x_1(t-\tau_1(t)) - \overline{a}_{22}(t)x_2(t) - a_{23}(t)x_3(t-\tau_3(t))\Big) - \overline{h}_2(t),\\
&x_3'(t) = x_3(t)\Big(r_3(t) + \overline{a}_{32}(t)x_2(t-\tau_2(t)) - \overline{a}_{33}(t)x_3(t)\Big) - \overline{h}_3(t),
\end{aligned}
\tag{2.1}
$$

where

$$\overline{a}_{11}(t)=a_{11}(t)\prod_{0<t_k<t}(1+\Gamma_{1k}),\qquad \overline{a}_{12}(t)=a_{12}(t)\prod_{0<t_k<t}(1+\Gamma_{2k}),$$

$$\overline{h}_1(t)=h_1(t)\prod_{0<t_k<t}(1+\Gamma_{1k})^{-1},\qquad \overline{a}_{21}(t)=a_{21}(t)\prod_{0<t_k<t}(1+\Gamma_{1k}),$$

$$\overline{a}_{22}(t)=a_{22}(t)\prod_{0<t_k<t}(1+\Gamma_{2k}),\qquad \overline{a}_{23}(t)=a_{23}(t)\prod_{0<t_k<t}(1+\Gamma_{3k}),$$

$$\overline{h}_2(t)=h_2(t)\prod_{0<t_k<t}(1+\Gamma_{2k})^{-1},\qquad \overline{a}_{32}(t)=a_{32}(t)\prod_{0<t_k<t}(1+\Gamma_{2k}),$$

$$\overline{a}_{33}(t)=a_{33}(t)\prod_{0<t_k<t}(1+\Gamma_{3k}),\qquad \overline{h}_3(t)=h_3(t)\prod_{0<t_k<t}(1+\Gamma_{3k})^{-1}.$$

Lemma 2.5 *For systems* (1.2) *and* (2.1), *the following results hold:*

(1)if $(x_1(t),x_2(t),x_3(t))^T$ *is a solution of (1.2), then*

$$(\overline{x}_1(t),\overline{x}_2(t),\overline{x}_3(t))^T=\Big(\prod_{0<t_k<t}(1+\Gamma_{1k})^{-1}x_1(t),\prod_{0<t_k<t}(1+\Gamma_{2k})^{-1}x_2(t),\prod_{0<t_k<t}(1+\Gamma_{3k})^{-1}x_3(t)\Big)^T$$

is a solution of (2.1).

(2)if $(\overline{x}_1(t),\overline{x}_2(t),\overline{x}_3(t))^T$ *is a solution of (2.1), then*

$$(x_1(t),x_2(t),x_3(t))^T=\Big(\prod_{0<t_k<t}(1+\Gamma_{1k})\overline{x}_1(t),\prod_{0<t_k<t}(1+\Gamma_{2k})\overline{x}_2(t),\prod_{0<t_k<t}(1+\Gamma_{2k})\overline{x}_3(t)\Big)^T$$

is a solution of (1.2).

Proof. (1) Suppose that $(x_1(t),x_2(t),x_3(t))^T$ is a solution of (1.2). Let

$$\overline{x}_1(t)=\prod_{0<t_k<t}(1+\Gamma_{1k})^{-1}x_1(t),\overline{x}_2(t)=\prod_{0<t_k<t}(1+\Gamma_{2k})^{-1}x_2(t),\overline{x}_3(t)=\prod_{0<t_k<t}(1+\Gamma_{3k})^{-1}x_3(t),$$

we first show that $\bar{x}_1(t),\bar{x}_2(t),\bar{x}_3(t)$ are continuous. Since $\bar{x}_1(t),\bar{x}_2(t),\bar{x}_3(t)$ are continuous on each interval $(t_k,t_{k+1}]$, it is sufficient to check the continuity of $\bar{x}_1(t),\bar{x}_2(t),\bar{x}_3(t)$ at the impulse points $t_k, k\in Z^+$. Since

$$\bar{x}_1(t_k^+) = \prod_{0<t_s\le t_k}(1+\Gamma_{1s})^{-1}x_1(t_k^+) = (1+\Gamma_{1k})^{-1}\prod_{0<t_s<t_k}((1+\Gamma_{1s})^{-1}(1+\Gamma_{1k})x_1(t_k) = \bar{x}_1(t_k)$$

and

$$\bar{x}_1(t_k^-) = \prod_{0<t_s<t_k^-}(1+\Gamma_{1s})^{-1}x_1(t_k^-) = \prod_{0<t_s<t_k}(1+\Gamma_{1s})^{-1}x_1(t_k) = \bar{x}_1(t_k),$$

thus $\bar{x}_1(t_k)$ is continuous on $[0,+\infty)$. Using the same method, we get $\bar{x}_2(t_k),\bar{x}_3(t_k)$ is continuous on $[0,+\infty)$. By substituting

$$x_1(t) = \prod_{0<t_k<t}(1+\Gamma_{1k})\bar{x}_1(t),\; x_2(t) = \prod_{0<t_k<t}(1+\Gamma_{2k})\bar{x}_2(t),\; x_3(t) = \prod_{0<t_k<t}(1+\Gamma_{3k})\bar{x}_3(t)$$

into the equation of system (1.2), we obtain

$$\begin{aligned}
\bar{x}_1'(t) &= \bar{x}_1(t)\Big(r_1(t) - \bar{a}_{11}(t)\bar{x}_1(t) - \bar{a}_{12}(t)\bar{x}_2(t-\tau_2(t))\Big) - \bar{h}_1(t),\\
\bar{x}_2'(t) &= \bar{x}_2(t)\Big(r_2(t) + \bar{a}_{21}(t)\bar{x}_1(t-\tau_1(t)) - \bar{a}_{22}(t)\bar{x}_2(t) - a_{23}(t)\bar{x}_3(t-\tau_3(t))\Big) - \bar{h}_2(t),\\
\bar{x}_3'(t) &= \bar{x}_3(t)\Big(r_3(t) + \bar{a}_{32}(t)\bar{x}_2(t-\tau_2(t)) - \bar{a}_{33}(t)\bar{x}_3(t)\Big) - \bar{h}_3(t).
\end{aligned}$$

Therefore, $(\bar{x}_1(t),\bar{x}_2(t),\bar{x}_3(t))^T$ is a solution of (2.1).

(2) Suppose that $(\bar{x}_1(t),\bar{x}_2(t),\bar{x}_3(t))^T$ is a solution of (2.1). Let

$$x_1(t) = \prod_{0<t_k<t}(1+\Gamma_{1k})\bar{x}_1(t),\; x_2(t) = \prod_{0<t_k<t}(1+\Gamma_{2k})\bar{x}_2(t),\; x_3(t) = \prod_{0<t_k<t}(1+\Gamma_{3k})\bar{x}_3(t),$$

then for any $t\neq t_k, k\in Z^+$, by substituting

$$\bar{x}_1(t) = \prod_{0<t_k<t}(1+\Gamma_{1k})^{-1}x_1(t),\; \bar{x}_2(t) = \prod_{0<t_k<t}(1+\Gamma_{2k})^{-1}x_2(t),\; \bar{x}_3(t) = \prod_{0<t_k<t}(1+\Gamma_{3k})^{-1}x_3(t)$$

into the equation of system (2.1), we obtain

$$\begin{aligned}
x_1'(t) &= x_1(t)\Big(r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t-\tau_2(t))\Big) - h_1(t),\\
x_2'(t) &= x_2(t)\Big(r_2(t) + a_{21}(t)x_1(t-\tau_1(t)) - a_{22}(t)x_2(t) - a_{23}(t)x_3(t-\tau_3(t))\Big) - h_2(t),\\
x_3'(t) &= x_3(t)\Big(r_3(t) + a_{32}(t)x_2(t-\tau_2(t)) - a_{33}(t)x_3(t)\Big) - h_3(t).
\end{aligned}$$

And for $t = t_k, k \in Z^+$, we obtain

$$\begin{aligned} x_1(t_k^+) &= \lim_{t\to t_{k+}} \prod_{0<t_k<t} (1+\Gamma_{1k})\overline{x}_1(t) = \prod_{0<t_s\le t_k} (1+\Gamma_{1s})\overline{x}_1(t_k) \\ &= (1+\Gamma_{1k}) \prod_{0<t_s<t_k} (1+\Gamma_{1s})\overline{x}_1(t_k) = (1+\Gamma_{1k})x_1(t_k). \end{aligned}$$

Similarly, we have $x_2(t_k^+) = (1+\Gamma_{2k})x_2(t_k), x_3(t_k^+) = (1+\Gamma_{2k})x_3(t_k)$. Therefore, $(x_1(t), x_2(t), x_3(t))^T$ is a solution of (1.2).

Lemma 2.6 [2] *Let $x > 0, y > 0, z > 0$ and $x > 2\sqrt{yz}$, for the functions $f(x,y,z) = \dfrac{x+\sqrt{x^2-4yz}}{2z}$ and $g(x,y,z) = \dfrac{x-\sqrt{x^2-4yz}}{2z}$, the following assertions hold.*

(1) $f(x,y,z)$ and $g(x,y,z)$ are monotonically increasing and monotonically decreasing on the variable $x \in (0,+\infty)$, respectively.

(2) $f(x,y,z)$ and $g(x,y,z)$ are monotonically decreasing and monotonically increasing on the variable $y \in (0,+\infty)$, respectively.

(3) $f(x,y,z)$ and $g(x,y,z)$ are monotonically decreasing and monotonically increasing on the variable $z \in (0,+\infty)$, respectively.

For the sake of convenience, we denote $f^l = \inf_{t\in[0,\omega]} f(t)$, $f^M = \sup_{t\in[0,\omega]} f(t)$ here $f(t)$ is a continuous almost periodic function.

Throughout this paper, we need the following assumptions:

(H_1) $r_1^l - \overline{a}_{12}^M l_2^+ > 2\sqrt{\overline{a}_{11}^M \overline{h}_1^M}$, $r_2^l - \overline{a}_{23}^M l_3^+ > 2\sqrt{\overline{a}_{22}^M \overline{h}_2^M}$, $r_3^l > 2\sqrt{\overline{a}_{33}^M \overline{h}_3^M}$,

where

$$\begin{aligned} l_1^{\pm} &= \frac{r_1^M \pm \sqrt{(r_1^M)^2 - 4\overline{a}_{11}^l \overline{h}_1^l}}{2\overline{a}_{11}^l}, \\ l_2^{\pm} &= \frac{(r_2^M + \overline{a}_{22}^M l_1^+) \pm \sqrt{(r_2^M + \overline{a}_{22}^M l_1^+)^2 - 4\overline{a}_{22}^l \overline{h}_2^l}}{2\overline{a}_{22}^l}, \\ l_3^{\pm} &= \frac{(r_3^M + \overline{a}_{32}^M l_2^+) \pm \sqrt{(r_3^M + \overline{a}_{32}^M l_2^+)^2 - 4\overline{a}_{33}^l \overline{h}_3^l}}{2\overline{a}_{33}^l}. \end{aligned}$$

(H_2) *The set of sequences* $\{t_k^j = t_{k+j} - t_k, k, j \in Z\}$ *is uniformly almost periodic.*

(H_3) $\prod_{0<t_k<t}(1+\Gamma_{ik}), (i=1,2)$ *is almost periodic.*

For simplicity, we need to introduce some notations as follows:

$$A_1^{\pm} = \frac{(r_1^l - \overline{a}_{12}^M l_2^+) \pm \sqrt{(r_1^l - \overline{a}_{12}^M l_2^+)^2 - 4\overline{a}_{11}^M \overline{h}_1^M}}{2\overline{a}_{11}^M},$$

$$A_2^{\pm} = \frac{(r_2^l - \overline{a}_{23}^M l_3^+) \pm \sqrt{(r_2^l - \overline{a}_{23}^M l_3^+)^2 - 4\overline{a}_{22}^M \overline{h}_2^M}}{2\overline{a}_{22}^M},$$

$$A_3^{\pm} = \frac{r_3^l \pm \sqrt{(r_3^l)^2 - 4\overline{a}_{33}^M \overline{h}_3^M}}{2\overline{a}_{33}^M}.$$

Lemma 2.7 *For the following equation*

$$r_1(t) - \overline{a}_{11}(t)e^{u_1(t)} - \overline{h}_1(t)e^{-u_1(t)} = 0,$$

$$r_2(t) - \overline{a}_{22}(t)e^{u_2(t)} - \overline{h}_2(t)e^{-u_2(t)} = 0,$$

$$r_3(t) - \overline{a}_{33}(t)e^{u_3(t)} - \overline{h}_3(t)e^{-u_3(t)} = 0,$$

by the assumption H_1 *and lemma* 2.6*, we have the following inequalities*

$$\ln l_1^- < \ln u_1^- < \ln A_1^- < \ln A_1^+ < \ln u_1^+ < \ln l_1^+,$$

$$\ln l_2^- < \ln u_2^- < \ln A_2^- < \ln A_2^+ < \ln u_2^+ < \ln l_2^+,$$

$$\ln l_3^- < \ln u_3^- < \ln A_3^- < \ln A_3^+ < \ln u_3^+ < \ln l_3^+,$$

where

$$u_1^{\pm} = \frac{r_1(t) \pm \sqrt{(r_1(t))^2 - 4\overline{a}_{11}(t)\overline{h}_1(t)}}{2\overline{a}_{11}(t)},$$

$$u_2^{\pm} = \frac{r_2(t) \pm \sqrt{(r_2(t))^2 - 4\overline{a}_{22}(t)\overline{h}_2(t)}}{2\overline{a}_{22}(t)},$$

$$u_3^{\pm} = \frac{r_3(t) \pm \sqrt{(r_3(t))^2 - 4\overline{a}_{33}(t)\overline{h}_3(t)}}{2\overline{a}_{33}(t)}.$$

3. Existence of multiple positive almost periodic solutions

In this section, by using Mawhins continuation theorem, we will show a theorem about eight positive almost periodic solutions for system (1.2).

Let X and Z be real normed vector spaces. Let L: $DomL \subset X \to Z$ be a linear mapping and $N : X \times [0,1] \to Z$ be a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if dim $KerL = codimImL < +\infty$ and ImL is closed in Z. If L is a Fredholm mapping of index zero, then there exist continuous projectors $P : X \to X$ and $Q : Z \to Z$ such that $ImP = KerL$ and $KerQ = ImL = Im(I-Q)$, and $X = KerL \oplus KerP$, $Z = ImL \oplus ImQ$. It follows that $L|_{DomL \bigcap kerP} : (I-P)X \to ImL$ is invertible and its inverse is denoted by K_p. If Ω is a bounded open subset of X, the mapping N is called $L-$ compact on $\overline{\Omega} \times [0,1]$, if $QN(\overline{\Omega} \times [0,1])$ is bounded and $K_p(I-Q)N : \overline{\Omega} \times [0,1] \to X$ is compact. Because ImQ is isomorphic to $KerL$, there exists an isomorphism$J$:$ImQ \to KerL$.

Lemma 3.1 [23] *Let L be a Fredholm mapping of index zero and let N be $L-$compact on $\overline{\Omega} \times [0,1]$ Assume that:*

(a) for each $\lambda \in (0,1)$, every solution x of $Lx = \lambda N(x,\lambda)$ is such that $x \notin \partial\Omega \cap DomL$;

(b)$QN(x,0)x \neq 0$ for each $x \in \partial\Omega \cap KerL$;

(c)$deg(JQN(x,0),\Omega \cap KerL, 0) \neq 0$.

Then $Lx = N(x,1)$ has at least one solution in $\overline{\Omega} \bigcap DomL$.

In what follows, we always assume that (H_3) holds.

Consider $X = Z = V_1 \bigoplus V_2$, $V_1 = \{z(t) = (z_1(t), z_2(t), z_3(t))^T : z_i(t) \in AP(R)$, mod $(z_i(t)) \subseteq$ mod$(F_i), \forall \mu \in \Lambda(z_i(t))$ satisfies $|\mu| \geq \alpha, (i = 1,2,3)\}$, satisfies that $V_1 \cup \{r_i(t), \overline{a}_{i2}(t), \overline{a}_{11}(t), \overline{a}_{21}(t),$ $\overline{a}_{23}(t), \overline{a}_{33}(t), \tau_i(t), \overline{h}_i(t), (i = 1,2,3)\}$ is equipotentially almost periodic. $V_2 = \{z(t) \equiv (c_1, c_2, c_3) \in R^3\}$, where

$$F_1(t,\varphi_1,\varphi_2,\varphi_3) = r_1(t) - \overline{a}_{11}(t)e^{\varphi_1(0)} - \overline{a}_{12}(t)e^{\varphi_2(-\tau_2(t))} - \overline{h}_1(t)e^{-\varphi_1(0)},$$

$$F_2(t,\varphi_1,\varphi_2,\varphi_3) = r_2(t) + \overline{a}_{21}(t)e^{\varphi_1(-\tau_1(t))} - \overline{a}_{22}(t)e^{\varphi_2(0)} - \overline{a}_{23}(t)e^{\varphi_3(-\tau_3(t))} - \overline{h}_2(t)e^{-\varphi_2(0)},$$

$$F_3(t,\varphi_1,\varphi_2,\varphi_3) = r_3(t) + \overline{a}_{32}(t)e^{\varphi_2(-\tau_2(t))} - \overline{a}_{33}(t)e^{\varphi_3(0)} - \overline{h}_3(t)e^{-\varphi_3(0)}.$$

in which $\varphi_i \in C([-\tau,0),R), i=1,2,3, \tau = max\sup_{t\in R}\{\tau_1(t),\tau_2(t),\tau_3(t)\}$ and α is given positive constant. Define

$$\|z\| = \sup_{t\in R}|z_1(t)| + \sup_{t\in R}|z_2(t)| + \sup_{t\in R}|z_3(t)| \quad for\,all\, z \in X = Z.$$

Similar to the proofs of Lemma 3.1, Lemma 3.2 in [19] and Lemma 3.3 in [20], one can easily prove the following three lemmas, respectively.

Lemma 3.2 *X and Z are Banach spaces equipped with the norm* $\|.\|$.

Lemma 3.3 *Let* $L : X \to Z, Lx = x' = (x_1', x_2', x_3')^T$, *then L is a Fredholm mapping of index zero.*

Lemma 3.4 *Let* $N : X\times[0,1] \to Z$, $N(x(t),\lambda) = (N(x_1(t),\lambda), N(x_2(t),\lambda),$ *and* $N(x_3(t),\lambda))^T = (G_1^x, G_2^x, G_3^x)^T$, *where*

$$G_1^x = N(x_1(t),\lambda) = r_1(t) - \overline{a}_{11}(t)e^{x_1(t)} - \lambda\overline{a}_{12}(t)e^{x_2(t-\tau_2(t))} - \overline{h}_1(t)e^{-x_1(t)},$$

$$G_2^x = N(x_2(t),\lambda) = r_2(t) + \lambda\overline{a}_{21}(t)e^{x_1(t-\tau_1(t))} - \overline{a}_{22}(t)e^{x_2(t)} - \lambda\overline{a}_{23}(t)e^{x_3(t-\tau_3(t))} - \overline{h}_2(t)e^{-x_2(t)},$$

$$G_3^x = N(x_3(t),\lambda) = r_3(t) + \lambda\overline{a}_{32}(t)e^{x_2(t-\tau_2(t))} - \overline{a}_{33}(t)e^{x_3(t)} - \overline{h}_3(t)e^{-x_3(t)}$$

and

$$P : X \to X, Px = \Big(m(x_1), m(x_2), m(x_3)\Big)^T,\;\; Q : Z \to Z, Qz = \Big(m(z_1), m(z_2), m(z_3)\Big)^T.$$

Then N is L−compact on $\overline{\Omega}$, *where* Ω *is an open bounded subset of X.*

Theorem 3.1 *Assume that* $(H_1)-(H_3)$ *hold, then system* (1.2) *has at least eight positive almost periodic solutions.*

Proof. By making the substitutions $x_1(t) = exp(u_1(t)), x_2(t) = exp(u_2(t)), x_3(t) = exp(u_3(t))$ then system (2.1) is reformulated as

$$\begin{aligned} u_1'(t) &= r_1(t) - \overline{a}_{11}(t)e^{u_1(t)} - \overline{a}_{12}(t)e^{u_2(t-\tau_2(t))} - \overline{h}_1(t)e^{-u_1(t)},\\ u_2'(t) &= r_2(t) + \overline{a}_{21}(t)e^{u_1(t-\tau_1(t))} - \overline{a}_{22}(t)e^{u_2(t)} - \overline{a}_{23}(t)e^{u_3(t-\tau_3(t))} - \overline{h}_2(t)e^{-u_2(t)}, \qquad (3.1)\\ u_3'(t) &= r_3(t) + \overline{a}_{32}(t)e^{u_2(t-\tau_2(t))} - \overline{a}_{33}(t)e^{u_3(t)} - \overline{h}_3(t)e^{-u_3(t)}. \end{aligned}$$

Then if there exists almost periodic solution $(u_1(t),u_2(t),u_3(t))^T$ of (3.1) , We can get at least one positive almost periodic solutions $(x_1(t),x_2(t),x_3(t))^T$ of (2.1).

In order to use Lemma 3.1, we have to find at least eight appropriate open bounded subsets X. Corresponding to the operator equation $Lx = \lambda N(x,\lambda), \lambda \in (0,1)$, we have

$$\begin{aligned} u_1'(t) &= \lambda\Big(r_1(t) - \overline{a}_{11}(t)e^{u_1(t)} - \lambda\overline{a}_{12}(t)e^{u_2(t-\tau_2(t))} - \overline{h}_1(t)e^{-u_1(t)}\Big),\\ u_2'(t) &= \lambda\Big(r_2(t) + \lambda\overline{a}_{21}(t)e^{u_1(t-\tau_1(t))} - \overline{a}_{22}(t)e^{u_2(t)} - \lambda\overline{a}_{23}(t)e^{u_3(t-\tau_3(t))} - \overline{h}_2(t)e^{-u_2(t)}\Big),\\ u_3'(t) &= \lambda\Big(r_3(t) + \lambda\overline{a}_{32}(t)e^{u_2(t-\tau_2(t))} - \overline{a}_{33}(t)e^{u_3(t)} - \overline{h}_3(t)e^{-u_3(t)}\Big). \end{aligned} \tag{3.2}$$

Assume that $u = (u_1,u_2,u_3)^T \in X$ is an almost periodic solution of system (3.2) for some $\lambda \in (0,1)$. Then by Lemma 2.4, for any $\varepsilon > 0$ and $a \in R$ there exist $\xi_i, \eta_i \in [a, a+l(\varepsilon)] \cap T(u,\varepsilon), i = 1,2,3$ such that $u_i(\xi_i) > u_i^M - \varepsilon, u_i(\eta_i) < u_i^l + \varepsilon$ and $u_i'(\xi_i) = u_i'(\eta_i) = 0$. From this and (3.2), we have

$$\begin{aligned} &r_1(\xi_1) - \overline{a}_{11}(\xi_1)e^{u_1(\xi_1)} - \lambda\overline{a}_{12}(\xi_1)e^{u_2(\xi_1-\tau_2(\xi_1))} - \overline{h}_1(\xi_1)e^{-u_1(\xi_1)} = 0, (a)\\ &r_2(\xi_2) + \lambda\overline{a}_{21}(\xi_2)e^{u_1(\xi_2-\tau_1(\xi_2))} - \overline{a}_{22}(\xi_2)e^{u_2(\xi_2)} - \lambda\overline{a}_{23}(\xi_2)e^{u_3(\xi_2-\tau_3(\xi_2))} - \overline{h}_2(\xi_2)e^{-u_2(\xi_2)} = 0, (b)\\ &r_3(\xi_3) + \lambda\overline{a}_{32}(\xi_3)e^{u_2(\xi_3-\tau_2(\xi_3))} - \overline{a}_{33}(\xi_3)e^{u_3(\xi_3)} - \overline{h}_3(\xi_3)e^{-u_3(\xi_3)} = 0, (c) \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &r_1(\eta_1) - \overline{a}_{11}(\eta_1)e^{u_1(\eta_1)} - \lambda\overline{a}_{12}(\eta_1)e^{u_2(\eta_1-\tau_2(\eta_1))} - \overline{h}_1(\eta_1)e^{-u_1(\eta_1)} = 0, (a)\\ &r_2(\eta_2) + \lambda\overline{a}_{21}(\eta_2)e^{u_1(\eta_2-\tau_1(\eta_2))} - \overline{a}_{22}(\eta_2)e^{u_2(\eta_2)} - \lambda\overline{a}_{23}(t)e^{u_3(\eta_2-\tau_3(\eta_2))} - \overline{h}_2(\eta_2)e^{-u_2(\eta_2)} = 0, (b)\\ &r_3(\eta_3) + \lambda\overline{a}_{32}(\eta_3)e^{u_2(\eta_3-\tau_2(\eta_3))} - \overline{a}_{33}(\eta_3)e^{u_3(\eta_3)} - \overline{h}_3(\eta_3)e^{-u_3(\eta_3)} = 0. (c) \end{aligned} \tag{3.4}$$

On the one hand, according to equation (a) of (3.3), we have

$$r_1(\xi_1) - \overline{a}_{11}(\xi_1)e^{u_1(\xi_1)} - \overline{h}_1(\xi_1)e^{-u_1(\xi_1)} = \lambda\overline{a}_{12}(\xi_1)e^{u_2(\xi_1-\tau_2(\xi_1))} > 0,$$

then, we have

$$\overline{a}_{11}^l e^{2u_1(\xi_1)} - r_1^M e^{u_1(\xi_1)} + \overline{h}_1^l \le \overline{a}_{11}(\xi_1)e^{2u_1(\xi_1)} - r_1(\xi_1)e^{u_1(\xi_1)} + \overline{h}_1(\xi_1) < 0,$$

namely,

$$\overline{a}_{11}^l e^{2u_1(\xi_1)} - r_1^M e^{u_1(\xi_1)} + \overline{h}_1^l < 0,$$

which implies that

$$\ln\frac{r_1^M-\sqrt{(r_1^M)^2-4\overline{a}_{11}^l\overline{h}_1^l}}{2\overline{a}_{11}^l}<u_1(\xi_1)<\ln\frac{r_1^M+\sqrt{(r_1^M)^2-4\overline{a}_{11}^l\overline{h}_1^l}}{2\overline{a}_{11}^l},$$

namely,

$$\ln l_1^- < u_1(\xi_1) < \ln l_1^+. \tag{3.5}$$

Similarly, from the equation (a) of (3.4), we obtain

$$\ln l_1^- < u_1(\eta_1) < \ln l_1^+. \tag{3.6}$$

From (3.5) and the equation (b) of (3.3), we obtain

$$\begin{aligned}\overline{a}_{22}^l e^{u_2(\xi_2)}+\overline{h}_2^l e^{-u_2(\xi_2)} &\leq \overline{a}_{22}(\xi_2)e^{u_2(\xi_2)}+\overline{h}_2(\xi_2)e^{-u_2(\xi_2)}\\ &< r_2(\xi_2)+\lambda\overline{a}_{21}(\xi_2)e^{u_1(\xi_2-\tau_1(\xi_2))}\\ &< r_2^M+a_{21}^M l_1^+,\end{aligned}$$

that is

$$\overline{a}_{22}^l e^{2u_2(\xi_2)}-(r_2^M+\overline{a}_{21}^M l_1^+)e^{u_2(\xi_2)}+\overline{h}_2^l<0,$$

which implies that

$$\ln\frac{(r_2^M+\overline{a}_{22}^M l_1^+)-\sqrt{(r_2^M+\overline{a}_{22}^M l_1^+)^2-4\overline{a}_{22}^l\overline{h}_2^l}}{2\overline{a}_{22}^l}<u_2(\xi_2)<\ln\frac{(r_2^M+\overline{a}_{22}^M l_1^+)+\sqrt{(r_2^M+\overline{a}_{22}^M l_1^+)^2-4\overline{a}_{22}^l\overline{h}_2^l}}{2\overline{a}_{22}^l},$$

namely,

$$\ln l_2^- < u_2(\xi_2) < \ln l_2^+. \tag{3.7}$$

Similarly from the equation (b) of (3.4), we obtain

$$\ln l_2^- < u_2(\eta_2) < \ln l_2^+. \tag{3.8}$$

From (3.7) and the equation (c) of (3.3), we obtain

$$\begin{aligned}\overline{a}_{33}^l e^{u_3(\xi_3)}+\overline{h}_3^l e^{-u_3(\xi_3)} &\leq \overline{a}_{33}(\xi_3)e^{u_3(\xi_3)}+\overline{h}_3(\xi_3)e^{-u_3(\xi_3)}\\ &= r_3(\xi_3)+\lambda\overline{a}_{32}(\xi_3)e^{u_2(\xi_3-\tau_2(\xi_3))}\\ &< r_3^M+a_{32}^M l_2^+,\end{aligned}$$

that is

$$\overline{a}_{33}^{l}e^{2u_3(\xi_3)}-(r_3^M+\overline{a}_{32}^{M}l_2^{+})e^{u_3(\xi_3)}+\overline{h}_3^{l}<0,$$

which implies that

$$\ln\frac{(r_3^M+\overline{a}_{32}^{M}l_2^{+})-\sqrt{(r_3^M+\overline{a}_{32}^{M}l_2^{+})^2-4\overline{a}_{33}^{l}\overline{h}_3^{l}}}{2\overline{a}_{33}^{l}}<u_3(\xi_3)<\ln\frac{(r_3^M+\overline{a}_{32}^{M}l_2^{+})+\sqrt{(r_3^M+\overline{a}_{32}^{M}l_2^{+})^2-4\overline{a}_{33}^{l}\overline{h}_3^{l}}}{2\overline{a}_{33}^{l}},$$

namely,

$$\ln l_3^{-}<u_3(\xi_3)<\ln l_3^{+}. \tag{3.9}$$

Similarly from the equation (c) of (3.4), we obtain

$$\ln l_3^{-}<u_3(\eta_3)<\ln l_3^{+}. \tag{3.10}$$

On the other hand, by the equation (a) of (3.3) and (3.7), we obtain

$$\begin{aligned} r_1^l\le r_1(\xi_1) &= \overline{a}_{11}(\xi_1)e^{u_1(\xi_1)}+\lambda\overline{a}_{12}(\xi_1)e^{u_2(\xi_1-\tau_2(\xi_1))}+\overline{h}_1(\xi_1)e^{-u_1(\xi_1)}\\ &< \overline{a}_{11}^{M}e^{u_1(\xi_1)}+\overline{a}_{12}^{M}l_2^{+}+\overline{h}_1^{M}e^{-u_1(\xi_1)}, \end{aligned}$$

so, we have

$$\overline{a}_{11}^{M}e^{2u_1(\xi_1)}-(r_1^l-\overline{a}_{12}^{M}l_2^{+})e^{u_1(\xi_1)}+\overline{h}_1^{M}>0,$$

which imply that

$$\begin{aligned} u_1(\xi_1)&>\ln\frac{(r_1^l-\overline{a}_{12}^{M}l_2^{+})+\sqrt{(r_1^l-\overline{a}_{12}^{M}l_2^{+})^2-4\overline{a}_{11}^{M}\overline{h}_1^{M}}}{2\overline{a}_{11}^{M}}=\ln A_1^{+},\\ u_1(\xi_1)&<\ln\frac{(r_1^l-\overline{a}_{12}^{M}l_2^{+})-\sqrt{(r_1^l-\overline{a}_{12}^{M}l_2^{+})^2-4\overline{a}_{11}^{M}\overline{h}_1^{M}}}{2\overline{a}_{11}^{M}}=\ln A_1^{-}. \end{aligned} \tag{3.11}$$

Similarly, we can obtain from the equation (a) of (3.4) that

$$\begin{aligned} u_1(\eta_1)&>\ln\frac{(r_1^l-\overline{a}_{12}^{M}l_2^{+})+\sqrt{(r_1^l-\overline{a}_{12}^{M}l_2^{+})^2-4\overline{a}_{11}^{M}\overline{h}_1^{M}}}{2\overline{a}_{11}^{M}}=\ln A_1^{+},\\ u_1(\eta_1)&<\ln\frac{(r_1^l-\overline{a}_{12}^{M}l_2^{+})-\sqrt{(r_1^l-\overline{a}_{12}^{M}l_2^{+})^2-4\overline{a}_{11}^{M}\overline{h}_1^{M}}}{2\overline{a}_{11}^{M}}=\ln A_1^{-}. \end{aligned} \tag{3.12}$$

By the equation (b) of (3.3) and (3.9), we obtain

$$\begin{aligned} r_2^l &\leq r_2(\xi_2) \\ &= -\lambda \overline{a}_{21}(\xi_2)e^{u_1(\xi_2-\tau_1(\xi_2))}+\overline{a}_{22}(\xi_2)e^{u_2(\xi_2)}+\lambda \overline{a}_{23}(\xi_2)e^{u_3(\xi_2-\tau_3(\xi_2))}+\overline{h}_2(\xi_2)e^{-u_2(\xi_2)} \\ &< \overline{a}_{22}^M e^{u_2(\xi_2)}+\overline{a}_{23}^M l_3^+ +\overline{h}_2^M e^{-u_2(\xi_2)}, \end{aligned}$$

so, we have

$$\overline{a}_{22}^M e^{2u_1(\xi_1)}-(r_2^l-\overline{a}_{23}^M l_3^+)e^{u_2(\xi_2)}+\overline{h}_2^M>0,$$

which imply that

$$\begin{aligned} u_2(\xi_1) &> \ln\frac{(r_2^l-\overline{a}_{23}^M l_3^+)+\sqrt{(r_2^l-\overline{a}_{23}^M l_3^+)^2-4\overline{a}_{22}^M\overline{h}_2^M}}{2\overline{a}_{22}^M}=\ln A_2^+, \\ u_2(\xi_1) &< \ln\frac{(r_2^l-\overline{a}_{23}^M l_3^+)-\sqrt{(r_2^l-\overline{a}_{23}^M l_3^+)^2-4\overline{a}_{22}^M\overline{h}_2^M}}{2\overline{a}_{22}^M}=\ln A_2^-. \end{aligned} \tag{3.13}$$

Similarly, we can obtain from the equation (b) of (3.4) that

$$\begin{aligned} u_2(\eta_2) &> \ln\frac{(r_2^l-\overline{a}_{23}^M l_3^+)+\sqrt{(r_2^l-\overline{a}_{23}^M l_3^+)^2-4\overline{a}_{22}^M\overline{h}_2^M}}{2\overline{a}_{22}^M}=\ln A_2^+, \\ u_2(\eta_2) &< \ln\frac{(r_2^l-\overline{a}_{23}^M l_3^+)-\sqrt{(r_2^l-\overline{a}_{23}^M l_3^+)^2-4\overline{a}_{22}^M\overline{h}_2^M}}{2\overline{a}_{22}^M}=\ln A_2^-. \end{aligned} \tag{3.14}$$

By the equation (c) of (3.3), we obtain

$$\begin{aligned} r_3^l \leq r_3(\xi_3) &= -\lambda \overline{a}_{32}(\xi_3)e^{u_2(\xi_3-\tau_2(\xi_2))}+\overline{a}_{33}(\xi_3)e^{u_3(\xi_3)}+\overline{h}_3(\xi_3)e^{-u_3(\xi_3)} \\ &< \overline{a}_{33}^M e^{u_3(\xi_3)}+\overline{h}_3^M e^{-u_3(\xi_3)}, \end{aligned}$$

so, we have

$$\overline{a}_{33}^M e^{2u_3(\xi_3)}-r_3^l e^{u_3(\xi_3)}+\overline{h}_3^M>0,$$

which imply that

$$\begin{aligned} u_3(\xi_3) &> \ln\frac{r_3^l+\sqrt{(r_3^l)^2-4\overline{a}_{33}^M\overline{h}_3^M}}{2\overline{a}_{33}^M}=\ln A_3^+, \\ u_3(\xi_3) &< \ln\frac{r_3^l-\sqrt{(r_3^l)^2-4\overline{a}_{33}^M\overline{h}_3^M}}{2\overline{a}_{33}^M}=\ln A_3^-. \end{aligned} \tag{3.15}$$

Similarly, we can obtain from the equation (c) of (3.4) that

$$\begin{aligned} u_3(\eta_3) &> \ln \frac{r_3^l + \sqrt{(r_3^l)^2 - 4\overline{a}_{33}^M \overline{h}_3^M}}{2\overline{a}_{33}^M} = \ln A_3^+, \\ u_3(\eta_3) &< \ln \frac{r_3^l - \sqrt{(r_3^l)^2 - 4\overline{a}_{33}^M \overline{h}_3^M}}{2\overline{a}_{33}^M} = \ln A_3^-. \end{aligned} \tag{3.16}$$

It follows from (3.5)-(3.10),(3.11)-(3.16) and Lemma 2.7, we get

$$\ln l_1^- < u_1(t) < \ln A_1^- \quad or \quad \ln A_1^+ < u_1(t) < \ln l_1^+,$$

$$\ln l_2^- < u_2(t) < \ln A_2^- \quad or \quad \ln A_2^+ < u_2(t) < \ln l_2^+,$$

$$\ln l_3^- < u_3(t) < \ln A_3^- \quad or \quad \ln A_3^+ < u_3(t) < \ln l_3^+.$$

Clearly, $\ln l_1^\pm, \ln l_2^\pm, \ln l_3^\pm, \ln A_1^\pm, \ln A_2^\pm, \ln A_3^\pm$, are independent of λ. We denote

$$\Omega_1 = \{u = (u_1,u_2,u_3)^T \in X | u_1(t) \in (\ln l_1^-, \ln A_1^-), u_2(t) \in (\ln l_2^-, lnA_2^-), u_3(t) \in (\ln l_3^-, lnA_3^-)\},$$

$$\Omega_2 = \{u = (u_1,u_2,u_3)^T \in X | u_1(t) \in (\ln l_1^-, \ln A_1^-), u_2(t) \in (\ln l_2^-, lnA_2^-), u_3(t) \in (\ln l_3^+, lnA_3^+)\},$$

$$\Omega_3 = \{u = (u_1,u_2,u_3)^T \in X | u_1(t) \in (\ln l_1^-, \ln A_1^-), u_2(t) \in (\ln l_2^+, lnA_2^+), u_3(t) \in (\ln l_3^-, lnA_3^-)\},$$

$$\Omega_4 = \{u = (u_1,u_2,u_3)^T \in X | u_1(t) \in (\ln l_1^-, \ln A_1^-), u_2(t) \in (\ln l_2^+, lnA_2^+), u_3(t) \in (\ln l_3^+, lnA_3^+)\},$$

$$\Omega_5 = \{u = (u_1,u_2,u_3)^T \in X | u_1(t) \in (\ln l_1^+, \ln A_1^+), u_2(t) \in (\ln l_2^+, lnA_2^+), u_3(t) \in (\ln l_3^-, lnA_3^-)\},$$

$$\Omega_6 = \{u = (u_1,u_2,u_3)^T \in X | u_1(t) \in (\ln l_1^+, \ln A_1^+), u_2(t) \in (\ln l_2^-, lnA_2^-), u_3(t) \in (\ln l_3^-, lnA_3^-)\},$$

$$\Omega_7 = \{u = (u_1,u_2,u_3)^T \in X | u_1(t) \in (\ln l_1^+, \ln A_1^+), u_2(t) \in (\ln l_2^-, lnA_2^-), u_3(t) \in (\ln l_3^+, lnA_3^+)\},$$

$$\Omega_8 = \{u = (u_1,u_2,u_3)^T \in X | u_1(t) \in (\ln l_1^+, \ln A_1^+), u_2(t) \in (\ln l_2^+, lnA_2^+), u_3(t) \in (\ln l_3^+, lnA_3^+)\}.$$

Thus $\Omega_k, k = 1,2,3,4,5,6,7,8$ are bounded open subsets of X, $\Omega_i \cap \Omega_j = \emptyset, i \neq j$. Thus Ω_k satisfies the requirement (a) in Lemma 3.1.

Now we show that (b) of Lemma 3.1 holds, i.e., we prove when $u \in \partial\Omega_i \cap KerL = \partial\Omega_i \cap R^3, QN(u,0) \neq (0,0,0)^T, i = 1,2,3,4,5,6,7,8$. If it is not true, then when $u \in \partial\Omega_i \cap KerL =$

$\partial\Omega_i \cap R^3, i = 1,2,3,4,5,6,7,8$. constant vector $u = (u_1, u_2, u_3)^T$ with $u \in \partial\Omega_i, i = 1,2,3,4,5,6,7,8$ satisfies

$$m\Big(r_1(t) - \bar{a}_{11}(t)e^{u_1} - \bar{h}_1(t)e^{-u_1}\Big) = 0,$$
$$m\Big(r_2(t) - \bar{a}_{22}(t)e^{u_2} - \bar{h}_2(t)e^{-u_2}\Big) = 0,$$
$$m\Big(r_3(t) - \bar{a}_{33}(t)e^{u_3} - \bar{h}_3(t)e^{-u_3}\Big) = 0.$$

In view of the mean value theorem of calculous, there exist three points $\zeta_1, \zeta_2, \zeta_3$ such that

$$\begin{aligned} &r_1(\zeta_1) - \bar{a}_{11}(\zeta_1)e^{u_1} - \bar{h}_1(\zeta_1)e^{-u_1} = 0, \\ &r_2(\zeta_2) - \bar{a}_{22}(\zeta_2)e^{u_2} - \bar{h}_2(\zeta_2)e^{-u_2} = 0, \\ &r_3(\zeta_3) - \bar{a}_{33}(\zeta_3)e^{u_3} - \bar{h}_3(\zeta_3)e^{-u_3} = 0. \end{aligned} \tag{3.17}$$

From (3.17), we have

$$\begin{aligned} u_1^{\pm} &= \ln \frac{r_1(\zeta_1) \pm \sqrt{(r_1(\zeta_1))^2 - 4\bar{a}_{11}(\zeta_1)\bar{h}_1(\zeta_1)}}{2\bar{a}_{11}(\zeta_1)}, \\ u_2^{\pm} &= \ln \frac{r_2(\zeta_2) \pm \sqrt{(r_2(\zeta_2))^2 - 4\bar{a}_{22}(\zeta_2)\bar{h}_2(\zeta_2)}}{2\bar{a}_{22}(\zeta_2)}, \\ u_3^{\pm} &= \ln \frac{r_3(\zeta_3) \pm \sqrt{(r_3(\zeta_3))^2 - 4\bar{a}_{33}(\zeta_3)\bar{h}_3(\zeta_3)}}{2\bar{a}_{33}(\zeta_3)}. \end{aligned} \tag{3.18}$$

According to Lemma 2.7, we obtain

$$\begin{aligned} &\ln l_1^- < \ln u_1^- < \ln A_1^- < \ln A_1^+ < \ln u_1^+ < \ln l_1^+, \\ &\ln l_2^- < \ln u_2^- < \ln A_2^- < \ln A_2^+ < \ln u_2^+ < \ln l_2^+, \\ &\ln l_3^- < \ln u_3^- < \ln A_3^- < \ln A_3^+ < \ln u_3^+ < \ln l_3^+. \end{aligned} \tag{3.19}$$

Then u belongs to one of $\Omega_i \cap R^3, i = 1,2,3,4,5,6,7,8$. This contradicts the fact that $u \in \partial\Omega_i \cap R^3, i = 1,2,3,4,5,6,7,8$. This proves (b) in Lemma 3.1 holds.

Finally, we show that (c) in Lemma 3.1 holds. Note that the system of algebraic equations

$$r_1(\zeta_1)-\overline{a}_{11}(\zeta_1)e^{x}-\overline{h}_1(\zeta_1)e^{-x}=0,$$

$$r_2(\zeta_2)-\overline{a}_{22}(\zeta_2)e^{y}-\overline{h}_2(\zeta_2)e^{-y}=0,$$

$$r_3(\zeta_3)-\overline{a}_{33}(\zeta_3)e^{z}-\overline{h}_3(\zeta_3)e^{-z}=0,$$

has eight distinct solutions since H_1 holds:

$$(x_1^*,y_1^*,z_1^*)=(\ln x_-,\ln y_-,\ln z_-),\ \ (x_2^*,y_2^*,z_2^*)=(\ln x_-,\ln y_-,\ln y_+),$$

$$(x_3^*,y_3^*,y_3^*)=(\ln x_-,\ln y_+,\ln z_-),\ \ (x_4^*,y_4^*,z_4^*)=(\ln x_-,\ln y_+,\ln z_+),$$

$$(x_5^*,y_5^*,z_5^*)=(\ln x_+,\ln y_+,\ln z_-),\ \ (x_6^*,y_6^*,z_6^*)=(\ln x_+,\ln y_-,\ln y_-),$$

$$(x_7^*,y_7^*,y_7^*)=(\ln x_+,\ln y_-,\ln z_+),\ \ (x_8^*,y_8^*,z_8^*)=(\ln x_+,\ln y_+,\ln z_+),$$

where

$$x_\pm=\frac{r_1(\zeta_1)\pm\sqrt{(r_1(\zeta_1))^2-4\overline{a}_{11}(\zeta_1)\overline{h}_1(\zeta_1)}}{2\overline{a}_{11}(\zeta_1)},$$

$$y_\pm=\frac{r_2(\zeta_2)\pm\sqrt{(r_2(\zeta_2))^2-4\overline{a}_{22}(\zeta_2)\overline{h}_2(\zeta_2)}}{2\overline{a}_{22}(\zeta_2)},$$

$$z_\pm=\frac{r_3(\zeta_3)\pm\sqrt{(r_3(\zeta_3))^2-4\overline{a}_{33}(\zeta_3)\overline{h}_3(\zeta_3)}}{2\overline{a}_{33}(\zeta_3)}.$$

From $(3.18),(3.19)$, we have

$$(x_1^*,y_1^*,z_1^*)\in\Omega_1,\ \ (x_2^*,y_2^*,z_2^*)\in\Omega_2,\ \ (x_3^*,y_3^*,z_3^*)\in\Omega_3,\ \ (x_4^*,y_4^*,z_4^*)\in\Omega_4,$$

$$(x_5^*,y_5^*,z_5^*)\in\Omega_5,\ \ (x_6^*,y_6^*,z_6^*)\in\Omega_6,\ \ (x_7^*,y_7^*,z_7^*)\in\Omega_7,\ \ (x_8^*,y_8^*,z_8^*)\in\Omega_8.$$

Since $KerL = ImQ$, we can take $J = I$. A direct computation gives, we get

$$deg\{JQN(u,0),\Omega_i \cap KerL,(0,0,0)^T\}$$
$$= sign\begin{vmatrix} -\overline{a}_{11}(\zeta_1)x^* + \dfrac{\overline{h}_1(\zeta_1)}{x^*} & 0 & 0 \\ 0 & -\overline{a}_{22}(\zeta_2)y^* + \dfrac{\overline{h}_2(\zeta_2)}{y^*} & 0 \\ 0 & 0 & -\overline{a}_{33}(\zeta_3)z^* + \dfrac{\overline{h}_3(\zeta_3)}{z^*} \end{vmatrix}$$

$$= sign[(-\overline{a}_{11}(\zeta_1)x^* + \frac{\overline{h}_1(\zeta_1)}{x^*})(-\overline{a}_{22}(\zeta_2)y^* + \frac{\overline{h}_2(\zeta_2)}{y^*})(-\overline{a}_{33}(\zeta_3)z^* + \frac{\overline{h}_3(\zeta_3)}{z^*})].$$

Since

$$r_1(\zeta_1) - \overline{a}_{11}(\zeta_1)x^* - \frac{\overline{h}_1(\zeta_1)}{x^*} = 0,$$

$$r_2(\zeta_2) - \overline{a}_{22}(\zeta_2)y^* - \frac{\overline{h}_2(\zeta_2)}{y^*} = 0,$$

$$r_3(\zeta_3) - \overline{a}_{33}(\zeta_3)z^* - \frac{\overline{h}_3(\zeta_3)}{z^*} = 0,$$

then

$$deg\{JQN(u,0),\Omega_i \cap KerL,(0,0,0)^T\}$$

$$= sign[(r_1(\zeta_1) - 2\overline{a}_{11}(\zeta_1)x^*)(r_2(\zeta_2) - 2\overline{a}_{22}(\zeta_2)y^*)(r_3(\zeta_3) - 2\overline{a}_{33}(\zeta_3)y^*)], i = 1,2,3,4,5,6,7,8.$$

Thus

$$deg\{JQN(u,0),\Omega_1 \cap KerL,(0,0,0)^T\} = -1, i = 1,4,5,7,$$
$$deg\{JQN(u,0),\Omega_2 \cap KerL,(0,0,0)^T\} = 1,\ i = 2,3,6,8.$$

namely,

$$deg\{JQN(u,0),\Omega_i \cap KerL,(0,0,0)^T\} \neq 0, i = 1,2,3,4,5,6,7,8.$$

So far, we have proved that $\Omega_k, k = 1,2,3,4,5,6,7,8$ satisfies all the assumptions in Lemma 3.1. Hence, system (3.1) has at least 8 different almost periodic solutions. So, system (2.1) has at least 8 different positive almost periodic solutions. If $(\overline{x}(t),\overline{y}(t),\overline{z}(t))^T$ is an almost periodic solution of system (2.1), by using Lemma 2.5, we know that

$$\Big(x(t) = \prod_{0<t_k<t}(1+\Gamma_{1k})\overline{x}(t), y(t) = \prod_{0<t_k<t}(1+\Gamma_{2k})\overline{y}(t), z(t) = \prod_{0<t_k<t}(1+\Gamma_{3k})\overline{z}(t)\Big)^T$$

is a solution of system (1.2). Therefore, system (1.2) has at least 8 different positive almost periodic solutions. This completes the proof of Theorem 3.1.

Consider the following non-autonomous Lotka-Volterra predator-prey system with harvesting terms

$$\begin{aligned} x_1'(t) &= x_1(t)\Big(r_1(t) - a_{11}(t)x_1(t) - a_{12}(t)x_2(t-\tau_2(t))\Big) - h_1(t), \\ x_2'(t) &= x_2(t)\Big(r_2(t) + a_{21}(t)x_1(t-\tau_1(t)) - a_{22}(t)x_2(t) - a_{23}(t)x_3(t-\tau_3(t))\Big) - h_2(t), \\ x_3'(t) &= x_3(t)\Big(r_3(t) + a_{32}(t)x_2(t-\tau_2(t)) - a_{33}(t)x_3(t)\Big) - h_3(t). \end{aligned} \tag{3.20}$$

Similar to the proof of Theorem 3.1, one can easily obtain

Corollary 3.1 *Assume that the following condition holds*

(H_1') $r_1^l - a_{12}^M l_2^+ > 2\sqrt{a_{11}^M h_1^M}$, $r_2^l - a_{23}^M l_3^+ > 2\sqrt{a_{22}^M h_2^M}$, $r_3^l > 2\sqrt{a_{33}^M h_3^M}$. *Then system(3.20) has at least* 8 *different positive almost periodic solutions. Since condition unrelated to delays, thus, if* $\tau_i(t) \equiv 0 (i = 1,2,3)$, *our results also supplement the results of Liu and Wei (see[1]).*

4. An example

Consider the following three species non-autonomous Lotka-Volterra predator-prey with with impulsive and harvesting terms:

$$\begin{aligned} x_1'(t) &= x_1(t)\Big(3 + \sin\sqrt{2}t - \frac{8+2\cos t}{17}x_1(t) - \frac{2+\cos t}{90}x_2(t-5|\sin t|)\Big) - \frac{153+17\cos\sqrt{5}t}{400}, t \neq t_k, \\ x_2'(t) &= x_2(t)\Big(4 + \cos\sqrt{3}t + \frac{4+2\sin t}{17}x_1(t-3|\sin t|) - \frac{6+\cos t}{9}x_2(t) - \frac{2+\cos t}{88}x_3(t-2|\sin t|)\Big) \\ &\quad - \frac{18+9\cos t}{50}, t \neq t_k, \\ x_3'(t) &= x_3(t)\Big(3 + \cos\sqrt{2}t + \frac{2+\sin t}{9}x_2(t-5|\sin t|) - \frac{30+5\cos t}{44}x_3(t)\Big) - \frac{44+22\cos\sqrt{3}t}{125}, t \neq t_k, \\ x_1(t_k^+) &= (1+(-0.15))x_1(t_k), t = t_k, \\ x_2(t_k^+) &= (1+(-0.1))x_2(t_k), t = t_k, \\ x_3(t_k^+) &= (1+(-0.12))x_3(t_k), t = t_k. \end{aligned} \tag{4.1}$$

In this case, $r_1(t)=3+\sin\sqrt{2}t$, $a_{11}(t)=\dfrac{8+2\cos t}{17}$, $a_{12}(t)=\dfrac{2+\cos t}{90}$, $h_1(t)=\dfrac{153+17\cos\sqrt{5}t}{400}$, $r_2(t)=4+\cos\sqrt{3}t$, $a_{21}(t)=\dfrac{4+2\sin t}{17}$, $a_{22}(t)=\dfrac{6+\cos t}{9}$, $a_{23}(t)=\dfrac{2+\cos t}{88}$, $h_2(t)=\dfrac{18+9\cos t}{50}$, $r_3(t)=3+\cos\sqrt{2}t$, $a_{32}(t)=\dfrac{2+\sin t}{9}$, $a_{33}(t)=\dfrac{30+5\cos t}{44}$, $h_3(t)=\dfrac{44+22\cos\sqrt{3}t}{125}$. $\tau_1(t)=3|\sin t|$, $\tau_2(t)=5|\sin t|$, $\tau_3(t)=2|\sin t|$. Then, we have $\overline{a}_{11}(t)=a_{11}(t)\prod\limits_{0<t_k<t}(1+\Gamma_{1k})=\dfrac{8+2\cos t}{17}(1+(-0.15))=\dfrac{4+\cos t}{10}$, $\overline{a}_{12}(t)=a_{12}(t)\prod\limits_{0<t_k<t}(1+\Gamma_{2k})=\dfrac{2+\cos t}{90}(1+(-0.1))=\dfrac{2+\cos t}{100}$, $\overline{h}_1(t)=h_1(t)\prod\limits_{0<t_k<t}(1+\Gamma_{1k})^{-1}=\dfrac{153+17\cos\sqrt{5}t}{400}(1+(-0.15))^{-1}=\dfrac{9+\cos\sqrt{5}t}{20}$, $\overline{a}_{21}(t)=a_{21}(t)\prod\limits_{0<t_k<t}(1+\Gamma_{1k})=\dfrac{4+2\sin t}{17}(1+(-0.15))=\dfrac{2+\sin t}{10}$, $\overline{a}_{22}(t)=a_{22}(t)\prod\limits_{0<t_k<t}(1+\Gamma_{2k})=\dfrac{6+\cos t}{9}(1+(-0.1))=\dfrac{6+\cos t}{10}$, $\overline{a}_{23}(t)=a_{23}(t)\prod\limits_{0<t_k<t}(1+\Gamma_{3k})=\dfrac{2+\cos t}{88}(1+(-0.12))=\dfrac{2+\cos t}{100}$, $\overline{h}_2(t)=h_2(t)\prod\limits_{0<t_k<t}(1+\Gamma_{2k})^{-1}=\dfrac{18+9\cos t}{50}(1+(-0.1))^{-1}=\dfrac{2+\cos t}{5}$, $\overline{a}_{32}(t)=a_{32}(t)\prod\limits_{0<t_k<t}(1+\Gamma_{2k})=\dfrac{2+\sin t}{9}(1+(-0.1))=\dfrac{2+\sin t}{10}$, $\overline{a}_{33}(t)=a_{33}(t)\prod\limits_{0<t_k<t}(1+\Gamma_{3k})=\dfrac{30+5\cos t}{44}(1+(-0.12))=\dfrac{6+\cos t}{10}$, $\overline{h}_3(t)=h_3(t)\prod\limits_{0<t_k<t}(1+\Gamma_{3k})^{-1}=\dfrac{44+22\cos\sqrt{3}t}{125}(1+(-0.12))^{-1}=\dfrac{2+\cos\sqrt{3}t}{5}$. Since

$$l_1^+=\frac{r_1^M+\sqrt{(r_1^M)^2-4\overline{a}_{11}^l\overline{h}_1^l}}{2\overline{a}_{11}^l}=\frac{4+\sqrt{4^2-4\times\frac{3}{10}\times\frac{8}{20}}}{2\times\frac{3}{10}}=\frac{20+2\sqrt{97}}{3}<\frac{40}{3},$$

$$\begin{aligned}l_2^+&=\frac{(r_2^M+\overline{a}_{22}^M l_1^+)+\sqrt{(r_2^M+\overline{a}_{22}^M l_1^+)^2-4\overline{a}_{22}^l\overline{h}_2^l}}{2\overline{a}_{22}^l}\\&<\frac{(5+\frac{7}{10}\times\frac{40}{3})+\sqrt{(5+\frac{7}{10}\times\frac{40}{3})^2-4\times\frac{5}{10}\times\frac{1}{5}}}{2\times\frac{5}{10}}<30,\end{aligned}$$

$$\begin{aligned}l_3^+&=\frac{(r_3^M+\overline{a}_{32}^M l_2^+)+\sqrt{(r_3^M+\overline{a}_{32}^M l_2^+)^2-4\overline{a}_{33}^l\overline{h}_3^l}}{2\overline{a}_{33}^l}\\&=\frac{(4+\frac{3}{10}\times 20)+\sqrt{(4+\frac{3}{10}\times 20)^2-4\times\frac{5}{10}\times\frac{1}{5}}}{2\times\frac{5}{10}}<26,\end{aligned}$$

then

$$r_1^l - \overline{a}_{12}^M l_2^+ > 2 - \frac{3}{100} \times 30 > 2\sqrt{\frac{5}{10} \times \frac{10}{20}} = 2\sqrt{\overline{a}_{11}^M \overline{h}_1^M},$$

$$r_2^l - \overline{a}_{23}^M l_3^+ > 3 - \frac{3}{100} \times 26 > 2\sqrt{\frac{7}{10} \times \frac{3}{5}} = 2\sqrt{\overline{a}_{22}^M \overline{h}_2^M},$$

$$r_3^l = 2 > 2\sqrt{\frac{7}{10} \times \frac{3}{5}} = 2\sqrt{\overline{a}_{33}^M \overline{h}_3^M}.$$

Hence, all conditions of Theorem 3.1 are satisfied, then, the system (4.1) has at least eight positive almost periodic solutions.

Conflict of Interests

The authors declare that there is no conflict of interests.

References

[1] J.F. Liu , F.Y. Wei, Existence of multiple periodic solutions to a periodic predator-prey system with harvesting terms and Lotka-Volterra type functional response, J. Fuqing. Fujian. Norm. Univ. 5 (2010), 1-8. (in Chinese)

[2] K.H. Zhao, Y. Ye, Four positive periodic solutions to a periodic Lotka-Volterra predatory-prey system with harvesting terms, Nonlinear Anal. Real World Appl. 11 (2010), 2448-2455.

[3] G.D. Zhang, Y. Shen, B.S. Chen, Positive periodic solutions in a non-selective harvesting predator-prey model with multiple delays, J. Math. Anal. Appl. 395 (2012), 298-306.

[4] H. Fang, Existence of eight positive periodic solutions for a food-limited two-species cooperative patch system with periodic terms, Commun. Nonlinear Sci. Numer. Simulat. 18 (2013), 1857-1869.

[5] C.J. Wei, L.S. Chen, Periodic solution and heteroclinic bifurcation in a predator-prey system with Allee effect and impulsive harvesting, Nonlinear Dyn. 76 (2014), 1109-1117.

[6] Y.H. Fan, L.L. Wang, Multiplicity of periodic solutions for a delayed ratio-dependent predator-prey model with monotonic functional response and harvesting terms, Appl. Math. Comput. 244 (2014), 878-894.

[7] Y.K. Li, T.W. Zhang, Global asymptotical stability of a unique almost periodic solution for enterprise clusters based on ecology theory with time-varying delays and feedback controls, Commun. Nonlinear Sci. Numer. Simulat. 17 (2012), 904-913.

[8] C. Zhang, N.J. Huang, D. O'Regan, Almost periodic solutions for a Volterra model with mutual interference and Holling type III functional response, Appl. Math. Comput. 225 (2013), 503-511.

[9] Z.J. Liu, Q.L. Wang, An almost periodic competitive system subject to impulsive perturbations, Appl. Math. Comput. 231 (2014), 377-385.

[10] H. Zhang, Y.Q. Li, B. Jing, W.Z. Zhao, Global stability of almost periodic solution of multispecies mutualism system with time delays and impulsive effects, Appl. Math. Comput. 232 (2014), 1138-50.

[11] Z.J. Du, Y.S. Lv, Permanence and almost periodic solution of a Lotka-Volterra model with mutual interference and time delays, Appl. Math. Model. 37 (2013), 1054-1068.

[12] Y.K. Li, L. Yang, W.Q. Wu, Almost periodic solutions for a class of discrete systems with Allee-effect, Appl. Math. 59 (2014), 191-203.

[13] Z. Li, F.D. Chen, M.X. He, Almost periodic solutions of a discrete Lotka-Volterra competition system with delays, Nonlinear Anal. Real World Appl. 12 (2011), 2344-2355.

[14] Z. Li, M.A. Han, F.D. Chen, Almost periodic solutions of a discrete almost periodic logistic equation with delay, Appl. Math. Comput. 232 (2014), 743-751.

[15] C.Y. Niu, X.X. Chen, Almost periodic sequence solutions of a discrete Lotka-Volterra competitive system with feedback control, Nonlinear Anal. Real World Appl. 10 (2009), 3152-3161.

[16] Y.J. Wang, Periodic and almost periodic solutions of a nonlinear single species discrete model with feedback control, Appl. Math. Comput. 219 (2013), 5480-5486.

[17] T.W. Zhang, X.R. Gan, Almost periodic solutions for a discrete fishing model with feedback control and time delays, Commun. Nonlinear Sci. Numer. Simulat. 19 (2014), 150-163.

[18] Y.K. Li, Y. Ye, Multiple positive almost periodic solutions to an impulsive non-autonomous Lotka-Volterra predator-prey system with harvesting terms, Commun. Nonlinear Sci. Numer Simulat. 18 (2013), 3190-3201.

[19] Y. Xie, X.G. Li, Almost periodic solutions of single population model with hereditary, Appl. Math. Comput. 203 (2008), 690-697.

[20] K.H. Zhao, Existence of almost-periodic solutions for Lotka-Volterra cooperative systems with time delay, J. Appl. Math. 2012 (2012), Article ID 270104.

[21] A.M. Samoilenko, N.A. Perestyuk, Impulsive differential equations, Singapore: World Scientific. 1995.

[22] C.Y. He, Almost periodic differential equations, Beijing: Higher Education Publishing House. 1992. (in Chinese)

[23] R. Gaines, J. Mawhin, Coincidence degree and nonlinear differential equations, Berlin: Springer Verlag. 1977.