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AGE-STRUCTURED MATHEMATICAL MODEL FOR HIV/AIDS IN A TWO-DIMENSIONAL HETEROGENEOUS POPULATION

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Abstract. In this paper, a two-dimensional age-structured mathematical model for HIV/AIDS in three different communities of a heterogeneous population is proposed. The two distinct classes of each community lead to a set of model equations with single ordinary and partial differential equations. Different rates of contact are considered among the individuals of the communities. The equilibrium states and the corresponding characteristic equations are obtained which are analyzed by Bellman and Cookes theorem, especially non-zero equilibrium state.

Keywords: Age-structured population; HIV/AIDS; Equilibrium state; Bellman and Cooke's theorem.

2010 AMS Subject Classification: 93A30.

1. Introduction

AIDS (Acquired Immune Deficiency Syndrome or Acquired Immunodeficiency Syndrome) is a disease induced by a virus designated HIV (Human Immunodeficiency Virus). HIV is found in the body fluids containing such as white blood cells, blood, placental fluid, semen, spinal fluid, vaginal fluid, breast milk and cerebrospinal fluid of an infected person. The virus is passed

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from one person to another through blood-to-blood and sexual contact. Both the virus and the disease are often referred to HIV/AIDS. HIV is responsible for the development of AIDS in common individual. The development of HIV infections in an AIDS patient can conclusively lead to death [11]. Initially, there were no vaccines and medicines available that can prevent HIV/AIDS. HIV infected individual required high nutrient foods, rest, recreation and regular exercise. Awareness of the danger of HIV/AIDS among common people is necessarily required those have a high risk of HIV [14].

Biomathematics is the application of mathematical methods to solve the problems originating in biology and life sciences [4]. Mathematics has been used to understand and prognosticate the spread of diseases cognate to public health quandaries over the past one hundred years [2]. Some of the infectious diseases depend on the various factors viz. Age, time, contact rates etc. Age is one of the most consequential characteristics in the mathematical modelling of populations and infectious diseases. Some of the childhood diseases, such as measles, chicken pox, rubella, influenza are spread between homogeneous ages of children. Younger people may be more active in interaction between populations and in disease transmissions [17, 19]. Consequently, most of the AIDS cases occur in the group of incipient adults. Diversified age groups have different reproduction and survival capacities and altered behavior. Subsequently, mortality rates, infection rates and behavioral change are different which plays a paramount role in persistence, control and prevention of many infectious diseases. Sexually-transmitted diseases (STDs) are spread through partner interactions with pair-formations which are age-dependent in most of the cases. The contact rates between two members of populations variably correlate with the age of couples in the modelling of age-structured population.

The common age-structured population model is described and investigated by using partial differential and integral equations [10]. Sharpe and Lotka [18] studied the disease transmission of single species population. Gurtin and MacCamy [8] evaluated the general sufficient condition for stability of the model. Cushing [6] developed the dynamics of hierarchical age-structured population model. Cushing and Saleem [5] formulated a general prey-predator model with age-dependent predator population. Zhonghua and Jigen [20] considered a SIRS epidemic model with age-dependent infectivity and general nonlinear contact rate. Mathematical models have become consequential implements in analyzing the spread and control of HIV/AIDS disease dynamics [7, 9]. The first simple mathematical model for HIV had been developed by Anderson and May [2]. Nowadays, sundry mathematical models have been concentrated for the age-dependent HIV/AIDS transmission dynamics. M. O. Okongo et al. [15] introduced a comprehensive deterministic mathematical model for HIV/AIDS transmission model incorporating gregarious behaviour, treatment, vaccination, stages of infection, age structures, discrete time delay and vertical transmission. Recently, Musa [12, 13] considered two dimensional and three dimensional mathematical model of HIV/AIDS disease with age-structured infectives.

The deterministic mathematical model of HIV/AIDS has been proposed for disease among heterogeneous population which termed into system of ordinary and partial differential equations. It is especially for sexually active members of the core population [16]. The equilibrium states and the corresponding characteristic equations are obtained which are analyzed by Bellman and Cookes theorem.

The present paper is organized as follows: the basic model equations for susceptibles and HIV/AIDS infected population of three different communities are introduced in section 2. In section 3, equilibrium states of the model equations are discussed. The analysis and the stability of zero and non-zero equilibrium states of the model has been studied in Section 4. In Section 5, the discussion and numerical results are demonstrated through graphs.

2. The basic model

A model with three sub populations $(i = 1, 2, 3)$ with different sexual and social practices are considered. Community C_1 includes those whose sexual preferences, degree of sexual activity and social practices can facilitate the transmission of disease. Community C_2 and C_3 include those heterosexual individuals who have multiple sexual partners, whose risk of infection arises from social and sexual contact with the individuals of C_1. In order to construct a compartmental model of disease spread in three communities, we assume that

(1) There is an intra-interaction of the individuals of each community.
(2) There is an inter-interaction of members of community C_1 and C_2, C_1 and C_3, but no interaction between the individuals of communities C_2 and C_3.

The infected class in the community C_1, C_2 and C_3 is structured by the infection age with the density function $\rho_1(t,a)$, $\rho_2(t,a)$ and $\rho_3(t,a)$ respectively, where $'t'$ is the time parameter and $'a'$ is the infection age. There is a maximum infection age $'T'$ in each communities at which a member of the infected class must quit the compartment via death and so $0 \leq a \leq T$. The gross death rate via infection is given by $\sigma_1(a)$, $\sigma_2(a)$ and $\sigma_3(a)$; δ_1, δ_2 and δ_3 are the additional burden from infection, while K is a control parameter associated with the measure of slowing down the death of the infected member in the community C_1, C_2 and C_3 respectively such that the effectiveness of the anti-retroviral drugs gives the longer life span of infected peoples. A high level of control parameter will imply high rate of effectiveness of anti-retroviral drugs and vice-versa. The model equations for C_1, C_2 and C_3 are given as follows:

for C_1,

$$\frac{dS_1}{dt} = (\beta_1 - \mu_1) S_1(t) + \theta_1 \beta_1 I_1(t) - \alpha_{11} S_1(t) I_1(t) - \alpha_{12} S_1(t) I_2(t) - \alpha_{13} S_1(t) I_3(t), \tag{1}$$

$$I_1(t) = \int_0^T \rho_1(t,a)\, da, 0 \leq a \leq T, \tag{2}$$

$$\frac{\partial \rho_1(t,a)}{\partial t} + \frac{\partial \rho_1(t,a)}{\partial a} + \sigma_1(a) \rho_1(t,a) = 0, \tag{3}$$

$$\rho_1(t,0) = B_1(t) = \alpha_{11} S_1(t) I_1(t) + \alpha_{12} S_1(t) I_2(t) + \alpha_{13} S_1(t) I_3(t) + (1-\theta_1) \beta_1 I_1(t), \tag{4}$$

$$\rho_1(0,t) = \varphi_1(a). \tag{5}$$

For C_2,

$$\frac{dS_2}{dt} = (\beta_2 - \mu_2) S_2(t) + \theta_2 \beta_2 I_2(t) - \alpha_{21} S_2(t) I_1(t) - \alpha_{22} S_2(t) I_2(t), \tag{6}$$

$$I_2(t) = \int_0^T \rho_2(t,a)\, da, 0 \leq a \leq T, \tag{7}$$

$$\frac{\partial \rho_2(t,a)}{\partial t} + \frac{\partial \rho_2(t,a)}{\partial a} + \sigma_2(a) \rho_2(t,a) = 0, \tag{8}$$

$$\rho_2(t,0) = B_2(t) = \alpha_{21} S_2(t) I_1(t) + \alpha_{22} S_2(t) I_2(t) + (1-\theta_2) \beta_2 I_2(t), \tag{9}$$

$$\rho_2(0,t) = \varphi_2(a). \tag{10}$$

For C_3,

$$\frac{dS_3}{dt} = (\beta_3 - \mu_3) S_3(t) + \theta_3 \beta_3 I_3(t) - \alpha_{31} S_3(t) I_1(t) - \alpha_{33} S_3(t) I_3(t), \tag{11}$$

$$I_3(t) = \int_0^T \rho_3(t,a)\, da, 0 \leq a \leq T, \tag{12}$$

$$\frac{\partial \rho_3(t,a)}{\partial t} + \frac{\partial \rho_3(t,a)}{\partial a} + \sigma_3(a)\,\rho_3(t,a) = 0, \tag{13}$$

$$\rho_3(t,0) = B_3(t) = \alpha_{31}\,S_3(t)\,I_1(t)\; + \alpha_{33}\,S_3(t)\,I_3(t) + (1-\theta_3)\,\beta_3\,I_3(t), \tag{14}$$

$$\rho_3(0,t) = \varphi_3(a). \tag{15}$$

Here, $S_i(i=1,2,3)$ and $I_i(i=1,2,3)$ denote the member of susceptibles and AIDS infected population in the community $C_i(i=1,2,3)$, $\sigma_i(a) = \mu_i + \delta_i tan\frac{\pi a}{2TK}$; $i=1,2,3$, and the parameters $\rho_i(t,a)$, β_i, μ_i, $\sigma_i(a)$, δ_i, K, θ_i and T be the age-specific function, natural birth rate, natural death rate, gross death rate of the infected class, additional burden from infection, measure of the effectiveness of efforts at slowing down the death of infected members, the proportion of the off-springs of the infected class which are virus free at birth and maximum infection age in the community $C_i(i=1,2,3)$ respectively. Also, α_{ii} be the transmission rate of HIV within the community $C_i(i=1,2,3)$ and α_{ij} be the transmission rate of HIV between the communities $C_i(i=1,2,3)$ and $C_j(j=1,2,3)$. Now, we have to study the zero and non-zero equilibrium states of HIV/AIDS in each of these communities.

3. Equilibrium states

At the equilibrium state, let us assume that

$$S_i(0) = x_i,\; I_i(0) = y_i,\; \rho_i(t,a) = \varphi_i(a). \tag{16}$$

To find the zero and non-zero equilibrium states of the above model eqs (1)-(15), we have to substitute (16) in (1)-(5), we get

$$(\beta_1 - \mu_1)\,x_1 + \theta_1\,\beta_1\,y_1 - \alpha_{11}\,x_1\,y_1 - \alpha_{12}x_1\,y_2 - \alpha_{13}x_1\,y_3 = 0, \tag{17}$$

$$y_1 = \int_0^T \varphi_1(a)\,da, \tag{18}$$

$$\frac{d\varphi_1(a)}{da} + \sigma_1(a)\,\varphi_1(a) = 0, \tag{19}$$

$$\varphi_1(0) = \alpha_{11}\,x_1\,y_1 + \alpha_{12}x_1\,y_2 + \alpha_{13}x_1\,y_3 + (1-\theta_1)\,\beta_1\,\theta_1, \tag{20}$$

Integrating (19) from '0' to 'a', we get

$$\varphi_1(a) = \varphi_1(0)\,\exp\{-\int_0^a \sigma_1(s)\,ds\}. \tag{21}$$

It can also be written as

$$\varphi_1(a) = \varphi_1(0)\;\pi_1(a),\; where\; \pi_1(a) = \exp\{-\int_0^a \sigma_1(s)\,ds\}. \tag{22}$$

Therefore, (18) becomes

$$y_1 = \varphi_1(0)\,\bar{\pi}_1(a),\, where\; \bar{\pi}_1(a) = \int_0^T \pi_1(a)\,da. \tag{23}$$

By using (20) in (23), it implies

$$y_1 = (\alpha_{11}\,x_1\,y_1 + \alpha_{12}\,x_1\,y_2 + \alpha_{13}\,x_1\,y_3\; + (1-\theta_1)\,\beta_1\,y_1)\,\bar{\pi}_1(a). \tag{24}$$

Substituting (16) in (6)-(10), we have

$$(\beta_2 - \mu_2)\,x_2 + \theta_2\,\beta_2\,y_2 - \alpha_{21}\,x_2\,y_1 - \alpha_{22}\,x_2\,y_2 = 0, \tag{25}$$

$$y_2 = \int_0^T \varphi_2(a)da, \tag{26}$$

$$\frac{d\,\varphi_2(a)}{da} + \sigma_2(a)\,\varphi_2(a) = 0, \tag{27}$$

$$\varphi_2(0) = \alpha_{21}\,x_2\,y_1 + \alpha_{22}\,x_2\,y_2 + (1-\theta_2)\,\beta_2\,\theta_2. \tag{28}$$

Integrating (27) from '0' to 'a', we get

$$\varphi_2(a) = \varphi_2(0)\,\exp\{-\int_0^a \sigma_2(s)\,ds\}. \tag{29}$$

It can also be written as

$$\varphi_2(a) = \varphi_2(0)\,\pi_2(a),\, where\; \pi_2(a) = \exp\{-\int_0^a \sigma_2(s)\,ds\}. \tag{30}$$

Therefore, (26) becomes

$$y_2 = \varphi_2(0)\,\bar{\pi}_2(a),\, where\; \bar{\pi}_2(a) = \int_0^T \pi_2(a)\,da. \tag{31}$$

By using (28) in (31), it implies

$$y_2 = (\alpha_{21}\,x_2\,y_1 + \alpha_{22}\,x_2\,y_2 + (1-\theta_2)\,\beta_2\,y_2)\,\bar{\pi}_2(a). \tag{32}$$

Substituting (16) in (11)-(15), we have

$$(\beta_3 - \mu_3)\,x_3 + \theta_3\,\beta_3\,y_3 - \alpha_{31}\,x_3\,y_1 - \alpha_{33}\,x_3\,y_3 = 0, \tag{33}$$

$$y_3 = \int_0^T \varphi_3(a)\,da, \tag{34}$$

$$\frac{d\,\varphi_3(a)}{da} + \sigma_3(a)\,\varphi_3(a) = 0, \tag{35}$$

$$\varphi_3(0) = \alpha_{31}\,x_3\,y_1 + \alpha_{33}\,x_3\,y_3 + (1-\theta_3)\,\beta_3\,\theta_3. \tag{36}$$

Integrating (35) from '0' to 'a', we get

$$\varphi_3(a) = \varphi_3(0)\,\exp\{-\int_0^a \sigma_3(s)\,ds\}. \tag{37}$$

It can also be written as

$$\varphi_3(a) = \varphi_3(0)\,\pi_3(a),\, where\, \pi_3(a) = \exp\{-\int_0^a \sigma_3(s)\,ds\}. \tag{38}$$

Therefore, (34) becomes

$$y_3 = \varphi_3(0)\,\bar{\pi}_3(a),\, where\, \bar{\pi}_3(a) = \int_0^T \pi_3(a)\,da. \tag{39}$$

By using (36) in (39), it implies

$$y_3 = (\alpha_{31}\,x_3\,y_1 + \alpha_{33}\,x_3\,y_3 + (1-\theta_3)\,\beta_3\,y_3)\,\bar{\pi}_3(a). \tag{40}$$

Thus, from (24), (32) and (40), we get

$$x_1 = \frac{\left(\frac{y_1}{\bar{\pi}_1} - (1-\theta_1)\,\beta_1\,y_1\right)}{(\alpha_{11}\,y_1 + \alpha_{12}\,y_2 + \alpha_{13}\,y_3)}, \tag{41}$$

$$x_2 = \frac{\left(\frac{y_2}{\bar{\pi}_2} - (1-\theta_2)\,\beta_2\,y_2\right)}{(\alpha_{21}\,y_1 + \alpha_{22}\,y_2)}, \tag{42}$$

$$x_3 = \frac{\left(\frac{y_3}{\bar{\pi}_3} - (1-\theta_3)\,\beta_3\,y_3\right)}{(\alpha_{31}\,y_1 + \alpha_{33}\,y_3)}. \tag{43}$$

Substituting (41), (42) and (43) in (17), (25) and (33) respectively, we obtain

$$\alpha_{11}\,y_1+\alpha_{12}\,y_2+\alpha_{13}\,y_3=\frac{(\beta_1-\mu_1)\,\{1-(1-\theta_1)\,\beta_1\,\bar{\pi}_1\}}{(1-\beta_1\,\bar{\pi}_1)}, \tag{44}$$

$$\alpha_{21}\,y_1+\alpha_{22}\,y_2=\frac{(\beta_2-\mu_2)\{1-(1-\theta_2)\,\beta_2\,\bar{\pi}_2\}}{(1-\beta_2\,\bar{\pi}_2)} \tag{45}$$

and

$$\alpha_{31}\,y_1+\alpha_{33}\,y_3=\frac{(\beta_3-\mu_3)\,\{1-(1-\theta_3)\,\beta_3\,\bar{\pi}_3\}}{(1-\beta_3\,\bar{\pi}_3)}. \tag{46}$$

Solving (44), (45) and (46), we get

$$y_1=-\frac{1}{(-\alpha_{12}\alpha_{21}\alpha_{33}-\alpha_{22}\alpha_{13}\alpha_{31}+\alpha_{11}\alpha_{22}\alpha_{33})}\begin{bmatrix}\frac{\alpha_{12}\,\alpha_{33}\,(1-\beta_2\,\bar{\pi}_2-\beta_2\,\bar{\pi}_2\theta_2)(\beta_2-\mu_2)}{(1-\beta_2\bar{\pi}_2)}\\ -\frac{\alpha_{22}\alpha_{33}(1-\beta_1\bar{\pi}_1-\beta_1\bar{\pi}_1\theta_1)(\beta_1-\mu_1)}{(1-\beta_1\bar{\pi}_1)}\\ -\frac{\alpha_{22}\alpha_{13}(1-\beta_3\bar{\pi}_3-\beta_3\bar{\pi}_3\theta_3)(\beta_3-\mu_3)}{(1-\beta_3\bar{\pi}_3)}\end{bmatrix},$$

$$y_2=\frac{1}{(-\alpha_{12}\alpha_{21}\alpha_{22}\alpha_{33}-\alpha_{13}\alpha_{31}\alpha_{22}\alpha_{22}+\alpha_{11}\alpha_{22}\alpha_{22}\alpha_{33})}$$
$$\times\begin{bmatrix}\frac{\alpha_{12}\alpha_{21}\,\alpha_{33}\,(1-\beta_2\,\bar{\pi}_2-\beta_2\,\bar{\pi}_2\,\theta_2)\,(\beta_2-\mu_2)}{(1-\beta_2\,\bar{\pi}_2)}\\ -\frac{\alpha_{21}\alpha_{22}\,\alpha_{33}\,(1-\beta_1\,\bar{\pi}_1-\beta_1\,\bar{\pi}_1\,\theta_1\,)\,(\beta_1-\mu_1)}{(1-\beta_1\,\bar{\pi}_1)}\\ -\frac{\alpha_{21}\alpha_{22}\,\alpha_{13}\,(1-\beta_3\,\bar{\pi}_3-\beta_3\,\bar{\pi}_3\,\theta_3\,)\,(\beta_3-\mu_3)}{(1-\beta_3\,\bar{\pi}_3)}\end{bmatrix}+\frac{(1-\beta_2\,\bar{\pi}_2-\beta_2\,\bar{\pi}_2\,\theta_2)\,(\beta_2-\mu_2)}{\alpha_{22}\,(1-\beta_2\,\bar{\pi}_2)},$$

$$y_3=\frac{1}{(-\alpha_{12}\,\alpha_{21}\,\alpha_{33}\,\alpha_{33}\,-\,\alpha_{13}\,\alpha_{31}\,\alpha_{22}\,\alpha_{33}+\,\alpha_{11}\,\alpha_{22}\,\alpha_{33}\,\alpha_{33})}$$
$$\times\begin{bmatrix}\frac{\alpha_{12}\,\alpha_{31}\,\alpha_{33}\,(1-\beta_2\,\bar{\pi}_2-\beta_2\,\bar{\pi}_2\,\theta_2)\,(\beta_2-\mu_2)}{(1-\beta_2\,\bar{\pi}_2)}\\ -\frac{\alpha_{22}\,\alpha_{31}\,\alpha_{33}\,(1-\beta_1\,\bar{\pi}_1-\beta_1\,\bar{\pi}_1\,\theta_1)\,(\beta_1-\mu_1)}{(1-\beta_1\,\bar{\pi}_1)}\\ +\frac{\alpha_{22}\,\alpha_{31}\,\alpha_{13}\,(1-\beta_3\,\bar{\pi}_3-\beta_3\,\bar{\pi}_3\,\theta_3)\,(\beta_3-\mu_3)}{(1-\beta_3\bar{\pi}_3)}\end{bmatrix}+\frac{(1-\beta_3\,\bar{\pi}_3-\beta_3\,\bar{\pi}_3\,\theta_3)\,(\beta_3-\mu_3)}{\alpha_{33}\,(1-\beta_3\bar{\pi}_3)}.]$$

By using the above value of y_1, y_2, y_3 in (41), (42) and (43), we obtain

$$x_1=\frac{(1-\beta_1\,\bar{\pi}_1)}{\bar{\pi}_1\,(\beta_1-\mu_1)},\,x_2=\frac{(1-\beta_2\,\bar{\pi}_2)}{\bar{\pi}_2\,(\beta_2-\mu_2)},\,x_3=\frac{(1-\beta_3\,\bar{\pi}_3)}{\bar{\pi}_3\,(\beta_3-\mu_3)}.$$

Thus, the zero equilibrium state of the model is given by

$$(x_1,x_2,x_3,y_1,y_2,y_3)=(0,0,0,0,0,0)$$

and the non-zero equilibrium state is given by the above non-zero value of x_1, x_2, x_3, y_1, y_2 and y_3.

4. Analysis of the model

4.1 The Characteristics equation

To analyze the model, suppose that

$$S_i(t) = x_i + p_i(t), I_i(t) = y_i + q_i(t), \rho_i(t,a) = \varphi_i(a) + \eta_i(a)\, e^{\lambda t} : i = 1, 2, 3, \tag{47}$$

where

$$p_i(t) = \bar{p}_i e^{\lambda t}, q_i(t) = \bar{q}_i e^{\lambda t} = \int_0^T \eta_i(a)\, da. \tag{48}$$

Substituting (47) and (48) into (1)-(4), then (1) becomes

$$\frac{d}{dt}(x_1 + \bar{p}_1 e^{\lambda t}) = (\beta_1 - \mu_1)(x_1 + \bar{p}_1 e^{\lambda t}) + \theta_1\beta_1(y_1 + \bar{q}_1 e^{\lambda t}) - \alpha_{11}(x_1 + \bar{p}_1 e^{\lambda t})$$

$$(y_1 + \bar{q}_1 e^{\lambda t}) - \alpha_{12}(x_1 + \bar{p}_1 e^{\lambda t})(y_2 + \bar{q}_2 e^{\lambda t}) - \alpha_{13}(x_1 + \bar{p}_1 e^{\lambda t})(y_3 + \bar{q}_3 e^{\lambda t}),$$

it can also be written as

$$(\beta_1 - \mu_1 - \alpha_{11}y_1 - \alpha_{12}y_2 - \alpha_{13}y_3 - \lambda)\bar{p}_1 e^{\lambda t} + (\theta_1\beta_1 - \alpha_{11}x_1)\bar{q}_1 e^{\lambda t}$$

$$-\alpha_{12} x_1 \bar{q}_2 e^{\lambda t} - \alpha_{13} x_1 \bar{q}_3 e^{\lambda t} = 0. \tag{49}$$

(2) implies

$$y_1 + \bar{q}_1 e^{\lambda t} = \int_0^T (\varphi_1(a) + \eta_1(a) e^{\lambda t})\, da. \tag{50}$$

(3) implies

$$\frac{\partial}{\partial t}(\varphi_1(a) + \eta_1(a) e^{\lambda t}) + \frac{\partial}{\partial a}(\varphi_1(a) + \eta_1(a) e^{\lambda t}) + \sigma_1(a)(\varphi_1(a) + \eta_1(a) e^{\lambda t}) = 0$$

it can also be written as

$$\frac{d\,\eta_1(a)}{da} + (\lambda + \sigma_1(a))\,\eta_1(a) = 0. \tag{51}$$

Integrating (51) from ‘0’ to ‘a’, we have

$$\eta_1(a) = \eta_1(0)\, \exp\{-\int_0^a (\lambda + \sigma_1(s))ds\}.$$

Again integrating (51) over $[0, T]$, we get

$$\int_0^T \eta_1(a)da = \eta_1(0) \int_0^T [\exp\{-\int_0^a (\lambda + \sigma_1(s))ds\}]da. \tag{52}$$

Now, (4) becomes

$$\rho_1(t,0) = \alpha_{11}(x_1+\bar{p}_1 e^{\lambda t})(y_1+\bar{q}_1 e^{\lambda t}) + \alpha_{12}(x_1+\bar{p}_1 e^{\lambda t})(y_2+\bar{q}_2 e^{\lambda t})$$

$$+\alpha_{13}(x_1+\bar{p}_1 e^{\lambda t})(y_3+\bar{q}_3 e^{\lambda t}) + (1-\theta_1)\beta_1(y_1+\bar{q}_1 e^{\lambda t})$$

By using (20) and (47) at $a=0$ in the above equation, for $i=1$, it becomes

$$\alpha_{11}x_1y_1+\alpha_{12}x_1y_2+\alpha_{13}x_1y_3+(1-\theta_1)\beta_1y_1+\eta_1(0)e^{\lambda t}$$

$$=\alpha_{11}x_1y_1+\alpha_{11}x_1\bar{q}_1e^{\lambda t}+\alpha_{11}y_1\bar{p}_1e^{\lambda t}+\alpha_{11}\bar{p}_1\bar{q}_1e^{2\lambda t}+\alpha_{12}x_1y_2$$

$$+\alpha_{12}x_1\bar{q}_2e^{\lambda t}+\alpha_{12}y_2\bar{p}_1e^{\lambda t}+\alpha_{12}\bar{p}_1\bar{q}_2e^{2\lambda t}+\alpha_{13}x_1y_3$$

$$+\alpha_{13}x_1\bar{q}_3e^{\lambda t}+\alpha_{13}y_3\bar{p}_1e^{\lambda t}+(1-\theta_1)\beta_1(y_1+\bar{q}_1e^{\lambda t}),$$

(4) can also be written as

$$\eta_1(0) = (\alpha_{11}y_1+\alpha_{12}y_1+\alpha_{13}y_1)\,\bar{p}_1+(\alpha_{11}x_1+(1-\theta_1)\beta_1)\bar{q}_1+\alpha_{12}x_1\bar{q}_2+\alpha_{13}x_1\bar{q}_3. \tag{53}$$

Now, using (52) and (53) in (48) for $i=1$, thus we have

$$\bar{q}_1 = [(\alpha_{11}y_1+\alpha_{12}y_2+\alpha_{13}y_3)\bar{p}_1+(\alpha_{11}x_1+(1-\theta_1)\beta_1)\bar{q}_1$$

$$+\alpha_{12}x_1\bar{q}_2+\alpha_{13}x_1\bar{q}_3]\int_0^T[\exp\{-\int_0^a(\lambda+\sigma_1(s))ds\}]da.$$

Let $b_1(\lambda)=\int_0^T[\exp\{-\int_0^a(\lambda+\sigma_1(s))\,ds\}]\,da$. Then the above equation becomes

$$(\alpha_{11}y_1+\alpha_{12}y_2+\alpha_{13}y_3)b_1(\lambda)\bar{p}_1+\{(\alpha_{11}x_1+(1-\theta_1)\beta_1)b_1(\lambda)-1\}\bar{q}_1$$

$$+\alpha_{12}x_1b_1(\lambda)\bar{q}_2+\alpha_{13}x_1b_1(\lambda)\bar{q}_3=0. \tag{54}$$

Substituting (47) and (48) into (6)-(9), then (6) becomes

$$\frac{d}{dt}(x_2+\bar{p}_2e^{\lambda t}) = (\beta_2-\mu_2)(x_2+\bar{p}_2e^{\lambda t})+\theta_2\beta_2(y_2+\bar{q}_2e^{\lambda t})$$

$$-\alpha_{21}(x_2+\bar{p}_2e^{\lambda t})(y_1+\bar{q}_1e^{\lambda t})-\alpha_{22}(x_2+\bar{p}_2e^{\lambda t})(y_2+\bar{q}_2e^{\lambda t}),$$

it can also be written as

$$(\beta_2-\mu_2-\alpha_{21}y_1-\alpha_{22}y_2-\lambda)\bar{p}_2e^{\lambda t}-\alpha_{21}x_2\bar{q}_1e^{\lambda t}+(\theta_2\beta_2-\alpha_{22}x_2)\bar{q}_2e^{\lambda t}=0. \tag{55}$$

(7) implies

$$y_2 + \bar{q}_2\, e^{\lambda t} = \int_0^T (\varphi_2(a) + \eta_2(a)\, e^{\lambda t})\, da. \tag{56}$$

(8) implies

$$\frac{\partial}{\partial t}(\varphi_2(a) + \eta_2(a)e^{\lambda t}) + \frac{\partial}{\partial a}(\varphi_2(a) + \eta_2(a)e^{\lambda t}) + \sigma_2(a)(\varphi_2(a) + \eta_2(a)e^{\lambda t}) = 0,$$

it can also be written as

$$\frac{d\,\eta_2(a)}{da} + (\lambda + \sigma_2(a))\,\eta_2(a) = 0. \tag{57}$$

Integrating (57) from '0' to 'a', we have

$$\eta_2(a) = \eta_2(0)\,\exp\{-\int_0^a (\lambda + \sigma_2(s))\,ds\}.$$

Again integrating (57) over $[0, T]$, we get

$$\int_0^T \eta_2(a)\,da = \eta_2(0)\int_0^T [\exp\{-\int_0^a (\lambda + \sigma_2(s))\,ds\}]da. \tag{58}$$

Now, (9) becomes

$$\rho_2(t,0) = \alpha_{21}\,(x_2 + \bar{p}_2\, e^{\lambda t})\,(y_1 + \bar{q}_1\, e^{\lambda t}) + \alpha_{22}\,(x_2 + \bar{p}_2\, e^{\lambda t})$$

$$(y_2 + \bar{q}_2\, e^{\lambda t}) + (1-\theta_2)\,\beta_2\,(y_2 + \bar{q}_2 e^{\lambda t}).$$

By using (28) and (47) at $a = 0$ in the above equation for $i = 2$, it becomes

$$\alpha_{21}\,x_2\,y_1 + \alpha_{22}\,x_2\,y_2 + (1-\theta_2)\,\beta_2\,y_2 + \eta_2(0)\,e^{\lambda t} = \alpha_{21}\,x_2\,y_1 + \alpha_{21}\,x_2\,\bar{q}_1\,e^{\lambda t}$$

$$+\alpha_{21}\,y_1\,\bar{p}_2\,e^{\lambda t} + \alpha_{21}\,\bar{p}_2\,\bar{q}_1\,e^{2\lambda t} + \alpha_{22}\,x_2\,y_2\alpha_{22}\,x_2\,\bar{q}_2 e^{\lambda t}$$

$$+\alpha_{22}y_2\,\bar{p}_2 e^{\lambda t} + \alpha_{22}\,\bar{p}_2\bar{q}_2 e^{2\lambda t} + (1-\theta_2)\beta_2(y_2 + \bar{q}_2 e^{\lambda t}).$$

(9) can also be written as

$$\eta_2(0) = (\alpha_{21}\,y_1 + \alpha_{22}\,y_2)\,\bar{p}_2 + \alpha_{21}\,x_2\,\bar{q}_1 + (\alpha_{22}\,x_2 + (1-\theta_2)\,\beta_2)\,\bar{q}_2. \tag{59}$$

Now, using (58) and (59) in (48) for $i = 2$, thus we have

$$\bar{q}_2 = [\alpha_{21}x_2\bar{q}_1 + (\alpha_{21}\,y_1 + \alpha_{22}y_2)\bar{p}_2 + (\alpha_{22}x_2 + (1-\theta_2)\beta_2)\bar{q}_2]\int_0^T [\exp\{-\int_0^a (\lambda + \sigma_1(s))ds\}]da.$$

Let $b_2(\lambda) = \int\limits_0^T [\exp\{-\int\limits_0^a (\lambda + \sigma_2(s))\,ds\}]\,da$. Then the above equation becomes

$$(\alpha_{21}\,y_1 + \alpha_{22}\,y_2)\,b_2(\lambda)\,\bar{p}_2 + \alpha_{21}\,x_2\,b_2(\lambda)\,\bar{q}_1 + \{(\alpha_{22}x_2 + (1-\theta_2)\beta_2)b_2(\lambda) - 1\}\bar{q}_2 = 0. \quad (60)$$

Substituting (47) and (48) into (11)-(14), then (11) becomes

$$\frac{d}{dt}(x_3 + \bar{p}_3\,e^{\lambda\,t}) = (\beta_3 - \mu_3)\,(x_3 + \bar{p}_3\,e^{\lambda\,t}) + \theta_3\,\beta_3\,(y_3 + \bar{q}_3\,e^{\lambda\,t}) - \alpha_{31}(x_3 + \bar{p}_3 e^{\lambda t})$$

$$(y_1 + \bar{q}_1 e^{\lambda t}) - \alpha_{33}(x_3 + \bar{p}_3 e^{\lambda t})(y_3 + \bar{q}_3 e^{\lambda t}),$$

it can also be written as

$$(\beta_3 - \mu_3 - \alpha_{31}y_1 - \alpha_{33}\,y_3 - \lambda)\bar{p}_3 e^{\lambda t} - \alpha_{31}x_3\bar{q}_1 e^{\lambda t} + (\theta_3\beta_3 - \alpha_{33}x_3)\bar{q}_3 e^{\lambda t} = 0. \quad (61)$$

(12) implies

$$y_3 + \bar{q}_3\,e^{\lambda\,t} = \int_0^T (\varphi_3(a) + \eta_3(a)\,e^{\lambda\,t})\,da. \quad (62)$$

(13) implies

$$\frac{\partial}{\partial t}(\varphi_3(a) + \eta_3(a)e^{\lambda\,t}) + \frac{\partial}{\partial a}(\varphi_3(a) + \eta_3(a)\,e^{\lambda\,t}) + \sigma_3(a)\,(\varphi_3(a) + \eta_3(a)\,e^{\lambda t}) = 0,$$

it can also be written as

$$\frac{d\,\eta_3(a)}{da} + (\lambda + \sigma_3(a))\,\eta_3(a) = 0. \quad (63)$$

Integrating (63) from '0' to 'a', we get

$$\eta_3(a) = \eta_3(0)\,\exp\{-\int_0^a (\lambda + \sigma_3(s))ds\}.$$

Again integrating (63) over $[0, T]$, we get

$$\int_0^T \eta_3(a)da = \eta_3(0)\int_0^T [\exp\{-\int_0^a (\lambda + \sigma_3(s))ds\}]da. \quad (64)$$

Now, (14) becomes

$$\rho_3(t,0) = B_3(t) = \alpha_{31}\,(x_3 + \bar{p}_3\,e^{\lambda\,t})\,(y_1 + \bar{q}_1\,e^{\lambda\,t}) + \alpha_{33}\,(x_3 + \bar{p}_3\,e^{\lambda\,t})$$

$$(y_3 + \bar{q}_3\,e^{\lambda\,t}) + (1-\theta_3)\,\beta_3\,(y_3 + \bar{q}_3\,e^{\lambda\,t}).$$

By using (36) and (47) at $a=0$ in the above equation, for $i=3$, it becomes

$$\alpha_{31}x_3y_1+\alpha_{33}x_3y_3+(1-\theta_3)\beta_3y_3+\eta_3(0)e^{\lambda t}=\alpha_{31}x_3y_1+\alpha_{31}x_3\bar{q}_1e^{\lambda t}$$

$$+\alpha_{31}y_1\bar{p}_3e^{\lambda t}+\alpha_{31}\bar{p}_3\bar{q}_1e^{2\lambda t}+\alpha_{33}x_3y_3+\alpha_{33}x_3\bar{q}_3e^{\lambda t}$$

$$+\alpha_{33}y_3\bar{p}_3e^{\lambda t}+\alpha_{33}\bar{p}_3\bar{q}_3e^{2\lambda t}+(1-\theta_3)\beta_3(y_3+\bar{q}_3e^{\lambda t}).$$

(14) can also be written as

$$\eta_3(0)=(\alpha_{31}y_1+\alpha_{33}y_3)\bar{p}_3+\alpha_{31}x_3\bar{q}_1+(\alpha_{33}x_3+(1-\theta_3)\beta_3)\bar{q}_3. \tag{65}$$

Now, using (64) and (65) in (48) for $i=3$, thus we have

$$\bar{q}_3=[\alpha_{31}x_3\bar{q}_1+(\alpha_{31}y_1+\alpha_{33}y_3)\bar{p}_3+(\alpha_{33}x_3+(1-\theta_3)\beta_3)\bar{q}_3]\int_0^T[\exp\{-\int_0^a(\lambda+\sigma_3(s))ds\}]da.$$

Let $b_3(\lambda)=\int_0^T[\exp\{-\int_0^a(\lambda+\sigma_3(s))\,ds\}]da$. Then the above equation becomes

$$(\alpha_{31}y_1+\alpha_{33}y_3)b_3(\lambda)\bar{p}_3+\alpha_{31}x_3b_3(\lambda)\bar{q}_1+\{(\alpha_{33}x_3+(1-\theta_3)\beta_3)b_3(\lambda)-1\}\bar{q}_3=0. \tag{66}$$

Now, from (49), (54), (55), (60), (61) and (66), we obtain the Jacobian determinant for the system with the eigenvalue λ,

$$\left|\begin{array}{ccc}
\begin{matrix}(\beta_1-\mu_1-\alpha_{11}y_1-\\ \alpha_{12}y_2-\alpha_{13}y_3-\lambda)\end{matrix} & (\theta_1\beta_1-\alpha_{11}x_1) & 0 \\
\begin{matrix}(\alpha_{11}y_1+\alpha_{12}y_2\\ +\alpha_{13}y_3)b_1(\lambda)\end{matrix} & \begin{matrix}\{(\alpha_{11}x_1\\ +(1-\theta_1)\beta_1)\\ b_1(\lambda)-1\}\end{matrix} & 0 \\
0 & -\alpha_{21}x_2 & \begin{matrix}(\beta_2-\mu_2-\alpha_{21}y_1\\ -\alpha_{22}y_2-\lambda)\end{matrix} \\
0 & \alpha_{21}x_2b_2(\lambda) & \begin{matrix}(\alpha_{21}y_1+\alpha_{22}y_2)\\ b_2(\lambda)\end{matrix} \\
0 & -\alpha_{31}x_3 & 0 \\
0 & \alpha_{31}x_3b_2(\lambda) & 0
\end{array}\right.$$

$$\begin{array}{ccc|}
-\alpha_{12}x_1 & 0 & -\alpha_{13}x_1 \\
\alpha_{12}x_1 b_1(\lambda) & 0 & \alpha_{13}x_1 b_1(\lambda) \\
(\theta_2\beta_2-\alpha_{22}x_2) & 0 & 0 \\
\{(\alpha_{22}x_2+(1-\theta_2)\beta_2)\,b_2(\lambda)-1\} & 0 & 0 \\
0 & (\beta_3-\mu_3-\alpha_{31}y_1-\alpha_{33}y_3-\lambda) & (\theta_3\beta_3-\alpha_{33}x_3) \\
0 & (\alpha_{31}y_1+\alpha_{33}y_3)\,b_3(\lambda) & \{(\alpha_{33}x_3+(1-\theta_3)\beta_3)\,b_3(\lambda)-1\}
\end{array} = 0.$$

The characteristic equation is given by

$$\begin{aligned}
&(-b_3C\{R\{-b_2B\{-b_1x_1x_2\alpha_{12}\alpha_{21}(\lambda-\beta_1+\mu_1)+Q\{b_1AP-X\ (\lambda+A-\beta_1+\mu_1)\}\}\\
&+\{b_1b_2x_1x_2\alpha_{12}\alpha_{21}(\lambda-\beta_1+\mu_1)+Y\{b_1AP-X(\lambda+A-\beta_1+\mu_1)\}\}\\
&(-\lambda-B+\beta_2-\mu_2)\}-b_1x_1x_3\alpha_{13}\alpha_{31}(\lambda-\beta_1+\mu_1)(\lambda+B-\beta_2-\alpha_{21}y_1\beta_2b_2\\
&+\mu_2-b_2(\alpha_{22}x_2(\lambda-\beta_2+\mu_2)+\beta_2(\alpha_{22}y_2+(1-\theta_2)+\{Z(\lambda-\beta_2+\mu_2)))))\}\\
&\{-b_2B\{-b_1x_1x_2\alpha_{12}\alpha_{21}(\lambda-\beta_1+\mu_1)+Q\{b_1AP-X(\lambda+A-\beta_1+\mu_1)\}\}\\
&+\{b_1b_2x_1x_2\alpha_{12}\alpha_{21}(\lambda-\beta_1+\mu_1)+Y\{b_1AP-X(\lambda+A-\beta_1+\mu_1)\}\}\\
&(-\lambda-B+\beta_2-\mu_2)\}+b_1b_3x_1x_3\alpha_{13}\alpha_{31}(\lambda-\beta_1+\mu_1)\\
&(\lambda+B-\beta_2-\alpha_{21}y_1\beta_2b_2+\mu_2-b_2(\alpha_{22}x_2(\lambda-\beta_2+\mu_2)\\
&+\beta_2(\alpha_{22}y_2+(1-\theta_2)\lambda-\beta_2+\mu_2)))))\}(-\lambda-C+\beta_3-\mu_3))=0, \qquad (67)
\end{aligned}$$

where

$$A=(\alpha_{11}y_1+\alpha_{12}y_2+\alpha_{13}y_3), B=(\alpha_{21}y_1+\alpha_{22}y_2), C=(\alpha_{31}y_1+\alpha_{33}y_3),$$

$$P=-(\theta_1\beta_1-\alpha_{11}x_1), Q=(\theta_2\beta_2-\alpha_{22}x_2), R=(\theta_3\beta_3-\alpha_{33}x_3),$$

$$X=(-1+b_1(\alpha_{11}x_1+\beta_1(1-\theta_1))),$$

$$Y=(-1+b_2(\alpha_{22}x_2+\beta_2(1-\theta_2))), Z=(-1+b_3(\alpha_{33}x_3+\beta_3(1-\theta_3))).$$

4.2 Stability of the zero equilibrium state

At the zero equilibrium state, $(x_1, x_2, x_3, x_4, x_5, x_6) = (0, 0, 0, 0, 0, 0)$, then the characteristic equation becomes

$$\{-1+\beta_1 b_1(\lambda)(1-\theta_1)\}\{-1+\beta_2 b_2(\lambda)(1-\theta_2)\}\{-1+\beta_3 b_3(\lambda)(1-\theta_3)\}$$

$$(-\lambda+\beta_1-\mu_1)(-\lambda+\beta_2-\mu_2)(-\lambda+\beta_3-\mu_3) = 0. \tag{68}$$

This implies either

$$\{-1+\beta_i b_i(\lambda)(1-\theta_i)\} = 0, \tag{69}$$

or

$$(-\lambda+\beta_i-\mu_i) = 0. \tag{70}$$

Therefore, we have $\lambda = (\beta_i - \mu_i)$ and this shows that $\lambda < 0$ if $\beta_i < \mu_i$. Now, the nature of the roots of the transcendental equation (69) is now investigated for $i = 1, 2, 3$. Since $b_i(\lambda) = \int_0^T [\exp\{-\int_0^a (\lambda+\sigma_i(s))\,ds\}]da$, $i = 1, 2, 3$, therefore, it can be written as

$$b_i(\lambda) = \int_0^T (1-\lambda\, a)\,\pi_i(a)da.$$

This implies $b_i(\lambda) = \bar{\pi}_i(a) - \lambda A_i$, where $A_i = \int_0^T a\,\pi_i(a)\,da$ and therefore, (69) takes the form

$$(1-\theta_i)\,\beta_i\,(\bar{\pi}_i - \lambda A_i) - 1 = 0.$$

Thus,

$$\lambda = \frac{(1-\theta_i)\,\beta_i\,\bar{\pi}_i - 1}{(1-\theta_i)\,\beta_i A_i}, \; i = 1, 2, 3. \tag{71}$$

Let $D_i(K) = \{(1-\theta_i)\,\beta_i\,\bar{\pi}_i - 1\}$. So, the origin will be stable when $D_i(K) < 0$, i.e., $(1-\theta_i)\beta_i\,\bar{\pi}_i < 1$ and unstable otherwise. Some of the values are presented in Table 1 are as follows:

Table 1 Stability of origin for different values of parameters

K	β_i	μ_i	θ_i	T	δ_i	$D_i(K)$	Remark
0.3	0.01	0.05	0.4	10	0.003	-0.958142	stable
0.4	0.01	0.05	0.4	10	0.003	-0.957353	stable
0.5	0.01	0.05	0.4	10	0.003	-0.956885	stable
0.6	0.01	0.05	0.4	10	0.003	-0.956571	stable
0.7	0.19	0.01	0.4	10	0.003	0.0574185	unstable
0.8	0.19	0.01	0.4	10	0.003	0.0606162	unstable
0.9	0.19	0.01	0.4	10	0.003	0.0631032	unstable

Table 1 shows that if the birth rate is less than death rate, then $D_i(K) < 0$, which implies the stability of the origin. For the birth rate is greater than the death rate, then $D_i(K) > 0$, which implies the instability of the origin for different values of control parameter

4.3 Stability of the non-zero equilibrium state

To analyze the non-zero state for stability, we shall apply the result of Bellman and Cooke's theorem to the characteristic equation (67), taking it in the form $H(\lambda) = 0$, where $H(\lambda)$ is given by the L.H.S. of the (67). For $\lambda = iw$, we obtain

$$b_1(iw) = f_1(w) + i g_1(w),\, b_2(iw) = f_2(w) + i g_2(w),\, b_3(iw) = f_3(w) + i g_3(w),$$

where

$$f_1(w) = \int_0^T Cos(wa)\,\pi_1(a)\,da,\; g_1(w) = -\int_0^T Sin(wa)\,\pi_1(a)\,da,$$

$$f_2(w) = \int_0^T Cos(wa)\,\pi_2(a)\,da,\; g_2(w) = -\int_0^T Sin(wa)\,\pi_2(a)\,da,$$

and

$$f_3(w) = \int_0^T Cos(wa)\,\pi_3(a)\,da,\; g_3(w) = -\int_0^T Sin(wa)\,\pi_3(a)\,da.$$

For $w=0$, we have $f_i(0)=\bar{\pi}_i$, $g_i(0)=0$. Also $f_i^{'}(0)=0$, $g_i^{'}(0)=-A_i$, where $A_i=\int\limits_0^T a\pi_i(a)da$, $i=1,2,3$. Thus, $H(\lambda)$ takes the form

$$H(iw)=F(w)+iG(w),$$

where $F(w)$ is the real part and $G(w)$ is the imaginary part of $H(iw)$. Now, we need to obtain the condition for which the inequality

$$F(w)\,G'(w)-F'(w)\,G(w)>0 \tag{72}$$

holds at $w=0$, i.e., $F(0)\,G'(0)>0$. In this case, $D_i(K)=F(0)\,G'(0)$, then the non-zero state will be stable when $D_i(K)>0$.

Table 2 Stability of non-zero state for different values of parameters

K	α_{ii}	b_i	β_i	μ_i	θ_i	T	δ_i	$D_i(K)$	Remark
0.3	0.001	0.01	0.1	0.05	0.4	1	0.003	1.0233097	stable
0.4	0.001	0.01	0.1	0.05	0.4	1	0.003	1.0382323	stable
0.5	0.001	0.01	0.1	0.05	0.4	1	0.003	1.0471858	stable
0.6	0.001	0.01	0.1	0.05	0.4	1	0.003	1.0531549	stable
0.7	0.001	0.01	0.1	0.05	0.4	1	0.003	1.0574185	stable
0.8	0.001	0.01	0.1	0.05	0.4	1	0.003	1.0606162	stable
0.9	0.001	0.01	0.1	0.05	0.4	1	0.003	1.0631032	stable

Table 2 shows that the stability of the non-zero state i.e.,$D_i(K)>0$, for different values of parameters. In this table, we assume that the non-zero equilibrium state is stable only for $K\geq 0.3$, otherwise unstable.

5. Discussion

In this deterministic model for heterogeneous population of three different communities e-valuated after translating in ordinary and partial differential equations, the stability condition for the zero and non-zero equilibrium states are obtained. For the stability of zero equilibrium

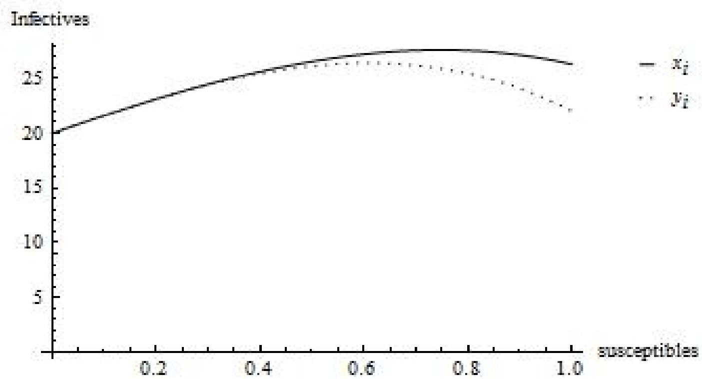

FIGURE 1. x_i versus y_i for $\beta_i = 0.5$, $\mu_i = 0.05$, $\alpha_{ii} = 0.002$, $\alpha_{ij} = 0.001$, $\delta_i = 0.003$, $\theta_i = 0.02$ and $\bar{\pi}_i = 0.01$.

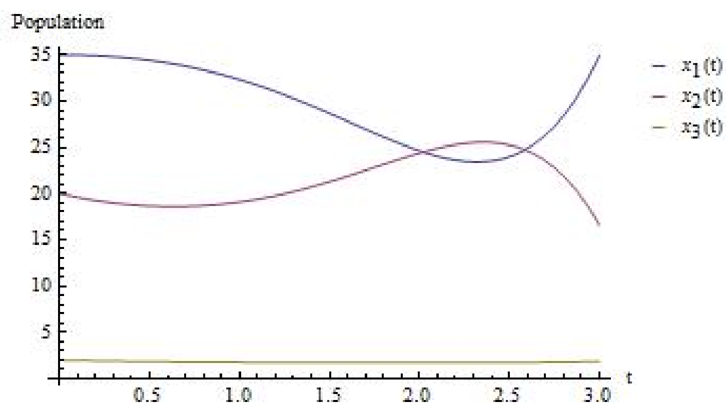

FIGURE 2. x_i versus t for $\beta_i = 0.5$, $\mu_i = 0.05$, $\alpha_{ii} = 0.002$, $\alpha_{ij} = 0.001$, $\delta_i = 0.003$, $\theta_i = 0.02$ and $\bar{\pi}_i = 0.01$.

state, the real part of the roots of the characteristic equation must be negative and thus we obtain $(1-\theta_i)\beta_i\bar{\pi}_i < 1$. In Table 1, all the conditions are satisfied for zero equilibrium state for the birth rate is less than the death rate. Now, by applying Bellman and Cooke s theorem at non-zero equilibrium state, the obtained transcendental equation $H(iw) = F(w) + iG(w)$, whose real and imaginary parts necessarily follow the inequality (72) for at least one value of w. In our case, the inequality satisfied at $w = 0$. Results of stability analysis for different values of parameters are shown in Table 2. As per our assumption, the behaviour of different values of parameters birth rate, death rate, control parameter and the numerical results are graphically shown in Fig. 1-3.

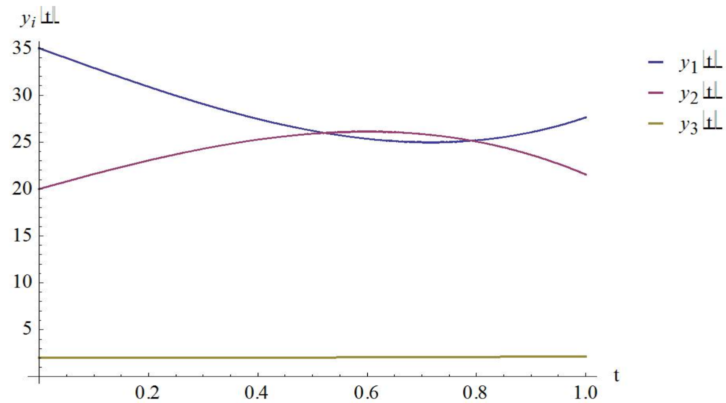

FIGURE 3. y_i versus t for $\beta_i = 0.5$, $\mu_i = 0.05$, $\alpha_{ii} = 0.002$, $\alpha_{ij} = 0.001$, $\delta_i = 0.003$, $\theta_i = 0.02$ and $\bar{\pi}_i = 0.01$.

6. Conclusion

The deterministic mathematical model is proposed with age-dependent HIV/AIDS infection for heterogeneous population of three different communities. Here, each community is divided into two classes: susceptible and HIV infectives. The infected individuals are age-dependent and structured with the age-density function $\rho(t, a)$ where 't' is the time and 'a' is the infection age. There is the maximum infection age 'T' at which the infected individuals must leave the compartment via death. The zero and non-zero equilibrium states are obtained for population suppression and maintenance in these communities which depends on the control parameter K. The zero equilibrium state is stable if $D_i(K) < 0$, which are analyzed in Table 1 for different values of parameters. The non-zero equilibrium state is stable if the inequality (72) holds which are verified in Table 2. The stability analysis of equilibrium states implies that the zero and non-zero equilibrium states must be bounded in these communities through public awareness campaign.

Conflict of Interests

The authors declare that there is no conflict of interests.

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