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DYNAMICS OF A DISCRETIZED *SIR* EPIDEMIC MODEL WITH DISTRIBUTED DELAY AND GENERALIZED SATURATION INCIDENCE

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Abstract. In this paper, a discrete *SIR* epidemic model is proposed, which distributed delay, generalized saturating incidence rate and disease-induced mortality are taken into consideration. This model is constructed from the discretization by the hybrid Euler method. The conditions for global asymptotical stability of the disease-free equilibrium and the permanence of our model are obtained. Finally, a numerical study is performed to illustrate the mathematical findings.

Keywords: Hybrid Euler method; Discrete epidemic model; Time delay; Generalized saturation incidence.

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1. Introduction

Recently, many authors have studied the dynamical behavior of epidemic models (see also [1-15] and the reference therein). In [1], a continuous *SIRS* epidemic model with distributed

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time delay was considered:

$$\begin{cases} S'(t) = \lambda - \mu_1 S(t) - \beta(I)S(t)\int_0^{\omega} I(t-s)d\eta(s) + \delta R(t), \\ I'(t) = \beta(I)S(t)\int_0^{\omega} I(t-s)d\eta(s) - (\mu_2+\gamma)I(t), \\ R'(t) = \gamma I(t) - (\mu_3+\delta)R(t), \end{cases} \tag{1.1}$$

where $S(t)$, $I(t)$ and $R(t)$ represent the numbers of susceptible, infectious and removed individuals at time t, respectively. The nonnegative constants μ_1, μ_2 and μ_3 denote the death rates of the susceptible, infected and recovered classes, respectively. The constant $\lambda > 0$ denotes the immigration rate, assuming all newborns to be susceptible. The constant $\gamma > 0$ is the recovery rate. The recovered class becomes susceptible again at a constant rate $\delta > 0$. Infectiousness is assumed to vary over time from the initial time of infection until a duration ω has passed. $\beta(I)$ is the probability per unit time and the incidence is used with the form $\beta(I)S(t)\int_0^{\omega} I(t-s)d\eta(s)$, which includes distributed time delay. The distributed delay allows infectivity to be a function of the duration since infection, up to some maximum duration. The function $\eta(s)$ is chosen so that it is nonnegative and continuous on $[0,\omega]$. The sufficient conditions for global stability of the equilibria and the permanence of system are obtained.

Owing to the statistical data of epidemic are collected and reported in discrete time, discrete model is more practical significance than continuous model. By applying a variation of backward Euler method, Sekiguchi [2] established the following discrete *SIRS* epidemic model which is derived from system (1.1).

$$\begin{cases} S_{n+1} = \lambda + (1-\mu_1)S_{n+1} - \beta(I_{n+1})S_{n+1}\sum_{k=0}^{\omega-1} I_{n-k}\eta_k + \delta R_n, \\ I_{n+1} = \beta(I_n)S_{n+1}\sum_{k=0}^{\omega-1} I_{n-k}\eta_k + (1-\mu_2)I_n - \gamma I_{n+1}, \\ R_{n+1} = \gamma I_n + (1-\mu_3)R_n - \delta R_{n+1}, \end{cases}$$

where $0 \le \eta_k \le 1$ and $\sum_{k=0}^{\omega-1}\eta_k = 1$.

Comparing with bilinear and proportionate mixing incidence, saturation incidence is may be more suitable for our real word, as contact rate affected by the number of infection individuals. Combining with incidence rate $\beta(I)SI$ which is proposed in [2], in this paper, we consider the generalized saturating incidence rate $\dfrac{\beta(I)S}{1+\alpha(I)S}$.

Motivated by the above facts, we propose the following discrete *SIR* epidemic model:

$$\begin{cases} S_{n+1}-S_n=\lambda-\mu_1 S_{n+1}-\dfrac{\beta(I_n)S_{n+1}}{1+\alpha(I_n)S_{n+1}}\sum\limits_{k=0}^{\omega-1}\eta_k I_{n-k}, \\ I_{n+1}-I_n=\dfrac{\beta(I_n)S_{n+1}}{1+\alpha(I_n)S_{n+1}}\sum\limits_{k=0}^{\omega-1}\eta_k I_{n-k}-(\mu_2+\gamma+c)I_{n+1}, \\ R_{n+1}-R_n=\gamma I_{n+1}-\mu_3 R_{n+1}. \end{cases} \tag{1.2}$$

It is natural to assume that $\mu_2 \geq max\{\mu_1,\mu_3\}$. Further, some assumptions are given as follows:

(H_1): $\beta(I)$ is a positive function and monotonically decreasing for I_n and $\beta(0)=\beta_0>0$.

(H_2): $\alpha(I)$ is a nonnegative function and monotonically increasing for I_n and $\alpha(0)=0$.

The initial conditions of the system (1.2) are given by

$$S_n=\phi_n^{(1)},\ I_n=\phi_n^{(2)},\ R_n=\phi_n^{(3)}\ \text{ for } n=-\omega+1,-\omega+2,\cdots,-1,$$

where $\phi_n^{(i)} \geq 0$ $(n=-\omega+1,-\omega+2,\cdots,-1, i=1,2,3)$ and $\phi_0^{(i)}>0 (i=1,2,3)$.

This paper is organized as follows. In Section 2, positivity and boundedness of the solution of system (1.2) are obtained. In Section 3, we establish conditions for the global asymptotic stability of disease-free equilibrium and discuss the existence of endemic equilibrium. In Section 4, we prove the permanence of the model by using discrete-time analogue of Lyapunov functional techniques. Finally, numerical examples for different epidemic parameters are shown in Section 5.

2. Basic properties

For system (1.2), since the variable R does not appear in the first and the second equations, it is sufficient to consider the following 2-dimensional system:

$$\begin{cases} S_{n+1}-S_n=\lambda-\mu_1 S_{n+1}-\dfrac{\beta(I_n)S_{n+1}}{1+\alpha(I_n)S_{n+1}}\sum\limits_{k=0}^{\omega-1}\eta_k I_{n-k}, \\ I_{n+1}-I_n=\dfrac{\beta(I_n)S_{n+1}}{1+\alpha(I_n)S_{n+1}}\sum\limits_{k=0}^{\omega-1}\eta_k I_{n-k}-dI_{n+1}, \end{cases} \tag{2.1}$$

with the initial conditions

$$S_n=\phi_n^{(1)},\ I_n=\phi_n^{(2)}\ \text{ for } n=-\omega+1,-\omega+2,\cdots,-1,$$

$$\phi_n^{(i)} \geq 0 \ (n = -\omega+1, -\omega+2, \cdots, -1, i = 1,2) \ \text{ and } \phi_0^{(i)} > 0 \ (i = 1,2), \tag{2.2}$$

where $d = \mu_2 + \gamma + c$.

The following results in Section 3 and Section 4 are obtained by applying techniques in Sekiguchi and Ishiwata [15] and Sekiguchi et al. [2] to system (2.1).

At first, we show that solutions of system (2.1) are positive and have upper bound, respectively. We have the following results.

Lemma 2.1. *Let (S_n, I_n) be the solution of model (2.1) with initial conditions (2.2), then (S_n, I_n) is positive for all $n > 0$.*

Proof. Model (2.1) is equivalent to the following form

$$\begin{cases} S_{n+1} = S_n + \lambda - \mu_1 S_{n+1} - \dfrac{\beta(I_n)S_{n+1}}{1+\alpha(I_n)S_{n+1}} \sum\limits_{k=0}^{\omega-1} \eta_k I_{n-k}, \\ I_{n+1} = \dfrac{1}{1+d} I_n + \dfrac{1}{1+d} \dfrac{\beta(I_n)S_{n+1}}{1+\alpha(I_n)S_{n+1}} \sum\limits_{k=0}^{\omega-1} \eta_k I_{n-k}. \end{cases} \tag{2.3}$$

In the following, we will use the induction method to prove the positivity of $(S(n), I(n))$. When $n = 0$, from system (2.3) we have

$$S_1 = S_0 + \lambda - \mu_1 S_1 - \frac{\beta(I_0)S_1}{1+\alpha(I_0)S_1} \sum_{k=0}^{\omega-1} \eta_k I_{-k}, \tag{2.4}$$

$$I_1 = \frac{1}{1+d} I_0 + \frac{1}{1+d} \frac{\beta(I_0)S_1}{1+\alpha(I_0)S_1} \sum_{k=0}^{\omega-1} \eta_k I_{-k}. \tag{2.5}$$

From (2.5), we see that as long as we determine S_1 then I_1 will be confirmed.

Firstly, we prove that if $S_1 > 0$ then it must be have $I_1 > 0$. Let $x = S_1$, from (2.4) we have

$$\phi(x) \equiv x - S_0 - (\lambda - \mu_1 x - \frac{\beta(I_0)x}{1+\alpha(I_0)x} \sum_{k=0}^{\omega-1} \eta_k I_{-k}) = 0.$$

It is obviously that $\phi(x)$ is monotonically increasing with respect to $x \geq 0$. Since $\phi(0) = -S_0 - h(\lambda) < 0$ and $\lim\limits_{x \to +\infty} \phi(x) = +\infty$. Therefore, there exists a unique $\bar{x} > 0$ such that $\phi(\bar{x}) = 0$. This shows that $S_1 = \bar{x} > 0$. From (2.5), we directly obtain $I_1 > 0$.

When $n = 1$, from the system of (2.3) we obtain

$$S_2 = S_1 + \lambda - \mu_1 S_2 - \frac{\beta(I_1)S_2}{1+\alpha(I_1)S_2} \sum_{k=0}^{\omega-1} \eta_k I_{1-k},$$

$$I_2 = \frac{1}{1+d} I_1 + \frac{1}{1+d} \frac{\beta(I_1)S_2}{1+\alpha(I_1)S_2} \sum_{k=0}^{\omega-1} \eta_k I_{1-k}.$$

A similar argument as in the above proof for $S_1 > 0$, $I_1 > 0$, we also can obtain that $S_2 > 0$, $I_2 > 0$. Lastly, by using the induction, we can finally obtain that $S_n > 0$, $I_n > 0$. The proof is completed.

□

Lemma 2.2. *For any solution (S_n, I_n) of system (2.1) with the initial conditions (2.2), the total number of the population $N_n = S_n + I_n$ satisfies*

$$\limsup_{n\to+\infty} N_n \leq \frac{\lambda}{\mu}, \ \ \textit{where} \ \ \mu = min\{\mu_1, d\}.$$

Proof. From system (2.1) we have

$$N_{n+1} - N_n = \lambda - \mu_1 S_{n+1} - dI_{n+1} \leq \lambda - \mu N_{n+1}.$$

Since the auxiliary equation

$$N_{n+1} = \frac{\lambda}{1+\mu} + \frac{1}{1+\mu} N_n,$$

has a globally asymptotically stable equilibrium $N^* = \frac{\lambda}{\mu}$. According to the comparison principle of the difference equations, we can obtain

$$\limsup_{n\to+\infty} N_n \leq \frac{\lambda}{\mu}.$$

This completes the proof.

□

According to Lemmas 2.1 and 2.2, we have the solutions of system (2.1) are positive and ultimately bounded.

3. Global stability of the disease-free equilibrium

In this section, we discuss the existence of equilibria and the stability of disease-free equilibrium of model (2.1).

Let

$$R_0 = \frac{\beta_0 \lambda}{d\mu_1}.$$

We easily verify that if $R_0 \leq 1$, then model (2.1) has only a disease-free equilibrium $E^0(\frac{\lambda}{\mu_1}, 0)$ and if $R_0 > 1$, then model (2.1) has a unique endemic equilibrium $E^*(S^*, I^*)$, except for E^0.

Theorem 3.1. *Assume that H_1 and H_2 hold.*

(i)If $R_0 \leq 1$, then model (2.1) has always a unique disease-free equilibrium $E^0 = (S^0, I^0) = (\dfrac{\lambda}{\mu_1}, 0)$.

(ii) If $R_0 > 1$ and $\beta(I) - d\alpha(I) > 0$, then model (2.1) has a unique endemic equilibrium $E^* = (S^*, I^*)$, except for E^0.

Proof. Obviously, model (2.1) always has a unique disease-free equilibrium $E^0 = (\dfrac{\lambda}{\mu_1}, 0)$. Clearly, we only need to consider the case (ii). We introduce some new marks, in order to simplify the operation of our process.

$$A(I) = 1 + \frac{d}{\dfrac{\beta(I)}{\alpha(I)} - d}, \tag{3.1}$$

$$B(I) = \frac{d}{1 - d\dfrac{\alpha(I)}{\beta(I)}}, \tag{3.2}$$

then we have $B(I) = dA(I)$.

Solving the equilibrium $(S,\ I)$ of system (2.1), from the second equation of model (2.1), we have

$$S(I) = \frac{d}{\beta(I) - d\alpha(I)}. \tag{3.3}$$

It is easy to see that $S(I)$ is increasingly for all $I \geq 0$ and $S(0) = \dfrac{d}{\beta_0} > 0$. From the first equation of model (2.1), we have that the equilibrium $(S,\ I)$ satisfies

$$\lambda - \mu_1 S(I) - \frac{\beta(I)S(I)}{1 + \alpha(I)S(I)} \sum_{k=0}^{\omega-1} I\eta_k = 0.$$

Then we can obtain

$$I = \frac{(\lambda - \mu_1 S(I))(1 + \alpha(I)S(I))}{\beta(I)S(I)}. \tag{3.4}$$

Substituting (3.1), (3.2) and (3.3) into (3.4), we have

$$I = \frac{1}{d}(\lambda - \mu_1 S(I)).$$

Furthermore, we consider the following auxiliary equation

$$f(I) = I - \frac{1}{d}(\lambda - \mu_1 S(I)).$$

Obviously, $f(I)$ is increasingly for all $I \geq 0$. Since $R_0 = \dfrac{\beta_0 \lambda}{\mu_1 d} > 1$, then we have $\beta_0 \lambda > \mu_1 d$. Therefore, we have

$$f(0) = -\frac{\lambda - \mu_1 S(0)}{d} = -\frac{\beta_0 \lambda - d\mu_1}{d\beta_0} < 0, \quad \lim_{I \to +\infty} f(I) = +\infty.$$

Hence, there exists a unique solution $I = I^* > 0$ such that $f(I^*) = 0$. Next we consider the equation (3.3) and $\beta(I) - d\alpha(I) > 0$, we can obtain there exists a unique S^* such that $S^* = S(I^*) = \dfrac{d}{\beta(I^*) - d\alpha(I^*)} > S_0 > 0$. Summarizing the above discussion, we obtain that model (2.1) has a unique endemic equilibrium $E^* = (S^*, I^*)$, except for E^0 when $R_0 > 1$. This completes the proof.

□

Theorem 3.2. *Assume that H_1 and H_2 hold. If $R_0 \leq 1$, then disease-free equilibrium $E^0 = (\frac{\lambda}{\mu_1}, 0)$ of model (2.1) is globally asymptotically stable.*

Proof. Define a function V_n :

$$V_n = I_n + C_1 \sum_{k=0}^{\omega-1} (\sum_{l=n-k}^{n} I_l)\eta_k + \frac{C_2}{2}(S_n - S^0)^2,$$

where C_1 and C_2 are positive constants to be determined later. Then the function V_n is positive defined, and

$$\begin{aligned} \Delta V_n = V_{n+1} - V_n = I_{n+1} - I_n + C_1 \sum_{k=0}^{\omega-1} (I_{n+1} - I_{n-k})\eta_k \\ + \frac{C_2}{2}\left((S_{n+1} - S^0)^2 - (S_n - S^0)^2\right). \end{aligned} \tag{3.5}$$

From the Lemma 2.2, there exists an integer $N^1 > 0$ such that

$$S_n \leq \frac{\lambda}{\mu_1} \text{ for all n} \geq N^1. \tag{3.6}$$

It follows from (3.5) and (3.6) that

$$\begin{aligned}\Delta V_n &\leq -\mu_1 C_2 (S_{n+1} - S^0)^2 + (C_1 - d) I_{n+1} \\ &\quad + \Big(\beta(I_n) S_{n+1} - C_2 \beta(I_n) S_{n+1} (S_{n+1} - S^0) - C_1\Big) \sum_{k=0}^{\omega-1} I_{n-k} \eta_k \\ &\leq -\mu_1 C_2 (S_{n+1} - S^0)^2 + (C_1 - d) I_{n+1} \\ &\quad + \Big(\beta_0 S_{n+1} - C_2 \beta_0 S_{n+1} (S_{n+1} - S^0) - C_1\Big) \sum_{k=0}^{\omega-1} I_{n-k} \eta_k.\end{aligned}$$

We choose $C_i > 0$ $(i = 1, 2)$ such that

$$C_1 - d < 0,$$

$$\beta_0 (1 + C_2 S^0)^2 < 4 C_1 C_2. \tag{3.7}$$

Obviously, the inequality (3.7) implies that

$$\beta_0 S_{n+1} - C_2 \beta_0 S_{n+1} (S_{n+1} - S^0) - C_1 < 0.$$

Since $R_0 = \dfrac{\beta_0 \lambda}{d \mu_1} \leq 1$, this implies that $\beta_0 S^0 \leq d$, we can choose $C_1 = \beta_0 S^0 - \varepsilon$. Here ε is a small positive number such that $C_1 = \beta_0 S^0 - \varepsilon < d$. Since $\beta_0 S^0 - 2C_1 < 0$ and $(\beta_0 S^0 - 2C_1)^2 > (\beta_0 S^0)^2$, we can choose $C_2 > 0$ to satisfy (3.7). Therefore, ΔV is negative definition and is equal to zero if and only if $S_n = S_0$, $I_n = 0$. This proof is completed. □

4. Permanence

In this section, we obtain the conditions for permanence of system (2.1). System (2.1) is said to be permanent if there exist positive constants m_S, M_S, m_I and M_I $(0 < m_S < M_S, 0 < m_I < M_I)$ such that for any solution (S_n, I_n) of model (2.1) has

$$m_S \leq \liminf_{n \to +\infty} S_n \leq \limsup_{n \to +\infty} S_n \leq M_S,\ m_I \leq \liminf_{n \to +\infty} I_n \leq \limsup_{n \to +\infty} I_n \leq M_I,$$

hold, and m_S, M_S, m_I, and M_I are independent of initial conditions (2.2).

Now we state the following theorem.

Theorem 4.1. *Assume that H_1 and H_2 hold. System (2.1) is permanent if $R_0 > 1$.*

Proof. Let (S_n, I_n) be any solution with initial conditions (2.2). From Lemma 2.2, we can choose $M_S = M_I = \dfrac{\lambda}{\mu}$.

First, we prove that there exists a constant $m_S > 0$ such that

$$\liminf_{n\to+\infty} S_n \geq m_S.$$

From the first equation of system (2.1), we can obtain

$$S_{n+1} + \mu_1 S_{n+1} + \frac{\beta(I_n)S_{n+1}}{1+\alpha(I_n)S_{n+1}} \sum_{k=0}^{\omega-1} \eta_k I_{n-k} = \lambda + S_n. \tag{4.1}$$

According to Lemma 2.2 and (4.1), we have there exists a sufficiently large number n_0 such that for $n > n_0$

$$\begin{aligned}
S_{n+1} &= \frac{\lambda + S_n}{1 + \mu_1 + \dfrac{\beta(I_n)}{1+\alpha(I_n)S_{n+1}} \sum\limits_{k=0}^{\omega-1} \eta_k I_{n-k}} \\
&\geq \frac{\lambda + S_n}{1 + \mu_1 + \beta(I_n) \sum\limits_{k=0}^{\omega-1} \eta_k I_{n-k}} \\
&\geq \frac{\lambda + S_n}{1 + \mu_1 + \beta_0 \dfrac{\lambda}{\mu}} \\
&= \frac{\lambda}{1 + \mu_1 + \beta_0 \dfrac{\lambda}{\mu}} + \frac{S_n}{1 + \mu_1 + \beta_0 \dfrac{\lambda}{\mu}}.
\end{aligned}$$

Since the auxiliary equation

$$W_{n+1} = \frac{\lambda}{1 + \mu_1 + \beta_0 \dfrac{\lambda}{\mu}} + \frac{W_n}{1 + \mu_1 + \beta_0 \dfrac{\lambda}{\mu}},$$

has a globally asymptotically stable equilibrium $W^* = \frac{\lambda\mu}{\mu_1\mu + \beta_0\lambda}$. According to the comparison principle of difference equation, we can obtain

$$\liminf_{n\to+\infty} S_n \geq \frac{\lambda\mu}{\mu_1\mu + \beta_0\lambda} \doteq m_S.$$

Second, we prove that there exists a positive constant $m_I > 0$ such that

$$\liminf_{n\to+\infty} I_n \geq m_I.$$

Since $R_0 = \dfrac{\beta_0 \lambda}{d\mu_1} > 1$, then there exist a arbitrarily small positive constant numbers ε and e ($e < \dfrac{\lambda}{\mu}$) such that

$$\frac{\beta(e)S^{\Delta}}{1+\alpha(e)S^{\Delta}} > d, \tag{4.2}$$

where $S^{\Delta} \doteq \frac{\lambda}{\mu_1+\beta_0 e} - \varepsilon$.

For any positive integer n_1, we claim that it is impossible for $I_n < e$ when $n \geq n_1$. Suppose the contrary, there exists a $n_1 > 0$, such that $I_n < e$ for all $n \geq n_1$. It follows from the first equation of (2.1) that for $n \geq n_1$,

$$S_{n+1} \geq \frac{\lambda + S_n}{1+\mu_1+\beta_0 e} = \frac{\lambda}{1+\mu_1+\beta_0 e} + \frac{S_n}{1+\mu_1+\beta_0 e}. \tag{4.3}$$

Then, we consider the following auxiliary equation of system (4.3)

$$G_{n+1} = \frac{\lambda}{1+\mu_1+\beta_0 e} + \frac{G_n}{1+\mu_1+\beta_0 e}. \tag{4.4}$$

Since system (4.4) has a globally asymptotically stable equilibrium $G^* = \frac{\lambda}{\mu_1+\beta_0 e}$. According to the comparison principle of difference equation, we can obtain

$$\liminf_{n\to+\infty} S_n \geq \frac{\lambda}{\mu_1+\beta_0 e}.$$

Therefore, for above mentioned ε, there exists an integer $N^* > n_1$ such that

$$S_{n+1} \geq \frac{\lambda}{\mu_1+\beta_0 e} - \varepsilon \doteq S^{\Delta}, \text{ for } n \geq N^*. \tag{4.5}$$

According the solution of system (2.1), we define

$$V_n = I_n + d\sum_{k=0}^{\omega-1}\Big(\sum_{l=n-k}^{n} I_l\Big)\eta_k, \tag{4.6}$$

where $\sum_{k=0}^{\omega-1} \eta_k = 1$.

It follows (2.1) and (4.6) that:

$$\begin{aligned} \Delta V_n &= V_{n+1} - V_n \\ &= I_{n+1} - I_n + d \sum_{k=0}^{\omega-1} (I_{n+1} - I_{n-k}) \eta_k \\ &= \frac{\beta(I_n) S_{n+1}}{1+\alpha(I_n) S_{n+1}} \sum_{k=0}^{\omega-1} I_{n-k} \eta_k - dI_{n+1} + dI_{n+1} - d \sum_{k=0}^{\omega-1} I_{n-k} \eta_k \\ &= \frac{\beta(I_n) S_{n+1}}{1+\alpha(I_n) S_{n+1}} \sum_{k=0}^{\omega-1} I_{n-k} \eta_k - d \sum_{k=0}^{\omega-1} I_{n-k} \eta_k . \end{aligned} \tag{4.7}$$

Hence, from (4.5) and (4.7) we have

$$\begin{aligned} \Delta V_n &= \frac{\beta(I_n) S_{n+1}}{1+\alpha(I_n) S_{n+1}} \sum_{k=0}^{\omega-1} I_{n-k} \eta_k - d \sum_{k=0}^{\omega-1} I_{n-k} \eta_k \\ &\geq \left(\frac{\beta(I_n) S^{\Delta}}{1+\alpha(I_n) S^{\Delta}} - d \right) \sum_{k=0}^{\omega-1} I_{n-k} \eta_k \\ &\geq \left(\frac{\beta(e) S^{\Delta}}{1+\alpha(e) S^{\Delta}} - d \right) \sum_{k=0}^{\omega-1} I_{n-k} \eta_k , \end{aligned}$$

for $j \in [N^*, N^* + n^*]$.

Set

$$\underline{m_I} = \min_{\theta} \{ I_{n_1+\omega+\theta} \} \ \text{ and } \ \theta = -\omega, -\omega+1, \cdots, -1, 0.$$

We will prove that $I_n \geq \underline{m_I}$ for $n \geq N^*$. Suppose the contrary, then there exists a nonnegative integer n^* such that

$$I_n \geq \underline{m_I}, \ \text{ for } \ N^* \leq n \leq N^* + n^*,$$

$$I_n < \underline{m_I}, \ \text{ for } \ n = N^* + n^* + 1.$$

Then there exists a positive integer j such that

$$I_j = \underline{m_I}, \ \text{ for } \ j \in [N^* \leq n \leq N^* + n^*].$$

On the other hand, by the second equation of (2.1) and (4.5), for $n = N^* + n^*$,

$$\begin{aligned}
I_{n+1} - I_j &= \frac{\frac{\beta(I_n)S_{n+1}}{1+\alpha(I_n)S_{n+1}} \sum_{k=0}^{\omega-1} I_{n-k}\eta_k + I_n}{1+d} - I_j \\
&\geq \frac{\frac{\beta(e)S^\Delta}{1+\alpha(e)S^\Delta} \sum_{k=0}^{\omega-1} I_{n-k}\eta_k + I_n - (1+d)I_j}{1+d} \\
&\geq \frac{\frac{\beta(e)S^\Delta}{1+\alpha(e)S^\Delta} - d}{1+d} I_j \\
&> 0.
\end{aligned}$$

This is a contradiction. Hence, $I_n \geq \underline{m_I}$ for $n \geq N^*$. Further, it follows from (4.2) that for $n \geq N^*$,

$$\begin{aligned}
\Delta V &\geq \left(\frac{\beta(e)S^\Delta}{1+\alpha(e)S^\Delta} - d \right) \sum_{k=0}^{\omega-1} I_{n-k}\eta_k \\
&\geq \left(\frac{\beta(e)S^\Delta}{1+\alpha(e)S^\Delta} - d \right) \underline{m_I} \\
&> 0,
\end{aligned}$$

which implies

$$\lim_{n\to+\infty} V(n) = +\infty.$$

However, in view of the positivity and upper boundedness of I_n for all $n \in N$,

$$\begin{aligned}
V_n &\leq \frac{\lambda}{\mu} + d \sum_{k=0}^{\omega-1} \Big(\sum_{l=n-k}^{n} I_l \Big) \eta_k \\
&\leq \frac{\lambda}{\mu} (1 + d\omega).
\end{aligned}$$

This is a contradiction. Hence, the claim is proved. From this claim, we discuss the following two possibilities.

(I) $I_n \geq e$ for all large n.

(II) I_n oscillates about e for all large n.

We show that $I_n \geq m_I$ as n is sufficiently large, where $m_I < e$ is a positive constant. Clearly, we only need to consider the case (II). Let positive integers n_2 and n_3 be sufficiently large such that

$$I_{n_2} \geq e, \;\; I_{n_3} \geq e,$$

$$I_n < e \;\text{ for }\; n_2 < n < n_3.$$

If $n_2 - n_3 \leq N^*$, from the second equation of system (2.1) and $I_{n_2} \geq e$, we obtain

$$\begin{aligned} I_n &\geq I_{n_2}(\frac{1}{1+d})^{N^*} \\ &\geq e(\frac{1}{1+d})^{N^*} \\ &\doteq m_I. \end{aligned}$$

This implies $I_n \geq m_I$ for $n_2 < n < n_3$.

If $n_2 - n_3 > N^*$, then it is clearly $I_n \geq e(\frac{1}{1+d})^{N^*}$ for $n_2 \leq n \leq n_2 + N^*$. Now, we assume that there is a nonnegative integer $\underline{n}$ such that

$$I_n \geq m_I, \;\text{ for }\; n_2 + N^* \leq n \leq n_2 + N^* + \underline{n},$$

$$I_n < m_I \;\text{ for }\; n = n_2 + N^* + \underline{n} + 1.$$

Moreover, there exists a positive integer q such that

$$I_q = \min_{n \in Q} I_n, \;\text{ for } Q = [n_2 + N^*, n_2 + N^* + \underline{n}],$$

then $I_q \geq m_I$.

However, for $n = n_2 + N^* + \underline{n}$, we have

$$\begin{aligned} I_{n+1} - I_q &> \frac{\dfrac{\beta(m_I)S^\Delta}{1+\alpha(m_I)S^\Delta} - d}{1+d} I_q \\ &> \frac{\dfrac{\beta(e)S^\Delta}{1+\alpha(e)S^\Delta} - d}{1+d} I_q \\ &> 0. \end{aligned}$$

This is a contradiction. So $I_n \geq m_I$ is valid for $n_2 < n < n_3$. Hence, we have $\lim_{n \to +\infty} I(n) \geq m_I$.

From the above discussion, we get

$$m_S \leq \liminf_{n \to +\infty} S_n \leq \limsup_{n \to +\infty} S_n \leq M_S,$$

$$m_I \leq \liminf_{n \to +\infty} I_n \leq \limsup_{n \to +\infty} I_n \leq M_I.$$

The proof is completed.

□

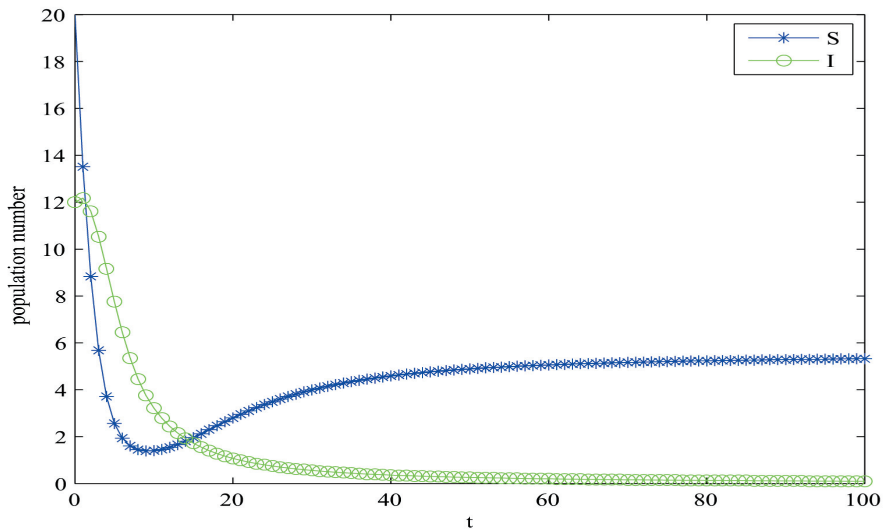

FIGURE 1. Numerical solution with $\omega = 10, \lambda = 1, \mu_1 = 0.18, d = 0.4$ and $R_0 = 0.9724$.

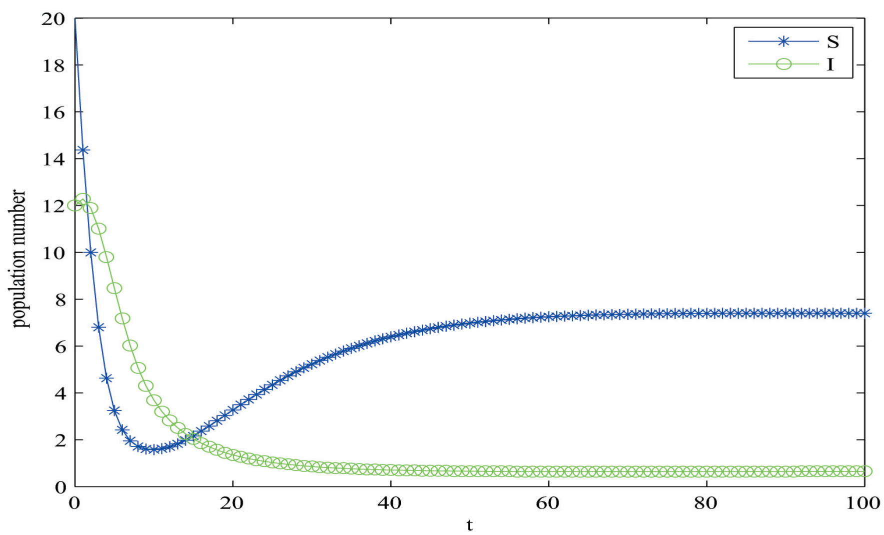

FIGURE 2. Numerical solution with $\omega = 10, \lambda = 1$, $\mu_1 = 0.1, d = 0.4$ and $R_0 = 1.7502$.

5. Numerical example

In order to simplify the operation of our process, we define B^*, where $B^* = 1+\mu_1-(S_n+\lambda)\alpha(I_n)+\beta(I_n)\sum_{k=0}^{\omega-1}\eta_k I_{n-k}$. Then, the system (2.1) are rearranged to the following explicit form:

$$\begin{cases} S_{n+1} = \dfrac{-B^* + \sqrt[2]{(B^*)^2+4(S_n+\lambda)(1+\mu_1)\alpha(I_n)}}{2(\mu_1+1)\alpha(I_n)} \\ I_{n+1} = I_n\dfrac{1}{1+d} + \dfrac{1}{1+d}\dfrac{\beta(I_n)S_{n+1}}{1+\alpha(I_n)S_{n+1}}\displaystyle\sum_{k=0}^{\omega-1}\eta_k I_{n-k}. \end{cases}$$

We choose $\beta(I_n) = 10^{-4}/(1+I_n)^2+0.2$ and $\alpha(I_n) = I_n/(10(1+I_n))$ with respect to system (2.1). In the following, we give some examples to illustrate our main theoretical results.

Example 1. In system (2.1), we choose $\lambda = 1$, $\mu_1 = 0.18, d = 0.4$, $\omega = 10$ and $\eta_k = 0.1(k = 0,1,\cdots,9)$. It is easy to see that $R_0 = 0.9724 < 1$. From Theorem 3.2, the disease-free equilibrium E^0 is globally asymptotically stable. Numerical simulation illustrates this fact (see Fig.1).

Example 2. In system (2.1), we choose $\lambda = 1$, $\mu_1 = 0.1, d = 0.4$, $\omega = 10$ and $\eta_k = 0.1(k = 0,1,\cdots,9)$. It is easy to see that $R_0 = 1.7502 > 1$. By Theorem 4.1, system (2.1) is permanent. Numerical simulation illustrates this fact (see Fig.2).

However, we are not able to analytically establish the local stability of the endemic equilibrium but numerical simulation suggest that it is stable if $R_0 > 1$. Therefore, this issue deserves to be investigated.

Conflict of Interests

The authors declare that there is no conflict of interests.

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