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GLOBAL DYNAMICS OF AN ECO-EPIDEMIOLOGICAL MODEL WITH BEDDINGTON-DEANGELIS FUNCTIONAL RESPONSE

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Abstract. In this paper, a predator-prey eco-epidemiological model with Beddington-DeAngelis functional response is investigated. In the model, it is assumed that the predator population suffers a transmissible disease. By means of Lyapunov functions and LaSalle's invariance principle, sufficient conditions are derived for the global stability of the endemic-coexistence equilibrium, the disease-free equilibrium and the predator-extinction equilibrium of the system, respectively.

Keywords: Eco-epidemiological model; LaSalle's invariance principle; Global stability analysis.

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1. Introduction

In recent years, great attention has been paid by many researchers to study the effect of disease transmission in ecological system (see, for example, [1-7]). Most of these works dealt with predator-prey models with disease in the prey. Recently, several authors proposed different eco-epidemiological predator-prey models by assuming that the predator population suffers

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a transmissible disease (see, for example, [8-11]). In [12], by assuming that a transmissible disease spreads among the predator population, Sun and Yuan considered the following eco-epidemiological model:

$$\begin{aligned} \dot{x}(t) &= x(t)(r - a_{11}x(t)) - a_{12}x(t)S(t), \\ \dot{S}(t) &= ka_{12}x(t)S(t) - r_1 S(t) - \beta S(t)I(t), \\ \dot{I}(t) &= \beta S(t)I(t) - r_2 I(t), \end{aligned} \tag{1.1}$$

where $x(t), S(t)$ and $I(t)$ represent the densities of the prey, susceptible (sound) predator and the infected predator population at time t, respectively. The parameters $k, a_{11}, a_{12}, r, r_1, r_2$ and β are positive constants. In system (1.1), the following assumptions have been made:

- (A1) In the absence of predation, the prey population $x(t)$ grows logistically with the intrinsic growth rate $r > 0$ and carrying capacity r/a_{11}, in which a_{11} measures intraspecific competition of the prey.
- (A2) The total predator population N is composed of two population classes: one is the class of susceptible (sound) predator, denoted by S, and the other is the class of infected predator, denoted by I.
- (A3) The disease spreads among the predator species only by contact and the disease can not be transmitted vertically. The infected predator population do not recover or become immune. The disease incidence is assumed to be the simple mass action incidence βSI, where $\beta > 0$ is called the disease transmission coefficient.
- (A4) The parameter a_{12} is the capturing rate of the sound predator, and k is the conversion rate of nutrients into the reproduction of the sound and infected predators by consuming prey, r_1 is the natural death rate of the sound predator, r_2 is the natural and disease-related mortality rate of the infected predator. Here, $r_1 \leq r_2$ is reasonable for biological meaning.

In [12], by using suitable Lyapunov function and LaSalle's invariance principle, the global asymptotic stability of the disease-free equilibrium and the endemic equilibrium were given.

In population dynamics, the functional response plays an important role in the research of predator-prey interactions. In [13], based on experiment, Holling suggested three kinds of functional response for different species to model the phenomena of predation as follows:

$$(1) \quad \phi_1(x) = ax, \quad (2) \quad \phi_2(x) = \frac{ax}{b+x}, \quad (3) \quad \phi_3(x) = \frac{ax^2}{b+x^2},$$

where x represents the density of prey. Functions $\phi_1(x), \phi_2(x)$ and $\phi_3(x)$ are now referred to as Holling type I, II and III functional responses. Here $a > 0$ denotes the search rate of the predator and $b > 0$ is the half-saturation constant. In classical predator-prey models, Holling type functional responses are frequently used. However, these functional responses fail to model the interference among predators, and have been facing challenges from the biology and physiology communities [14,15].

We note that many predators compete for prey. This can result in time wasted in interfering each other's effort of capturing and consuming prey [16]. For this reason, some works have shown that the functional response in a prey-predator model should be predator-dependent [17]. In 1975, Beddington [18] and DeAngelis et al. [19] introduced a generalisation of the Holling functional response as follows:

$$\varphi(x,y) = \frac{kx}{1+ax+by},$$

where the term by measures the mutual interference between predators. When $a > 0, b = 0$, the Beddington-DeAngelis functional response is simplified to Holling type II functional response [13]. Compared with Holling type II functional response, this functional response takes into account the delay in time incurred by the predators as a result of interspecific competition for the same prey species.

Motivated by the works of Sun and Yuan [12], Beddington [18] and DeAngelis et al. [19], in this paper, we are concerned with the combined effects of the disease transmission in the predator population and Beddington-DeAngelis functional response on the global dynamics of a predator-prey model. To this end, we study the following eco-epidemiological model:

$$\begin{aligned}
\dot{x}(t) &= x(t)(r - a_{11}x(t)) - \frac{a_{12}x(t)S(t)}{1 + ax(t) + bS(t)},\\
\dot{S}(t) &= \frac{a_{21}x(t)S(t)}{1 + ax(t) + bS(t)} - r_1 S(t) - \beta S(t)I(t),\\
\dot{I}(t) &= \beta S(t)I(t) - r_2 I(t),
\end{aligned} \tag{1.2}$$

where $x/(1 + ax + bS)$ describes the Beddington-DeAngelis functional response which incorporates mutual interference by predators. a_{21}/a_{12} is the conversion rate of nutrients into the reproduction of the predator by consuming prey. All other parameters are defined as in (A1)-(A4).

The initial conditions for system (1.2) takes the form

$$x(0) > 0, \quad S(0) > 0, \quad I(0) > 0. \tag{1.3}$$

It is easy to show that all solutions of system (1.2) with initial condition (1.3) are defined on $[0, +\infty)$ and remain positive for all $t \geq 0$.

The organization of this paper is as follows. In Section 2, by means of suitable Lyapunov functions and LaSalle's invariance principle, we establish sufficient conditions for the global stability of the endemic-coexistence equilibrium, the disease-free equilibrium and the predator-extinction equilibrium of system (1.2), respectively. In Section 3, numerical simulations are carried out to illustrate the main theoretical results in Section 2. A brief discussion is given in Section 4 to end this work.

2. Global stability

In this section, we are concerned with the global stability of each of feasible equilibria of system (1.2). The technique of proofs is to construct suitable Lyapunov functions and LaSalle's invariance principle.

Clearly, system (1.2) always has a trivial equilibrium $E_0(0,0,0)$ and a predator-extinction equilibrium $E_1(r/a_{11}, 0, 0)$. If the following holds:

(H1) $a_{21}r > r_1(a_{11} + ar)$,

then system (1.2) has a disease-free equilibrium $E_2(x_2, S_2, 0)$, where

$$x_2 = \frac{-(a_{12}a_{21} - aa_{12}r_1 - a_{21}br) + \sqrt{\triangle_1}}{2a_{11}a_{21}b}, \quad S_2 = \frac{a_{21}x_2(r - a_{11}x_2)}{a_{12}r_1},$$

here $\triangle_1 = (a_{12}a_{21} - aa_{12}r_1 - a_{21}br)^2 + 4a_{11}a_{12}a_{21}br_1$. Further, if the following holds:

$$\text{(H2)} \quad \frac{\beta[aa_{12}r_1r_2 + a_{21}(brr_2 + r\beta - a_{12}r_2)]^2 + a_{11}a_{12}a_{21}r_1r_2(br_2 + \beta)^2}{a_{21}\beta r(br_2 + \beta)[aa_{12}r_1r_2 + a_{21}(brr_2 + r\beta - a_{12}r_2)]} < 1,$$

in addition to the equilibria E_0, E_1 and E_2, system (1.2) has a unique endemic-coexistence equilibrium $E^*(x^*, S^*, I^*)$, where

$$x^* = \frac{-(a_{11}br_2 + a_{11}\beta - ar\beta) + \sqrt{\triangle_2}}{2aa_{11}\beta}, \quad S^* = \frac{r_2}{\beta}, \quad I^* = \frac{a_{21}\beta x^*(r - a_{11}x^*) - a_{12}r_1r_2}{a_{12}r_2\beta},$$

here $\triangle_2 = (a_{11}br_2 + a_{11}\beta - ar\beta)^2 - 4aa_{11}\beta(a_{12}r_2 - brr_2 - r\beta)$.

It is easy to show that the equilibrium $E_0(0,0,0)$ is always unstable.

We now give a result on the upper bound of positive solutions of system (1.2).

Lemma 2.1. *There are positive constants M_1 and M_2 such that for any positive solution $(x(t), S(t), I(t))$ of system (1.2) with initial conditions (1.3),*

$$\limsup_{t\to+\infty} x(t) < M_1, \quad \limsup_{t\to+\infty} S(t) < M_2, \quad \limsup_{t\to+\infty} I(t) < M_2. \tag{2.1}$$

Proof. Let $(x(t), S(t), I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). Define

$$V_0(t) = x(t) + \frac{a_{12}}{a_{21}}(S(t) + I(t)).$$

Calculating the derivative of $V_0(t)$ along positive solutions of system (1.2), it follows that

$$\begin{aligned}
\frac{d}{dt}V_0(t) =& x(t)(r - a_{11}x(t)) - \frac{a_{12}}{a_{21}}(r_1S(t) + r_2I(t)) \\
=& -r_1V_0(t) + x(t)(r + r_1 - a_{11}x(t)) + \frac{a_{12}}{a_{21}}(r_1 - r_2)I(t) \\
\leq& -r_1V_0(t) + \frac{(r + r_1)^2}{4a_{11}},
\end{aligned}$$

which yields $\limsup_{t\to+\infty} V_0(t) \leq k(r + r_1)^2/(4a_{11}r_1)$. If we choose

$$M_1 = \frac{(r + r_1)^2}{4a_{11}r_1}, \quad M_2 = \frac{a_{21}(r + r_1)^2}{4a_{11}a_{12}r_1}, \tag{2.2}$$

then (2.1) follows. This completes the proof.

Lemma 2.2. *Assume that* $4a_{11}rr_1 > a_{21}(r+r_1)^2$. *Then for any positive solution* $(x(t),S(t),I(t))$ *of system (1.2) with initial conditions (1.3), we have that*

$$\liminf_{t\to+\infty} x(t) > \underline{x} := \frac{r-a_{12}M_2}{a_{11}}, \tag{2.3}$$

where M_2 *is defined in (2.2).*

Proof. Let $(x(t),S(t),I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). By Lemma it follows that $\limsup_{t\to+\infty} S(t) \le M_2$. Hence, for $\varepsilon > 0$ being sufficiently small, there is a $T_0 > 0$ such that if $t > T_0, S(t) < M_2 + \varepsilon$. Accordingly, for $\varepsilon > 0$ being sufficiently small, we derive from the first equation of system (1.2) that, for $t > T_0$,

$$\dot{x}(t) \ge x(t)[r - a_{11}x(t) - a_{12}(M_2+\varepsilon)],$$

which yields

$$\liminf_{t\to+\infty} x(t) \ge \underline{x} := \frac{r-a_{12}M_2}{a_{11}}.$$

This completes the proof.

We now investigate the global stability of the endemic-coexistence equilibrium $E^*(x^*,S^*,I^*)$ of system (1.2).

Theorem 2.1. *If (H2) holds, then the endemic-coexistence equilibrium* $E^*(x^*,S^*,I^*)$ *of system (1.2) is globally asymptotically stable provided that*

(H3) $\underline{x} > r/(2a_{11})$.

Here, $\underline{x} > 0$ *is defined in (2.3).*

Proof. Let $(x(t),S(t),I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). Define

$$\begin{aligned} V_1(t) =& \frac{a_{21}}{a_{12}}\left(x(t)-x^* - (r_1S^*+r_2I^*)\int_{x^*}^{x(t)} \frac{1+a\tau+bS^*}{a_{21}\tau S^*}d\tau\right) \\ &+ S(t)-S^*-S^*\ln\frac{S(t)}{S^*} + I(t)-I^*-I^*\ln\frac{I(t)}{I^*}. \end{aligned} \tag{2.4}$$

Calculating the derivative of $V_1(t)$ along positive solutions to system (1.2), it follows that

$$\begin{aligned}\frac{d}{dt}V_1(t) =&\frac{a_{21}}{a_{12}}\left(1-(r_1S^*+r_2I^*)\frac{1+ax(t)+bS^*}{a_{21}x(t)S^*}\right)\\&\times\left(x(t)(r-a_{11}x(t))-\frac{a_{12}x(t)S(t)}{1+ax(t)+bS(t)}\right)\\&+\left(1-\frac{S^*}{S(t)}\right)\left(\frac{a_{21}x(t)S(t)}{1+ax(t)+bS(t)}-r_1S(t)-\beta S(t)I(t)\right)\\&+\left(1-\frac{I^*}{I(t)}\right)(\beta S(t)I(t)-r_2I(t)).\end{aligned}\tag{2.5}$$

Noting that $x^*(r-a_{11}x^*)=a_{12}x^*S^*/(1+ax^*+bS^*)$ and $r_1S^*+r_2I^*=a_{21}x^*S^*/(1+ax^*+bS^*)$, then Eq. (2.5) can be rewritten as

$$\begin{aligned}\frac{d}{dt}V_1(t) =&\frac{a_{21}}{a_{12}}\left(1-(r_1S^*+r_2I^*)\frac{1+ax(t)+bS^*}{a_{21}x(t)S^*}\right)\\&\times\left[x(t)(r-a_{11}x(t))-x^*(r-a_{11}x^*)+\frac{a_{12}x^*S^*}{1+ax^*+bS^*}\right]\\&-\frac{a_{21}}{a_{12}}\left(1-(r_1S^*+r_2I^*)\frac{1+ax(t)+bS^*}{a_{21}x(t)S^*}\right)\frac{a_{12}x(t)S(t)}{1+ax(t)+bS(t)}\\&+\frac{a_{21}x(t)S(t)}{1+ax(t)+bS(t)}-\frac{S^*}{S(t)}\frac{a_{21}x(t)S(t)}{1+ax(t)+bS(t)}\\&-r_1S(t)-\beta S(t)I^*+(r_1S^*+r_2I^*)\\=&\frac{a_{21}}{a_{12}}\left(1-\frac{x^*}{x(t)}\frac{1+ax(t)+bS^*}{1+ax^*+bS^*}\right)\\&\times\left[(x(t)-x^*)[r-a_{11}(x(t)+x^*)]+\frac{a_{12}x^*S^*}{1+ax^*+bS^*}\right]\\&+(r_1S^*+r_2I^*)\frac{S(t)}{S^*}\frac{1+ax(t)+bS^*}{1+ax(t)+bS(t)}-\frac{S^*}{S(t)}\frac{a_{21}x(t)S(t)}{1+ax(t)+bS(t)}\\&-r_1S^*\frac{S(t)}{S^*}-\beta S^*I^*\frac{S(t)}{S^*}+(r_1S^*+r_2I^*).\end{aligned}\tag{2.6}$$

On substituting $a_{21}x^*S^*/(1+ax^*+bS^*) = r_1S^* + \beta S^*I^*$ into Eq. (2.6), we obtain

$$\begin{aligned}
\frac{d}{dt}V_1(t) =& \frac{a_{21}(1+bS^*)(x(t)-x^*)^2}{a_{12}x(t)(1+ax^*+bS^*)}[r-a_{11}(x(t)+x^*)] \\
&+(r_1S^*+r_2I^*)\frac{(1+bS^*)(x(t)-x^*)}{x(t)(1+ax^*+bS^*)} \\
&+(r_1S^*+r_2I^*)\left[\frac{S(t)}{S^*}\frac{1+ax(t)+bS^*}{1+ax(t)+bS(t)}-\frac{S(t)}{S^*}-1+\frac{1+ax(t)+bS(t)}{1+ax(t)+bS^*}\right] \\
&+2(r_1S^*+r_2I^*)-\frac{S^*}{S(t)}\frac{a_{21}x(t)S(t)}{1+ax(t)+bS(t)}-(r_1S^*+r_2I^*)\frac{1+ax(t)+bS(t)}{1+ax(t)+bS^*} \\
=&\frac{a_{21}(1+bS^*)(x(t)-x^*)^2}{a_{12}x(t)(1+ax^*+bS^*)}[r-a_{11}(x(t)+x^*)] \\
&-\frac{b(r_1S^*+r_2I^*)(1+ax(t))(S(t)-S^*)^2}{S^*(1+ax(t)+bS^*)(1+ax(t)+bS(t))} \\
&+(r_1S^*+r_2I^*)\left[2-\frac{1+ax(t)+bS(t)}{1+ax(t)+bS^*}-\frac{x(t)}{x^*}\frac{1+ax^*+bS^*}{1+ax(t)+bS(t)}\right] \\
&+(r_1S^*+r_2I^*)\frac{1+bS^*}{1+ax^*+bS^*}-(r_1S^*+r_2I^*)\frac{x^*}{x(t)}\frac{1+bS^*}{1+ax^*+bS^*}.
\end{aligned} \tag{2.7}$$

It follows from Eqs. (2.5)-(2.7) that

$$\begin{aligned}
\frac{d}{dt}V_1(t) =& \frac{a_{21}(1+bS^*)(x(t)-x^*)^2}{a_{12}x(t)(1+ax^*+bS^*)}[r-a_{11}(x(t)+x^*)] \\
&-\frac{b(r_1S^*+r_2I^*)(1+ax(t))(S(t)-S^*)^2}{S^*(1+ax(t)+bS^*)(1+ax(t)+bS(t))} \\
&+(r_1S^*+r_2I^*)\left[3-\frac{x^*}{x(t)}\frac{1+ax(t)+bS^*}{1+ax^*+bS^*}\right. \\
&\left.-\frac{x(t)}{x^*}\frac{1+ax^*+bS^*}{1+ax(t)+bS(t)}-\frac{1+ax(t)+bS(t)}{1+ax(t)+bS^*}\right].
\end{aligned} \tag{2.8}$$

Since (H3) holds, there is a constant $T>0$ such that if $t \geq T$, $x(t) > r_1/(2a_{11})$. In this case, we have that, for $t \geq T$,

$$\frac{a_{21}(1+bS^*)(x(t)-x^*)^2}{a_{12}x(t)(1+ax^*+bS^*)}[r-a_{11}(x(t)+x^*)] \leq 0$$

with equality if and only if $x = x^*$. Further, since the arithmetic mean is greater than or equal to the geometric mean, it is clear that

$$\frac{x^*}{x}\frac{1+ax+bS^*}{1+ax^*+bS^*}+\frac{x}{x^*}\frac{1+ax^*+bS^*}{1+ax+bS}+\frac{1+ax+bS}{1+ax+bS^*} \geq 3,$$

and the equality holds only for $x = x^*, S = S^*$. Therefore, we have that if $t \geq T$, $V_1'(t) \leq 0$. By Theorem 4.1 in [20], solutions limit $\mathscr{M}$, the largest invariant subset of $\{V_1'(t)\} = 0$. Clearly, we see from (2.8) that $V_1'(t) = 0$ if and only if $x = x^*, S = S^*$. It therefore follows from the second equation of system (1.2) that

$$0 = \dot{S}(t) = S^* \left(\frac{a_{21} x^*}{1 + a x^* + b S^*} - r_1 - \beta I(t) \right),$$

which yields $I = I^*$. Hence, $V_1'(t) = 0$ if and only if $(x, S, I) = (x^*, S^*, I^*)$. Therefore, the global asymptotic stability of E^* follows from LaSalle's invariance principle. This completes the proof.

Theorem 2.2. *If (H1) holds, then the disease-free equilibrium* $E_2(x_2, S_2, 0)$ *of system (1.2) is globally asymptotically stable provided that*

(H3) $\underline{x} > r/(2a_{11})$.

Here, $\underline{x} > 0$ *is defined in (2.3).*

Proof. Let $(x(t), S(t), I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). Define

$$\begin{aligned} V_2(t) = & \frac{a_{21}}{a_{12}} \left(x(t) - x_2 - r_1 S_2 \int_{x_2}^{x(t)} \frac{1 + a\tau + bS_2}{a_{21} \tau S_2} d\tau \right) \\ & + S(t) - S_2 - S_2 \ln \frac{S(t)}{S_2} + I(t). \end{aligned} \tag{2.9}$$

Calculating the derivative of $V_2(t)$ along positive solutions to system (1.2), we obtain that

$$\begin{aligned} \frac{d}{dt} V_2(t) = & \frac{a_{21}}{a_{12}} \left(1 - r_1 S_2 \frac{1 + a x(t) + b S_2}{a_{21} x(t) S_2} \right) \left[x(t)(r - a_{11} x(t)) - \frac{a_{12} x(t) S(t)}{1 + a x(t) + b S(t)} \right] \\ & + \frac{a_{21} x(t) S(t)}{1 + a x(t) + b S(t)} - \frac{S_2}{S(t)} \frac{a_{21} x(t) S(t)}{1 + a x(t) + b S(t)} - r_1 S(t) + r_1 S_2 + (\beta S_2 - r_2) I(t). \end{aligned} \tag{2.10}$$

Noting that $x_2(r-a_{11}x_2)=a_{12}x_2S_2/(1+ax_2+bS_2)$, then Eq. (2.10) can be rewritten as

$$\begin{aligned}\frac{d}{dt}V_2(t)=&\frac{a_{21}}{a_{12}}\left(1-r_1S_2\frac{1+ax(t)+bS_2}{a_{21}x(t)S_2}\right)\\&\times\left[x(t)(r-a_{11}x(t))-x_2(r-a_{11}x_2)+\frac{a_{12}x_2S_2}{1+ax_2+bS_2}\right]\\&-\frac{a_{21}}{a_{12}}\left(1-r_1S_2\frac{1+ax(t)+bS_2}{a_{21}x(t)S_2}\right)\frac{a_{12}x(t)S(t)}{1+ax(t)+bS(t)}\\&+\frac{a_{21}x(t)S(t)}{1+ax(t)+bS(t)}-\frac{S_2}{S(t)}\frac{a_{21}x(t)S(t)}{1+ax(t)+bS(t)}-r_1S_2\frac{S(t)}{S_2}\\&+r_1S_2+(\beta S_2-r_2)I(t).\end{aligned}\tag{2.11}$$

On substituting $a_{21}x_2S_2/(1+ax_2+bS_2)=r_1S_2$ into Eq. (2.11), we derive that

$$\begin{aligned}\frac{d}{dt}V_2(t)=&\frac{a_{21}}{a_{12}}\left(1-\frac{x_2}{x(t)}\frac{1+ax(t)+bS_2}{1+ax_2+bS_2}\right)(x(t)-x_2)[r-a_{11}(x(t)+x_2)]\\&+r_1S_2\left(1-\frac{x_2}{x(t)}\frac{1+ax(t)+bS_2}{1+ax_2+bS_2}\right)\\&+r_1S_2\left(\frac{S(t)}{S_2}\frac{1+ax(t)+bS_2}{1+ax(t)+bS(t)}-\frac{S(t)}{S_2}\right)+r_1S_2\\&-r_1S_2\frac{x(t)}{x_2}\frac{1+ax_2+bS_2}{1+ax(t)+bS(t)}+(\beta S_2-r_2)I(t)\\=&\frac{a_{21}(1+bS_2)(x(t)-x_2)^2}{a_{12}x(t)(1+ax_2+bS_2)}[r-a_{11}(x(t)+x_2)]+(\beta S_2-r_2)I(t)\\&+r_1S_2\left(2-\frac{x_2}{x(t)}\frac{1+ax(t)+bS_2}{1+ax_2+bS_2}-\frac{x(t)}{x_2}\frac{1+ax_2+bS_2}{1+ax(t)+bS(t)}\right)\\&+r_1S_2\left(\frac{S(t)}{S_2}\frac{1+ax(t)+bS_2}{1+ax(t)+bS(t)}-\frac{S(t)}{S_2}\right).\end{aligned}\tag{2.12}$$

It follows from Eqs. (2.10)-(2.12) that

$$\begin{aligned}\frac{d}{dt}V_2(t)=&\frac{a_{21}(1+bS_2)(x(t)-x_2)^2}{a_{12}x(t)(1+ax_2+bS_2)}[r-a_{11}(x(t)+x_2)]+(\beta S_2-r_2)I(t)\\&-\frac{br_1(1+ax(t))(S(t)-S_2)^2}{(1+ax(t)+bS(t))(1+ax(t)+bS_2)}\\&+r_1S_2\left[3-\frac{x_2}{x(t)}\frac{1+ax(t)+bS_2}{1+ax_2+bS_2}\right.\\&\left.-\frac{x(t)}{x_2}\frac{1+ax_2+bS_2}{1+ax(t)+bS(t)}-\frac{1+ax(t)+bS(t)}{1+ax(t)+bS_2}\right].\end{aligned}\tag{2.13}$$

Since the arithmetic mean is greater than or equal to the geometric mean, it is clear that

$$\frac{x_2}{x}\frac{1+ax+bS_2}{1+ax_2+bS_2}+\frac{x}{x_2}\frac{1+ax_2+bS_2}{1+ax+bS}+\frac{1+ax+bS}{1+ax+bS_2}\geq 3,$$

and the equality holds only for $x=x_2, S=S_2$. If (H1) holds, then $\beta S_2-r_2<0$. Therefore, if (H1) holds, then $V_2'(t)\leq 0$ for $t\geq T$, with equality if and only if $x=x_2, S=S_2, I=0$. Using a similar argument as that in the proof of Theorem 2.1, we show that the only invariant set in $\mathscr{M}=\{(x,S,I):V_1'(t)=0\}$ is $\mathscr{M}=\{(x_2,S_2,0)\}$. Therefore, the global asymptotic stability of E_2 follows from LaSalle's invariance principle. This completes the proof.

Theorem 2.3. *If $a_{21}r<r_1(a_{11}+ar)$, then the predator-extinction equilibrium $E_1(r/a_{11},0,\ 0)$ of system (1.2) is globally asymptotically stable.*

Proof. Let $(x(t),S(t),I(t))$ be any positive solution of system (1.2) with initial conditions (1.3). Denote $x_1=r/a_{11}$.

Define

$$V_3(t)=\frac{a_{21}}{a_{12}(1+ax_1)}\left(x(t)-x_1-x_1\ln\frac{x(t)}{x_1}\right)+S(t)+I(t). \tag{2.14}$$

Calculating the derivative of $V_3(t)$ along positive solutions to system (1.2), it follows that

$$\begin{aligned}\frac{d}{dt}V_3(t)=&\frac{a_{21}}{a_{12}(1+ax_1)}\left(1-\frac{x_1}{x(t)}\right)\left(x(t)(r-a_{11}x(t))-\frac{a_{12}x(t)S(t)}{1+ax(t)+bS(t)}\right)\\&+\frac{a_{21}x(t)S(t)}{1+ax(t)+bS(t)}-r_1S(t)-r_2I(t).\end{aligned} \tag{2.15}$$

On substituting $r=a_{11}x_1$ into (2.15), one obtains

$$\begin{aligned}\frac{d}{dt}V_3(t)=&-\frac{a_{11}a_{21}}{a_{12}(1+ax_1)}(x(t)-x_1)^2-r_2I(t)-r_1S(t)+\frac{(1+ax(t))S(t)}{1+ax(t)+bS(t)}\frac{a_{21}x_1}{1+ax_1}.\\=&-\frac{a_{11}a_{21}}{a_{12}(1+ax_1)}(x(t)-x_1)^2-r_2I(t)-\frac{br_1S^2(t)}{1+ax(t)+bS(t)}\\&+\frac{r_1(1+ax(t))S(t)}{1+ax(t)+bS(t)}\frac{a_{21}r-r_1(a_{11}+ar)}{r_1(a_{11}+ar)}.\end{aligned} \tag{2.16}$$

Let $\mathscr{M}$ be the largest invariant subset of $\{V_3'(t)=0\}$. Clearly, if $a_{21}r<r_1(a_{11}+ar)$, we derive from (2.16) that $V_3'(t)\leq 0$, with equality if and only if $x=x_1, S=0, I=0$. Noting that $\mathscr{M}$ is invariant, for each element in $\mathscr{M}$, we have $x=x_1, S=0, I=0$. Hence, $V_3'(t)=0$ if and only if $(x,S,I)=(x_1,0,0)$. Therefore, the predator-extinction equilibrium E_1 is globally asymptotically stable. This completes the proof.

3. Numerical examples

In this section, we give some examples to illustrate the main theoretical results above.

Example 1. In system (1.2), let $a_{11} = 4.5, a_{12} = 0.1, a_{21} = 1.3, a = 0.1, b = 0.1, r = 4.5, r_1 = 1, r_2 = 1.2$ and $\beta = 0.9$. By calculation, we have

$$\frac{\beta[aa_{12}r_1r_2 + a_{21}(brr_2 + r\beta - a_{12}r_2)]^2 + a_{11}a_{12}a_{21}r_1r_2(br_2 + \beta)^2}{a_{21}\beta r(br_2 + \beta)[aa_{12}r_1r_2 + a_{21}(brr_2 + r\beta - a_{12}r_2)]} \approx 0.9751 < 1.$$

In this case, system (1.2) has an endemic-coexistence equilibrium $E^*(0.9759, 1.3333, 0.0341)$. Further, we obtain $\underline{x} - r/(2a_{11}) \approx 0.0145 > 0$. By Theorem 2.1, we see that E^* is globally asymptotically stable. Numerical simulation illustrates the above result (see, Fig. 1).

Example 2. In system (1.2), let $a_{11} = 12, a_{12} = 0.5, a_{21} = 3.5, a = 0.5, b = 0.5, r = 2, r_1 = 0.5, r_2 = 0.9$ and $\beta = 0.25$. It is easy to show that (H1) holds true. In this case, system (1.2) has a disease-free equilibrium $E_2(0.1625, 0.1129, 0)$. By calculation, we have $\underline{x} - r/(2a_{11}) \approx 0.0074 > 0$. By Theorem 2.2, we see that E_2 is globally asymptotically stable. Numerical simulation illustrates this fact (see, Fig. 2).

Example 3. In system (1.2), let $a_{11} = 1, a_{12} = 0.5, a_{21} = 0.5, a = 0.25, b = 0.25, r = 1, r_1 = 0.5, r_2 = 1$ and $\beta = 0.5$. Clearly, system (1.2) has a predator-extinction equilibrium $E_1(0.1250, 0, 0)$. By calculation, it is easy to show that $r_1(a_{11} + ar) - a_{21}r = 0.1250 > 0$. By Theorem 2.3, we see that E_1 is globally asymptotically stable. Numerical simulation illustrates the above result (see, Fig. 3).

4. Discussion

In this paper, we have investigated the delayed predator-prey model with Beddington-DeAngelis functional response and a transmissible disease spreading among the predator population. By using suitable Lyapunov functions and LaSalle's invariance principle, for system (1.2), the global asymptotic stability of each of feasible equilibria has been established.

By Theorem we see that if the prey population is always abundant enough, the coefficient of the mutual interference between predators b and the disease transmission coefficient β are large enough satisfying (H2), then the solutions of system (1.2) tend to the endemic-coexistence equilibrium which means that the disease becomes endemic in the predator population and the prey, sound predator and the infected predator populations coexist. By Theorem 2.2, we see that if the prey population is always abundant enough, the disease transmission coefficient β is small enough and the conversion rate of the sound predator is large enough, the disease among the predator population dies out and the prey and the sound predator populations coexist. We rewrite $a_{21}r < r_1(a_{11} + ar)$ as $a_{21} < r_1(a_{11}/r + a)$. By Theorem 2.3, we see that if the carrying capacity of the prey and the conversion rate of the sound predators are small enough, and the death rate of the infected predator is sufficiently large, then the prey population persists and the predator population goes to extinction.

We note that when $a = b = 0$, system (1.2) reduces to system (5) in [12] . In this case, it can be seen that the condition in Theorem 6 in [12] is a special case of Theorem 2.3. Furthermore, the condition (H2) in this paper can be rewritten as $\frac{a_{21}(r\beta - a_{12}r_2)^2 + a_{11}a_{12}r_1r_2\beta}{r\beta[a_{21}(r\beta - a_{12}r_2)]} < 1$ if $a = b = 0$. Compared with the result in Theorem 7 in [12] , our work can be viewed as a generalization and improvement of the work developed by Sun and Yuan [12].

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgements

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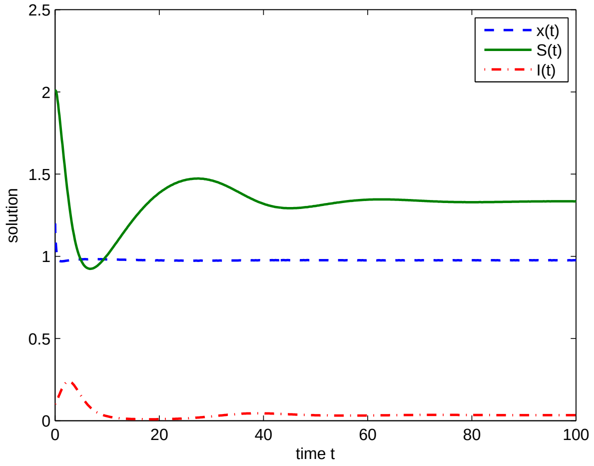

FIGURE 1. The solution and phase portrait found by numerical integration of the system (1.2) with $a_{11} = 4.5, a_{12} = 0.1, a_{21} = 1.3, a = 0.1, b = 0.1, r = 4.5, r_1 = 1, r_2 = 1.2, \beta = 0.9$ and $(\phi_1, \phi_2, \phi_3) = (1.2, 2, 0.1)$.

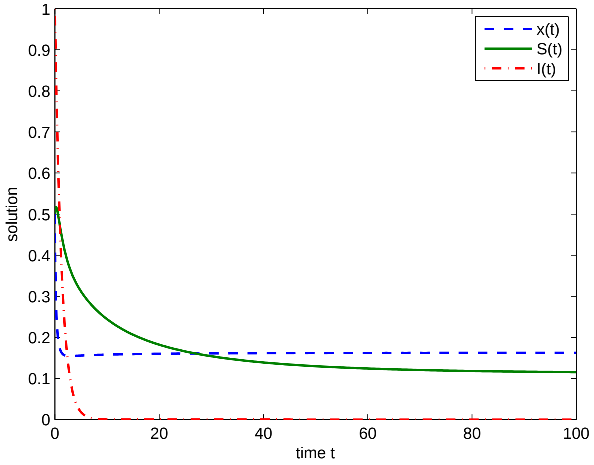

FIGURE 2. The solution and phase portrait found by numerical integration of the system (1.2) with $a_{11} = 12, a_{12} = 0.5, a_{21} = 3.5, a = 0.5, b = 0.5, r = 2, r_1 = 0.5, r_2 = 0.9, \beta = 0.25$ and $(\phi_1, \phi_2, \phi_3) = (0.5, 0.5, 1)$.

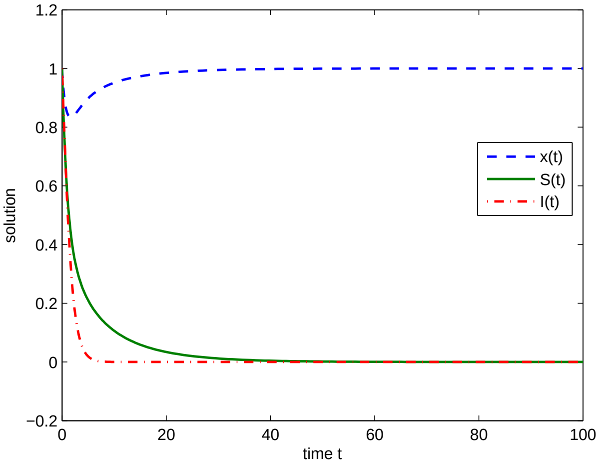

FIGURE 3. The solution and phase portrait found by numerical integration of the system (1.2) with $a_{11} = 1, a_{12} = 0.5, a_{21} = 0.5, a = 0.25, b = 0.25, r = 1, r_1 = 0.5, r_2 = 1, \beta = 0.5$ and $(\phi_1, \phi_2, \phi_3) = (1, 1, 1)$.