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NON-LINEAR HARVESTING OF PREY WITH DYNAMICALLY VARYING EFFORT IN A MODIFIED LESLIE-GOWER PREDATOR-PREY SYSTEM

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Abstract. In this paper, a three dimensional dynamical system incorporating non- linear harvesting effort for prey is investigated. The Holling type-II functional response is considered for prey while predator is assumed to follow Modified Leslie-Gower type dynamics. The steady states of the system are obtained and the local dynamics is explored. The sufficient condition is derived for global stability of its positive interior equilibrium point. The conditions for bionomic equilibrium and uniform persistence of the system have been investigated. It is also observed that the system exhibits transcritical bifurcation for a threshold level of taxation. A taxation policy is discussed with the help of Pontryagin's Maximum Principle as an effective control instrument to preserve the prey species from extinction and maintain a sustainable fishery.

Keywords: predator-prey model; modified Leslie-Gower type predation; stability, transcritical bifurcation; persistence; bionomic equilibrium; optimal taxation policy.

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1. Introduction

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The classical ecological non-linear models of interacting population have been discussed extensively by many authors. Predator- prey system is one of the most important population model. In population dynamics, the functional response of predator to prey density refers the attack rate per unit time w.r.t. prey density. Many researchers have analyzed mathematical models using functional response (e.g., Holling type I, II, III or IV) by considering different type of growth functions depending upon the species [4, 8, 10, 19]. A bio-economic modeling concerns with optimal management of renewable resources. Harvesting has a strong impact on the dynamics of biological resources. The severity of the impact depends on the nature of implementation of the strategy. Basically, there are three types of harvesting strategies that are being used mostly (i) a constant number of species are harvested per unit of time, (ii) proportional harvesting given by Schaefer catch- effort relation $H(x,E) = qEx$. Here, the number of species harvested per unit of time is proportional to the population stock x, q is the catch-ability coefficient and E is the effort applied to harvest, (iii) non-linear harvesting i.e., $H(x,E) = qEx/(m_1E + m_2x)$, where m_1 and m_2 are positive constants.

Effects of harvesting on variety of predator-prey models have been discussed by many researchers [2, 3, 4, 8, 19, 21, 22]. Mathematical modeling with harvesting of renewable resources and the policies related to its bionomic exploitations have been discussed extensively by Clark [3]. Brauer and Soudack [2] discussed the dynamical behavior of predator- prey system with constant rate of harvesting in prey. The problem of combined harvesting of two ecologically independent and logistically growing fish species was investigated by Clark [3]. Zhu and Lan [22] considered a Leslie-Gower model with constant harvesting in prey and analyzed the local dynamics of system in the neighborhood of predator free equilibrium point as well as interior equilibrium point. Zhang [21] proposed a Leslie-Gower predator- prey model with proportional harvesting in both prey and predator. He studied the persistence and global stability of this system. The global stability of the unique interior equilibrium of the system shows that harvesting has no influence on the persistence of the system. Das and Mukharjee et al. [4] studied a predator- prey model, where both species grow logistically and are subjected to a nonlinear harvesting. Recently, Gupta and Banerjee[8] studied a predator- prey model with non-linear harvesting of prey, considering logistically growing prey and Modified Leslie-Gower type predation. They studied that the system has a complex dynamical behavior and exhibits several local bifurcations.

Regulation of renewable resources is an essential and important part in the optimal management of renewable resources.The over exploitation of biological resources is controlled by imposing taxation and/or license fees. The extinction of species can also be controlled by creating reserve zones for harvesting or limiting the harvesting period. In fishery resource management, many investigations have been carried out with taxation as a control instrument. Harvesting problems with taxation have been studied by Clark [3]. Dubey analyzed a non-linear mathematical model to study a resource dependent fishery model with optimal harvesting policy by considering taxation as a control instrument[5]. They also proved that the fishery resources can be protected from over-exploitation by increasing the tax and discounted rate. Pradhan and Chaudhuri [17] also proposed and analyzed a dynamical reaction model of two species fishery with taxation as a control variable and then discussed its optimal harvesting policy. Recently, Huo et al.[9] discussed a dynamic model for fishery resource with reserve area and taxation as a control parameter. The present paper deals with a dynamic reaction model in the case of a predator- prey type fishery system, while the model considered here, is especially based on a modified version of the Leslie-Gower scheme, where only the prey species is subjected to non- linear harvesting. The harvesting effort is taken as a dynamical variable and taxation as a control instrument. This imposition of tax helps to control over harvesting of prey species and in turn, it helps the predator population to grow. The main aim of this paper is to find the proper taxation policy which gives the best possible benefit through harvesting.

2. The Mathematical Model And Its Qualitative Analysis

Let $x(t)$ denote the population density of a logistically growing prey with Holling type-II functional response and $y(t)$ be the density of predator assuming Modified Leslie-Gower type predation. Let the prey species be harvested with effort E and H(x,E) denote the harvesting function. The dynamics of system is governed by

$$\frac{dx}{dt} = rx\left(1-\frac{x}{k}\right)-\frac{\alpha xy}{a+x}-H(x,E), \quad (1)$$

$$\frac{dy}{dt} = sy\left(1-\frac{\beta y}{a+x}\right), \quad (2)$$

The parameter r is the intrinsic growth rate and k is the environmental carrying capacity for the prey. For the predator, s is the growth rate, α is its encounter rate with the prey and β is maximum rate of the reduction of predator population. All these parameters are assuming only positive values. The following more realistic non-linear harvesting function is considered instead of constant and proportional harvesting.

$$H(x,E) = \frac{qEx}{m_1E+m_2x}.$$

The net economic revenue of fishermen from harvesting of prey species is given by

$$Net \quad Revenue = T.R. - T.C. = E\left(\frac{qpx}{m_1E+m_2x}-c\right).$$

In order to control over exploitation of the species, the regulatory agencies impose a tax on harvested species. Let p and c are price and cost per unit mass, η is stiffness parameter and $\tau \in [\tau_{min}, \tau_{max}]$ be the imposed tax per unit harvested prey species. For a tax $\tau > 0$, the revenue of the fishermen will be reduced by $(p-\tau)$, assuming $p > \tau$ and $E(t)$ is taken as a dynamic harvesting effort at a time t. Thus, the effort dynamics is determined as follows[16]:

$$\frac{dE}{dt} = \eta E\left(\frac{q(p-\tau)x}{m_1E+m_2x}-c\right). \quad (3)$$

The coupled dynamical equations (1)-(3) constitute the model for the harvesting of prey. The model with associated initial conditions is as follows:

$$\begin{aligned}
\frac{dx}{dt} &= rx\left(1-\frac{x}{k}\right)-\frac{\alpha xy}{a+x}-\frac{qEx}{m_1E+m_2x} = xf(x,y,E),\\
\frac{dy}{dt} &= sy\left(1-\frac{\beta y}{a+x}\right) = yg(x,y), \qquad (4)\\
\frac{dE}{dt} &= \eta E\left(\frac{q(p-\tau)x}{m_1E+m_2x}-c\right) = Eh(x,E),.
\end{aligned}$$

$$x(0) = x_0, y(0) = y_0, E(0) = E_0; \quad (x_0,y_0,E_0) \in \mathbb{R}^3_+.$$

Theorem 1. *All the solutions* $(x(t), y(t), E(t))$ *of the system* (4) *with positive initial condition remain positive for all* $t > 0$.

Proof. The solution of the system (4) is obtained as follows:

$$x(t) = x(0)exp\left(\int_0^t f(x(p), y(p), E(p))dp\right) > 0,$$
$$y(t) = y(0)exp\left(\int_0^t g(x(p), y(p), E(p))dp\right) > 0,$$
$$E(t) = E(0)exp\left(\int_0^t h(x(p), y(p), E(p))dp\right) > 0.$$

This shows that the solution of the system (4) is positive for all $t > 0$. □

Theorem 2. *The system* (4) *has uniformly bounded solution.*

Proof. Consider a function $\psi(t)$ such that

$$\begin{aligned}
\psi(t) &= x(t) + y(t) + \frac{1}{\eta(p-\tau)}E(t),\\
\frac{d\psi(t)}{dt} &= x'(t) + y'(t) + \frac{1}{\eta(p-\tau)}E'(t),\\
&= rx\left(1 - \frac{x}{k}\right) - \frac{\alpha xy}{a+x} - \frac{qEx}{m_1E + m_2x} + sy - \frac{\beta sy}{a+x} + \frac{qEx}{m_1E + m_2x} - \frac{cE}{\eta(p-\tau)},\\
\frac{d\psi(t)}{dt} &\leq \left(rx - \frac{rx^2}{k}\right) + sy - \frac{\beta sy}{a+k} - \frac{cE}{\eta(p-\tau)}.
\end{aligned}$$

Introduce a positive constant N and rewrite the above equation as follows:

$$\frac{d\psi(t)}{dt} + N\psi(t) \leq \left((r+N)x - \frac{r}{k}x^2\right) + \left((s+N)y - \frac{s\beta}{a+k}y^2\right) - \frac{(c-N)}{\eta(p-\tau)}E.$$

For $c > N$, further simplification yields,

$$\begin{aligned}
\frac{d\psi(t)}{dt} + N\psi(t) &\leq -\frac{r}{k}\left(x - \frac{k(r+N)}{2r}\right)^2 - \frac{s\beta}{a+k}\left(y - \frac{(s+N)(k+a)}{2s\beta}\right)^2 + M,\\
\frac{d\psi(t)}{dt} + N\psi(t) &\leq M; \quad M = \left(\frac{k^2(r+N)^2}{4r^2} + \frac{(s+N)^2(k+a)^2}{4s^2\beta^2}\right).
\end{aligned}$$

Solution of above differential inequality gives,

$$\begin{aligned}
\psi(t) &\leq \frac{M}{N}\left(1 - e^{-Nt}\right) + \psi(0)e^{-Nt},\\
0 < \lim_{t\to\infty}\psi(t) &\leq \frac{M}{N}.
\end{aligned}$$

Accordingly, all the solutions of (4) initiating from $\mathbb{R}^3_+$ are confined in the region

$$R = \left\{ (x,y,E) \in \mathbf{R}; 0 < x(t) + y(t) + \frac{1}{\eta(p-\tau)} E(t) \le M + \phi \text{ for any } \phi > 0 \right\}$$

This proves the result. □

3. Existence, Stability and Bifurcation of Equilibria

The system has six feasible non-negative equilibrium states, namely

(i) $P_0(0,0,0)$ is a trivial equilibrium point.

(ii) $P_1(k,0,0)$ is the axial equilibrium point on x-axis.

(iii) $P_2(0,\frac{a}{\beta},0)$ is the axial point on y-axis. Predator exists due to presence of alternate food $a > 0$.

(iv) $P_3(\bar{x},\bar{y},0)$ is the boundary equilibrium point in xy-plane. The equilibrium level densities $\bar{x}$ and $\bar{y}$ are the positive solution of the following equations:

$$r\left(1 - \frac{\bar{x}}{k}\right) - \frac{\alpha \bar{y}}{a + \bar{x}} = 0,$$

$$1 - \frac{\beta \bar{y}}{a + \bar{x}} = 0.$$

The positive solution is obtained as

$$\bar{x} = k\left(1 - \frac{\alpha}{r\beta}\right) \quad and \quad \bar{y} = \frac{1}{\beta}\left(a + \bar{x}\right) \quad with \quad r\beta > \alpha. \tag{5}$$

(v) $P_4(\hat{x},0,\hat{E})$ is boundary equilibrium point in xE-plane. Here $\hat{x}$ and $\hat{E}$ are the positive solution of the following equations:

$$r\left(1 - \frac{\hat{x}}{k}\right) - \frac{q\hat{E}}{m_1\hat{E} + m_2\hat{x}} = 0,$$

$$\frac{q(p-\tau)\hat{x}}{m_1\hat{E} + m_2\hat{x}} - c = 0.$$

This gives

$$\hat{x} = k\left(\frac{(rm_1 - q)L + rm_2}{r(m_1L + m_2)}\right) \quad and \quad \hat{E} = L\hat{x}, \quad where \quad L = \frac{(p-\tau)q - cm_2}{cm_1}. \tag{6}$$

Accordingly, $\hat{x}$ is positive provided one of the following conditions is satisfied as follows:

$$m_1 \geq \frac{q}{r} \quad and \quad \tau < p - \frac{cm_2}{q}$$

or

$$m_1 < \frac{q}{r} \quad and \quad 0 < \frac{q}{r} - m_1 < \frac{rm_2}{L} \quad i.e., \tau > p - \left(\frac{cm_2}{q} + \frac{r^2 cm_1 m_2}{q(q - m_1 r)}\right).$$

(vi) $P_5(x^*, y^*, E^*)$ is the unique interior equilibrium point of the system (4) and is obtained by solving the following equations:

$$r\left(1 - \frac{x^*}{k}\right) - \frac{\alpha y^*}{a + x^*} - \frac{qE^*}{m_1 E^* + m_2 x^*} = 0,$$

$$1 - \frac{\beta y^*}{a + x^*} = 0,$$

$$\frac{q(p - \tau)x^*}{m_1 E^* + m_2 x^*} - c = 0.$$

These yield:

$$x^* = k\left(1 - \frac{\alpha}{r\beta} - \frac{qL}{r(m_1 L + m_2)}\right), \quad y^* = \frac{a + x^*}{\beta} \quad and \quad E^* = Lx^*. \tag{7}$$

The interior equilibrium point (x^*, y^*, E^*) is positive for the condition

$$p - \left(\frac{cm_2}{q} + \frac{(r - \frac{\alpha}{\beta})cm_1 m_2}{q(q - m_1(r - \frac{\alpha}{\beta}))}\right) < \tau < p - \frac{cm_2}{q}. \tag{8}$$

The condition (8) gives the range of tax for the existence of interior equilibrium and this range of tax can be useful for regulatory agency at the time of formulation of tax structure per unit biomass for controlling the fishery system.

From equation (7), the following can be derived:

$$\frac{dx^*}{d\tau} = \frac{rm_2 q^2}{(m_1 L + m_2)^2} > 0, \tag{9}$$

$$\frac{dy^*}{d\tau} > 0 \quad and \tag{10}$$

$$\frac{dE^*}{d\tau} = L\frac{dx^*}{d\tau} - x^* \frac{q}{cm_1} < 0 \quad for \quad \frac{dx^*}{d\tau} < \frac{qx^*}{Lcm_1}. \tag{11}$$

The equations (9) and (10) shows that as the value of taxation increasing, the prey and predator densities are also increasing. However, the harvesting effort decreases with increasing taxation for (11). This concludes that with the increase of taxation, the amount of effort will start decreasing. This means a fisherman's interest for investment in fishery will decrease and may be some of them will leave harvesting of species as it is no longer profitable. In resultant, this must help prey and predator population to grow. The enhance taxation can help in increasing higher equilibrium level of prey and predator densities. It can be concluded that equilibrium level of prey, predator populations can be increased by increasing value of taxation.

For the local stability, the Jacobian matrix of the system (4) at any point(x,y,E) is given by

$$J(x,y,E) = \begin{bmatrix} f + x\left(-\frac{r}{k} + \frac{\alpha y}{(a+x)^2} + \frac{qEm_2}{(m_1E+m_2x)^2}\right) & -\frac{\alpha x}{a+x} & -m_2 q\left(\frac{x}{m_1E+m_2x}\right)^2 \\ \frac{\beta s y^2}{(a+x)^2} & g - \frac{\beta s y}{a+x} & 0 \\ \frac{\eta q(p-\tau)m_1E^2}{(m_1E+m_2x)^2} & 0 & h - \frac{\eta q(p-\tau)m_1xE}{(m_1E+m_2x)^2} \end{bmatrix}$$

Using Routh- Hurwitz criterion [13, 14], the local stability analysis of all equilibrium states is gives as follows:

(1) The origin $P_0(0,0,0)$ is a saddle point with unstable manifold in y direction. It has a stable manifold in x as well as E direction.

(2) The equilibrium point $(k,0,0)$ is a saddle point with unstable manifold in y-direction and stable manifold in x-direction. The system has stable manifold in E-direction if

$$\tau > p - \frac{cm_2}{q}. \tag{12}$$

It may be noted that the equilibrium point $P_1(k,0,0)$ becomes non- hyperbolic and bifurcation may occur when

$$\frac{(p-\tau)q}{m_2} = c. \tag{13}$$

(3) The equilibrium point $P_2\left(0,\frac{a}{\beta},0\right)$ is locally asymptotically stable provided

$$r - \frac{\alpha}{\beta} < 0. \tag{14}$$

When the above condition (14) is violated, then the point P_2 is a saddle point with unstable manifold in x- direction.

(4) The characteristic equation corresponding to the equilibrium point $P_3(\bar{x},\bar{y},0)$ yields the eigen values:

$\lambda_{1,2} = \frac{1}{2}\left[\left(\frac{\alpha}{\beta} - r - s\right) + \frac{\alpha\bar{x}}{\beta(a+\bar{x})} \pm \sqrt{\left(\left(\frac{\alpha}{\beta} - r - s\right) + \frac{\alpha\bar{x}}{\beta(a+\bar{x})}\right)^2 - 4\frac{rs\bar{x}}{k}}\right]$ and

$\lambda_3 = \eta\left(\frac{(p-\tau)q}{m_2} - c\right)$.

Accordingly, the following conclusions can be drawn regarding the local stability of P_3:

(i) The equilibrium point P_3 is locally asymptotically stable when

$$\frac{\alpha\bar{x}}{\beta(a+\bar{x})} < r + s - \frac{\alpha}{\beta} \quad and \quad \frac{(p-\tau)q}{m_2} < c. \tag{15}$$

(ii) The point P_3 is a saddle point with an unstable manifold in E-direction provided

$$\frac{(p-\tau)q}{m_2} > c. \tag{16}$$

(iii) The bifurcation is possible when

$$\frac{(p-\tau)q}{m_2} = c. \tag{17}$$

(iv) If $\left(\frac{\alpha}{\beta} - r - s\right) + \frac{\alpha\bar{x}}{\beta(a+\bar{x})} = 0$, then a pair of purely imaginary eigen values exists. The transversality condition for Hopf bifurcation at the equilibrium point P_3 is given by

$$\frac{d}{ds}\left[\left(\frac{\alpha}{\beta} - r - s\right) + \frac{\alpha\bar{x}}{\beta(a+\bar{x})}\right] = -1 \neq 0 \quad for \quad s = \frac{(r\beta-\alpha)[k(\alpha - r\beta - \alpha\beta) - ar\beta]}{\beta(ar\beta + k(r\beta-\alpha))}$$

Thus, the existence of periodic solutions around the equilibrium point P_3 are possible. This will attract all small perturbations in the neighborhood of xy- plane when $\frac{(p-\tau)q}{m_2} < c$.

(5) Corresponding to the equilibrium point $P_4(\hat{x},0,\hat{E})$, one of the eigen value is $\lambda = s > o$, and the other two are obtained as eigenvalues of the following2×2 matrix

$$J_4^*(\hat{x},\hat{E}) = \begin{bmatrix} \hat{x}\left(-\frac{r}{k} + \frac{q\hat{E}m_2}{(m_1\hat{E}+m_2\hat{x})^2}\right) & -\frac{qm_2\hat{x}^2}{(m_1\hat{E}+m_2\hat{x})^2} \\ \frac{\eta\hat{E}^2q(p-\tau)m_1}{(m_1\hat{E}+m_2\hat{x})^2} & -\frac{\eta\hat{E}\hat{x}q(p-\tau)m_1}{(m_1\hat{E}+m_2\hat{x})^2} \end{bmatrix}$$

It can be observed that

$$det(J_4^*) = \frac{r\hat{x}}{k} > 0 \quad and$$

$$\begin{aligned} tr(J_4^*) &= \hat{x}\Big(-\frac{r}{k} + \frac{q\hat{E}m_2}{(m_1\hat{E}+m_2\hat{x})^2}\Big) + \frac{-\eta\hat{E}\hat{x}q(p-\tau)m_1}{(m_1\hat{E}+m_2\hat{x})^2} \\ &= \Big(\frac{qL}{m_1L+m_2} - r\Big) + \frac{qL(m_2-\eta m_1(p-\tau))}{(m_1L+m_2)^2}. \end{aligned}$$

Accordingly, P_4 is a saddle point when $tr(J_4^*) < 0$. Further, if $tr(J_4^*) = 0$, then it has pair of purely imaginary roots. The transversality condition for Hopf bifurcation at the equilibrium point P_3 is given by

$$\frac{d(tr)(J_4^*}{dr} = -1 \neq 0 \quad for \quad r = \frac{qL}{m_1L+m_2} + \frac{qL(m_2-\eta m_1(p-\tau))}{(m_1L+m_2)^2}.$$

Thus, there exists a family of an attracting periodic solutions through Hopf bifurcation from P_4 in the neighborhood of 'r', keeping other parameters fixed.

The necessary and sufficient condition for local stability of (x^*, y^*, E^*) is given by the following theorem:

Theorem 3. *The positive interior equilibrium point $P_5(x^*, y^*, E^*)$ is asymptotically locally stable provided condition* (19) *is satisfied.*

Proof. Let the jacobian matrix of the system (4) evaluated at the equilibrium point P_5 be $J_5(x^*, y^*, E^*) = (a_{ij})_{3\times 3}$.

$$\begin{aligned} a_{11} = x^*\Big(-\frac{r}{k} + \frac{\alpha y^*}{(a+x^*)^2} + \frac{qE^*m_2}{(m_1E^*+m_2x^*)^2}\Big), \quad a_{12} = \frac{-\alpha x^*}{a+x^*} > 0, \quad a_{13} = \frac{m_2q(x^*)^2}{(m_1E^*+m_2x^*)^2} > 0, \\ a_{21} = \frac{\beta s y^*}{(a+x^*)^2} > 0, \quad a_{22} = \frac{-\beta s y^*}{a+x^*} < 0; \quad a_{22} = -a_{21}(a+x^*) < 0, \quad a_{23} = a_{32} = 0, \\ a_{31} = \frac{\eta q(p-\tau)m_1(E^*)^2}{(m_1E^*+m_2x^*)^2} > 0, \quad a_{33} = -\frac{\eta q(p-\tau)m_1x^*E^*}{(m_1E^*+m_2x^*)^2} < 0, \quad a_{33} = -a_{31}\frac{x^*}{E^*} < 0. \end{aligned}$$

Thus, the characteristics equation of the jacobian matrix at $P_5(x^*, y^*, E^*)$ is obtained as

$$\lambda^3 + A_1\lambda^2 + A_2\lambda + A_3 = 0,$$

$$A_1 = -(a_{11} + a_{22} + a_{33}),$$

$$A_2 = a_{22}a_{33} + (a_{11}a_{33} - a_{13}a_{31}) + (a_{11}a_{22} - a_{21}a_{21}),$$

$$A_3 = a_{11}a_{22}a_{33} - a_{12}a_{21}a_{33} - a_{13}a_{31}a_{22}.$$

Using Routh- Hurwitz criteria, the condition for local stability of the equilibrium point $P_5(x^*, y^*, E^*)$ is

$$A_1 > 0, \quad A_2 > 0 \quad and \quad A_1A_2 - A_3 > 0. \tag{18}$$

Note that $A_1 > 0$ if

$$M = \frac{r}{k} - \frac{\alpha y^*}{(a + x^*)^2} - \frac{qE^* m_2}{(m_1E^* + m_2x^*)^2} > 0. \tag{19}$$

Also, $A_2 > 0$ and $A_1A_2 - A_3 > 0$ for the condition (19). Thus, the interior equilibrium point (x^*, y^*, E^*) is asymptotically stable provided $M > 0$. □

Theorem 4. *The interior equilibrium point* (x^*, y^*, E^*) *of the system* (4) *is globally asymptotically stable in the domain* $D = \{(x,y,E) : m_1E + m_2x > M_1, (x,y,E) \in \mathbb{R}^3_+\}$, *where* $M_1 = \dfrac{Lqka\beta}{(m_1L + m_2)(ra\beta - \alpha k)}$.

Proof. Consider a function $V(x,y,E)$ for arbitrary chosen positive constants d_0, d_1 and d_2 s.t:

$$V(x,y,E) = d_0\left[(x - x^*) - x^* \log\frac{x}{x^*}\right] + d_1\left[(y - y^*) - y^* \log\frac{y}{y^*}\right] + d_2\left[(E - E^*) - E^* \log\frac{E}{E^*}\right];$$

where, $V(x^*, y^*, E^*) = 0.$

Now differentiate V w.r.t. $'t'$,

$$
\begin{aligned}
\frac{dV}{dt} &= d_0(x-x^*)\frac{\dot{x}}{x}+d_1(y-y^*)\frac{\dot{y}}{y}+d_2(E-E^*)\frac{\dot{E}}{E},\\
&= d_0(x-x^*)\left(r\left(1-\frac{x}{k}\right)-\frac{\alpha y}{a+x}-\frac{qE}{m_1E+m_2x}\right)+d_1(y-y^*)\left(s-\frac{\beta sy}{a+x}\right)\\
&+ d_2(E-E^*)\eta\left(\frac{q(p-\tau)x}{m_1E+m_2x}-c\right).\\
\frac{dV}{dt} &=- d_0(x-x^*)^2\left[\frac{r}{k}-\frac{\alpha y^*}{(a+x)(a+x^*)}-\frac{qE^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}\right]+\frac{(x-x^*)(y-y^*)}{a+x}*\\
&\left[-d_0\alpha x^*+\frac{d_1s\beta E^*}{a+x^*}\right]+\frac{(-d_0qx^*+d_2q(p-\tau)E^*)(x-x^*)(E-E^*)}{(m_1E+m_2x)(m_1E^*+m_2x^*)}-\frac{d_1s\beta(y-y^*)^2}{a+x}\\
&- \frac{d_2q(p-\tau)x^*(E-E^*)^2}{(m_1E+m_2x)(m_1E^*+m_2x^*)}.
\end{aligned}
$$

Choosing $d_0=1, d_1=\dfrac{\alpha x^*(a+x^*)}{s\beta E^*}$ and $d_2=\dfrac{x^*}{(p-\tau)E^*}>0$ for $p>\tau$, the above equation becomes,

$$
\begin{aligned}
\frac{dV}{dt} &= -d_0(x-x^*)^2\left[\frac{r}{k}-\frac{\alpha y^*}{(a+x)(a+x^*)}-\frac{qE^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}\right]-\frac{d_1s\beta}{a+x}(y-y^*)^2\\
&- \frac{d_2q(p-\tau)x^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}(E-E^*)^2,\\
\frac{dV}{dt} &\le -(x-x^*)^2\left[\frac{r}{k}-\frac{\alpha y^*}{a(a+x^*)}-\frac{qE^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}\right]\frac{\alpha_1(a+x^*)x^*(y-y^*)^2}{(a+x)E^*}\\
&- \frac{qx^{2*}(E-E^*)^2}{(m_1E+m_2x)(m_1E^*+m_2x^*)}.
\end{aligned}
$$

$$\frac{dV}{dt}<0 \quad \text{if } \frac{r}{k}-\frac{\alpha y^*}{a(a+x^*)}-\frac{qE^*}{(m_1E+m_2x)(m_1E^*+m_2x^*)}>0.$$

Substituting the values of y^* and E^* in the above expression, a plane is obtained as follows:

$$(m_1E+m_2x)>\frac{Lqka\beta}{(m_1L+m_2)(ra\beta-\alpha k)}=M_1(say). \tag{20}$$

If $ra\beta<\alpha k$, the inequality (20) is trivially true and for $ra\beta>\alpha k$, a bound of plane (m_1E+m_2x) is obtained in positive octant. This shows that $\dfrac{dV}{dt}$ is negative definite for the condition (20).

Thus, the interior equilibrium point (x^*,y^*,E^*) is asymptotically globally stable for the sufficient condition (20). □

Theorem 5. *The system* (4) *exhibits a transcritical bifurcation around the axial equilibrium point* $P_1(k,0,0)$ *if*

$$\tau_c = p - \frac{cm_2}{q}. \tag{21}$$

Proof. The Jacobian of system (4) at equilibrium point $P_1(k,0,0)$ has a zero eigenvalue for the condition $\tau = p - \frac{cm_2}{q}$ and therefore, the equilibrium point $(k,0,0)$ becomes non-hyperbolic. So there is a chance of bifurcation around this equilibrium point. The threshold value of the bifurcation is $\tau_c = p - \frac{cm_2}{q}$.

The eigenvectors of $J(k,0,0)$ and $(J(k,0,0))^T$ corresponding to zero eigenvalue are obtained as

$$V = \left(1, 0, \frac{-rm_2}{q}\right)^T \quad \textit{and} \quad W = (0,0,1)^T, \quad \textit{respectively.} \tag{22}$$

Compute Δ_1, Δ_2 and Δ_3 as follows:

$$\Delta_1 = W^T F_\tau(P_1, \tau_c) = 0, \quad F = (F^1, F^2, F^3)^T = (xf, yg, Eh)^T.$$

$$\Delta_2 = W^T \left[DF_\tau(P_1, \tau_c)V\right] = r\eta \neq 0,$$

where

$$DF_\tau(P_1, \tau^{tc}) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{-\eta q}{m_2} \end{pmatrix}$$

.

$$\Delta_3 = W^T \left[D^2 F_\tau(P_1, \tau_c)(V,V)\right] = \frac{\eta m_1 r^2}{qk} \neq 0$$

. Since, $\Delta_1 = 0$, there is no chance of saddle- node bifurcation.

Thus, by the Sotomayor's theorem, the system (4) undergoes a transcritical bifurcation around the axial equilibrium point $(k,0,0)$ for the condition (23).

□

Remark 0.1. *Similarly, the system* (4) *exhibits a transcritical bifurcation around the axial equilibrium point* $(0, \frac{a}{\beta}, 0)$ *if* $r = \frac{\alpha}{\beta}$.

Further, the system (4) *has a transcritical bifurcation around the boundary equilibrium point* $(\bar{x}, \bar{y}, 0)$ *for* $\tau = p - \frac{cm_2}{q}$.

4. Persistence

Persistence ensures the long term co-existence of all species. The system is investigated near the boundaries of the positive octant. According to the approach of Freedman and Waltman [6][section-4], consider the system (4) along with the following assumptions:

(B1): $f_y = \dfrac{-\alpha}{a+x} < 0, \qquad f_E = \dfrac{-m_2 q x}{(m_1E+m_2x)^2} < 0,$

$g_x = \dfrac{s\beta y}{(a+x)^2} > 0,$

$h_x = \dfrac{\eta(p-\tau)m_1E}{(m_1E+m_2x)^2} > 0,$

$g(0,y,E) = 1 - \dfrac{\beta}{a}y < 0 \quad \text{if} \quad y > \dfrac{a}{\beta},$

$h(0,0,E) = -\eta c < 0.$

(B2): The prey species x grows to the carrying capacity in the absence of predator i.e.,

$$f(0,0,0) = r > 0 \quad \textit{and} \quad f(k,0,0) = 0.$$

While, due to the intra-specific competition within prey species, it is observed

$$\frac{\partial f}{\partial x}(x,0,0) = -\frac{r}{k} < 0.$$

(B3): There is no equilibrium point on yE plane.

(B4): In the absence of harvesting ($E=0$) and predator ($y=0$), there exist equilibrium points

$(\bar{x},\bar{y},0)$ and $(\hat{x},0,\hat{E})$ respectively, such that

$$f(\bar{x},\bar{y},0) = g(\bar{x},\bar{y},0) = 0,$$

$$f(\hat{x},0,\hat{E}) = h(\hat{x},0,\hat{E}) = 0.$$

Therefore, the following results represent the conditions for persistence of the system (4).

Theorem 6. *Let the hypotheses [B1]-[B4] hold. The system* (4) *persists in the absence of periodic solutions in the boundary planes provided*

$$h(\hat{x},0,\hat{E}) = \frac{(p-\tau)q}{m_2} - c > 0. \tag{23}$$

$$g(\bar{x},\bar{y},0) = s > 0. \tag{24}$$

Proof. For the boundary equilibrium point $P_3(\bar{x},\bar{y},0)$ in $xy-$ plane, the eigen value in E- direction is obtained as $\lambda_3 = \dfrac{(p-\tau)q}{m_2} - c$. The point $P_3(\bar{x},\bar{y},0)$ is unstable provided (23) holds. Similarly, $\lambda_2 = s > 0$ is eigen value in y-direction corresponding to the boundary equilibrium point $(\hat{x},0,\hat{E})$ in xE plane, which is unstable provided (24) holds. Also, the points $(0,0,0)$ and $(k,0,0)$ are unstable. This shows that all trajectories are bounded away from all boundaries of the system. Hence, if there are no limit cycles on the boundary planes and the conditions(23) and (24) are satisfied , then the system(4) persists. □

Theorem 7. *Let there be a finite number of periodic solution in xy and xE planes. Then, for each limit cycle $(u(t),v(t))$ in the xy plane and $(\omega_1(t),\omega_2(t))$ in xE plane, the persistence conditions for the system would take the form:*

$$\int_0^{\xi} h(u(t),v(t),0)\,dt > 0 \quad and \quad \int_0^{\omega} g(\hat{u}(t),0,\hat{v}(t))\,dt > 0,$$

where ξ and ω are the limit periods of the limit cycle.

Proof. Assume that there exists a limit cycle in the in the $xy-$ plane , then the variational matrix about the limit cycle $x(t) = u(t), y(t) = v(t), z(t) = 0$ take the form

$$V(u(t),v(t),0) = \begin{bmatrix} u(t)\left(-\dfrac{r}{k} + \dfrac{\alpha v(t)}{(a+u(t))^2}\right) & \dfrac{-\alpha u(t)}{a+u(t)} & -\dfrac{q}{m_2} \\ \dfrac{s\beta (v(t))^2}{(a+u(t))^2} & \dfrac{-s\beta v(t)}{a+u(t)} & 0 \\ 0 & 0 & \eta\left(\dfrac{(p-\tau)q}{m_2} - c\right) \end{bmatrix}$$

Consider the solution of given system with the initial condition (t,a_1,a_2,a_3) sufficiently close to the limit cycle. From the above variational matrix, it can be obtained that

$$\frac{dE}{dt} = \eta\left(\frac{(p-\tau)q}{m_2} - c\right) \quad with \quad E(0) = a_3,$$

$$E = a_3 exp\left[\int_0^{\xi} \eta\left(\frac{(p-\tau)q}{m_2} - c\right) dt\right],$$

$$\frac{\partial E}{\partial a_3} = exp\left[\int_0^{\xi} \eta\left(\frac{(p-\tau)q}{m_2} - c\right) dt\right].$$

Using Taylor's expansion theorem:

$$E(t,a_1,a_2,a_3) - E(t,a_1,a_2,0) \backsimeq a_3\frac{\partial E}{\partial a_3} \backsimeq a_3 exp\left[\int_0^{\xi} \eta\left(\frac{(p-\tau)q}{m_2} - c\right) dt\right] \backsimeq a_3 exp\left[\int_0^{\xi} h(u(t),v(t),0)\,dt\right].$$

Then E increases or decreases according to $\int_0^{\xi} h(u(t),v(t),0)\,dt$ is positive or negative. Hence the trajectories go away from the $xy-$ plane under the assumptions of the theorem.

Similarly, result can be proved for $xE-$ plane.

□

5. Bionomic Equilibrium and Optimal Harvesting Policy

The net economic revenue to the society is represented as the sum of net economic revenue to the fishermen and net economic revenue to the regulatory agency, i.e.,

$$P(t,x,y,E,\tau) = \left(\frac{q(p-\tau)x}{m_1E+m_2x} - c\right)E + \frac{q\tau xE}{m_1E+m_2x} = \left(\frac{qpx}{m_1E+m_2x} - c\right)E. \tag{25}$$

Clark [3] defined the bionomic equilibrium point as the point of intersection of the interior equilibrium of the system (4) along with zero net economic revenue. The bionomic equilibrium $P_{BE}(x_{BE},y_{BE},E_{BE})$ is obtained as the positive solution of the system

$$\frac{dx}{dt} = \frac{dy}{dt} = \frac{dE}{dt} = P = 0.$$

It gives

$$x_{BE} = \frac{k}{m_1 r}\left[\left(r - \frac{\alpha}{\beta}\right)m_1 - q + \frac{cm_2}{p}\right], \quad y_{BE} = \frac{a + x_{BE}}{\beta} \quad and \quad E_{BE} = \frac{(pq - cm_2)}{cm_1}x_{BE},$$

for

$$\frac{cm_2}{p} < q < \left(r - \frac{\alpha}{\beta}\right)m_1 + \frac{cm_2}{p}. \tag{26}$$

Next, an optimal harvesting policy for the system (4) is investigated to maximize the total discounted net revenue using taxation as a control instrument. The optimal control problem over an infinite time horizon is given by

$$\max_{\tau_{min}<\tau(t)<\tau_{max}} I = \int_0^{\infty} e^{-\delta t}\left(\frac{qpx}{m_1E+m_2x} - c\right)dt. \tag{27}$$

The constant δ is the instantaneous annual rate of discount decided by harvesting agencies. Let $X = (x,y,E)$ and $X^* = (x^*,y^*,E^*)$ are the positions such that there exist a tax policy $\tau(t)$. The system (4) with $X(t_1) = X^*$ has a positive solution for $t > t_1$ under the policy $\tau(t)$.

Therefore, the taxation policy[10] is assumed as follows:

$$\tau(t) = \begin{cases} \overline{\tau}(t) & for \quad t \in [0, t_1] \\ \tau^* & for \quad t > t_1. \end{cases}$$

The objective is to determine an optimal taxation policy $\tau = \tau(t)$ to maximize (27) subject to the state equations in the system (4) and the control constraints $\tau_{min} < \tau(t) < \tau_{max}$. Pontryagin's Maximum Principle is used to obtain the optimal level of the solution of the problem (27). Let $\lambda_1(t), \lambda_2(t)$ and $\lambda_3(t)$ are adjoint variables w.r.t. the time 't' corresponding to the variables x, y and E, respectively. The associated Hamiltonian function is given by

$$\begin{aligned} H(t,x,y,E,\tau) \quad = \quad & e^{-\delta t}\left(\frac{qpx}{m_1E + m_2x} - c\right) + \lambda_1\left[rx\left(1 - \frac{x}{k}\right) - \frac{\alpha xy}{a + x} - \frac{qEx}{m_1E + m_2x}\right] \\ & + \lambda_2\left[sy\left(1 - \frac{\beta y}{a + x}\right)\right] + \lambda_3\left[\eta E\left(\frac{q(p - \tau)x}{m_1E + m_2x} - c\right)\right]. \end{aligned} \tag{28}$$

Notice that Hamiltonian is linear in control variable τ. The optimal control problem involves singular and bang-bang controls. Also, the optimal control must satisfy the following conditions to maximize 'H':

$$\overline{\tau} = \begin{cases} \tau_{max} & \forall t \in [0, t_1] \quad with \quad \frac{dH}{d\tau} > 0 \\ \tau_{min} & \forall t \in [0, t_1] \quad with \quad \frac{dH}{d\tau} < 0. \end{cases}$$

The Hamiltonian in (28) must be maximized for $\tau \in [\tau_{min}, \tau_{max}]$. Assume that the control constraints are not binding (i.e., the optimal solution does not occur atτ_{min} or τ_{max}). Thus, the considered control problem admits a singular solution on the control set(τ_{min}, τ_{max}) if

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= 0, \\ i.e., \frac{-qx\lambda_3}{m_1E + m_2x} &= 0 \quad \Rightarrow \quad \lambda_3(t) = 0. \end{aligned} \tag{29}$$

In order to find a singular control, Pontryagin's Maximum Principle [15] is utilized and the adjoint variables must satisfy the adjoint equations given by

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x}, \quad \frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y}, \quad \frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial E}. \tag{30}$$

The optimal equilibrium point is the equilibrium point corresponding to the optimal tax. Such a path is called the optimal path and is a solution of the system (4). Now, the adjoint equations are

$$\frac{d\lambda_1}{dt} = -\frac{\partial H}{\partial x} = -\Bigg[e^{-\delta t}\left(\frac{pqm_1E^2}{(m_1E+m_2x)^2}\right)+\lambda_1\left(-\frac{rx}{k}+\frac{\alpha xy}{(a+x)^2}-\frac{qExm_2}{(m_1E+m_2x)^2}\right)$$
$$+\lambda_2\left(\frac{s\beta y^2}{(a+x)^2}\right)\Bigg], \quad (31)$$

$$\frac{d\lambda_2}{dt} = -\frac{\partial H}{\partial y} = -\left[-\lambda_1\left(\frac{\alpha x}{a+x}\right)-\lambda_2\left(\frac{s\beta y}{a+x}\right)\right], \quad (32)$$

$$\frac{d\lambda_3}{dt} = -\frac{\partial H}{\partial E} = -\left[e^{-\delta t}\left(\frac{pqm_2x^2}{c(m_1E+m_2x)^2}\right)+\left(\frac{-qm_2x^2}{c(m_1E+m_2x)^2}\right)\right]$$
$$= -\left[e^{-\delta t}\left(p-\frac{c(m_1E+m_2x)^2}{qm_2x^2}\right)-\lambda_1\right]. \quad (33)$$

Also, the considered control problem admits a singular solution on the control set $[0, E_{max}]$ if $\frac{\partial H}{\partial E} = 0$,

$$\Rightarrow \lambda_1(t) = e^{-\delta t}\left(p-\frac{c(m_1E+m_2x)^2}{qm_2x^2}\right). \quad (34)$$

Let $\lambda_i(t) = \mu_i(t)e^{-\delta t}$, where $\mu_i(t) = \lambda_i(t))e^{\delta t}$ for $i = 1,2,3$ are known as the shadow prices and they should remain constant over time. Solving (32), a linear differential equation is obtained in λ_2 and in the interior equilibrium (x^*, y^*, E^*) such that

$$\frac{d\lambda_2}{dt} - A_1\lambda_2 = -e^{-\delta t}A_2, \quad (35)$$

where

$$A_1 = \frac{s\beta y^*}{a+x^*} \quad and \quad A_2 = \frac{\alpha x^*}{a+x^*}\left(p-\frac{c(m_1E^*+m_2x^*)^2}{qm_2x^{*2}}\right).$$

Solving equation (35),

$$\lambda_2(t) = \frac{A_1}{A_2+\delta}e^{-\delta t}. \quad (36)$$

To solve (31), put the value of $\lambda_2(t)$ using (36) in (31),

$$\frac{d\lambda_1}{dt} = -e^{-\delta t}\left(\frac{pqm_1E^{*2}}{(m_1E^*+m_2x^*)^2}\right)+\lambda_1\left(\frac{rx^*}{k}-\frac{\alpha x^*y^*}{(a+x^*)^2}+\frac{qE^*x^*m_2}{(m_1E^*+m_2x^*)^2}\right)$$
$$- \frac{A_1}{A_2+\delta}e^{-\delta t}\left(\frac{s\beta y^{*2}}{(a+x^*)^2}\right).$$

$$\frac{d\lambda_1}{dt} - B_1\lambda_1 = -e^{-\delta t}B_2, \quad (37)$$

where

$$B_1 = \frac{rx^*}{k} - \frac{\alpha x^* y^*}{(a+x^*)^2} + \frac{qE^* x^* m_2}{(m_1 E^* + m_2 x^*)^2} \quad and \quad B_2 = \frac{pqm_1 E^{*2}}{(m_1 E^* + m_2 x^*)^2} + \frac{A_1}{(A_2+\delta)} \frac{s\beta y^{*2}}{(a+x^*)^2}.$$

Solving equation (37),

$$\lambda_1(t) = \frac{B_1}{B_2+\delta} e^{-\delta t}. \tag{38}$$

Using (34) and (38),

$$p - \frac{c(m_1 E^* + m_2 x^*)^2}{qm_2 x^{*2}} = \frac{B_1}{B_2+\delta}. \tag{39}$$

Thus,(39) gives us desired singular path. Now, for the optimal level of this singular solution, Arrow Sufficiency condition for infinite time horizon [7] is applied. It is observed that

$$\frac{\partial^2 H}{\partial x^2} = \frac{-r\lambda_1}{k} - \frac{m_1 E^{*2}}{(m_1 E^* + m_2 x^*) x^{*2}} \left[\frac{\partial H}{\partial E} + ce^{-\delta t} \right] - \frac{m_2^2 q E^* x^*}{(m_1 E^* + m_2 x^*)^3} - \frac{\alpha y^* (a - x^*) \lambda_1}{(a+x^*)^3} - \frac{s\beta \lambda_2 y^{*2}}{(a+x^*)^3}.$$

For $\frac{\partial^2 H}{\partial x^2} < 0$, it is observed that $\lambda_1 > 0, \lambda_2 > 0$ and $x^* < a$ with the singular control i.e., $\frac{\partial H}{\partial E} = 0$ and $\frac{\partial^2 H}{\partial y^2} = \frac{-\lambda_2 s\beta}{a+x^*} < 0$.

Therefore, $\frac{\partial^2 H}{\partial x^2} < 0$ and $\frac{\partial^2 H}{\partial y^2} < 0$ for all $t \in [0,\infty)$ This shows that the Hamiltonian 'H' is concave in both x and y for all $t \in [0,\infty)$ provided the required conditions are satisfied. Hence, the arrow sufficiency condition for infinite time horizon shows that the singular solution is the part of optimal solution.

6. Numerical Simulations

In this section, numerical simulations are carried out for suitable choices of parameters to investigate the dynamical behavior of the system, keeping all the parameters fixed except τ. Hence, τ is known as bifurcation parameter. Let the value of parameters $r = 0.3, \quad k = 100, \quad \alpha = 0.005, \quad m_1 = 0.5, \quad m_2 = 0.5, \quad q = 0.15, \quad a = 3, \quad s = 1, \quad \beta = 0.15, \quad \eta = 1, \quad p = 5, \quad c = 1$ in appropriate units.

As a regulatory is always interested in the interior states. So, for the above set of parameters, examine the condition of existence and the stability of the steady state $P_5(x^*, y^*, E^*)$. To ensure the existence of the non- trivial steady states P_5, the value of taxation τ can be obtained

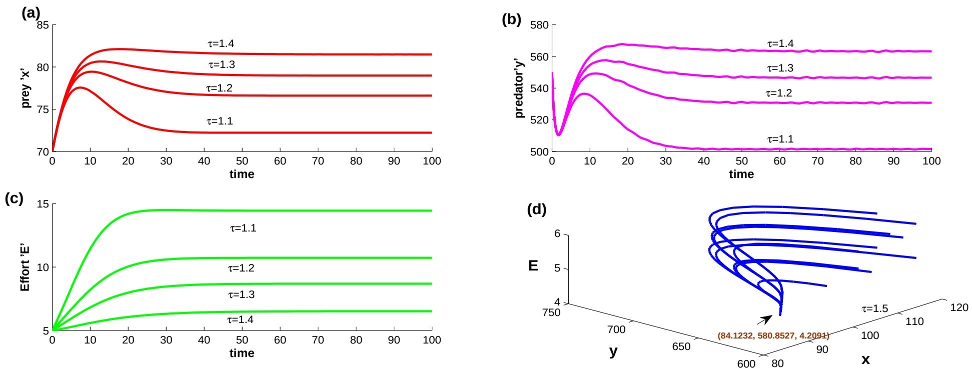

FIGURE 1. (a), (b) and (c) represent Solution curves of the prey population, predator population and Effort as a function of time for different low values of tax τ, for a fixed initial level $(70,550,5)$ and (d) represents phase plane trajectories of species x,y and effort E with the different initial levels for $\tau = 1.5$.

as $-0.7329 < \tau < 1.667$. For $\tau_{min} < 0$, there is a case of subsidies provided by government to the fishermen at the time of fishing. If there is no case of subsidy, then take $\tau_{min} = 0$ and $\tau_{max} = 1.6$(say). For the $\tau_{min} = 0$ and $\tau_{max} = 1.6$, the steady states can be obtained as $(55.557, 390.4073, 27.7777)$ and $(86.7306, 598.3166, 1.8993)$. It can be observed that when a fisherman have to pay no tax, he uses maximum amount of efforts to obtained the maximum benefits from fishery as compared in the case of taxation. The parameter values also satisfy the condition (19), which shows that steady state $P_5(x^*,y^*,E^*)$ is locally asymptotically stable. In the figure-1, diagrams (a), (b) and (c) give long term behavior of trajectories of prey-predator population and effort 'E' w.r.t. time 't' for the different low values of tax τ. This shows that for the fixed initial level $(70,550,5)$, all the trajectories converges to its interior equilibrium point in the positive octant. Also, it can be observed that as the value of taxation τ increases, the harvesting effort decreases. In resulted, prey population increases which helps predator population to grow. Figure-1(d) represents phase plane trajectories of species x,y and effort E with the different initial levels and it represents that the interior point (x^*,y^*,E^*)= $(84.1232, 580.8527, 4.2091)$ is globally stable corresponding to $\tau = 1.5$ for different initial levels in positive octant.

In figure-2, diagrams (a), (b) and (c) gives long term behavior of trajectories of prey- predator population and effort 'E' w.r.t. time 't' for the different high values of taxation τ. This shows

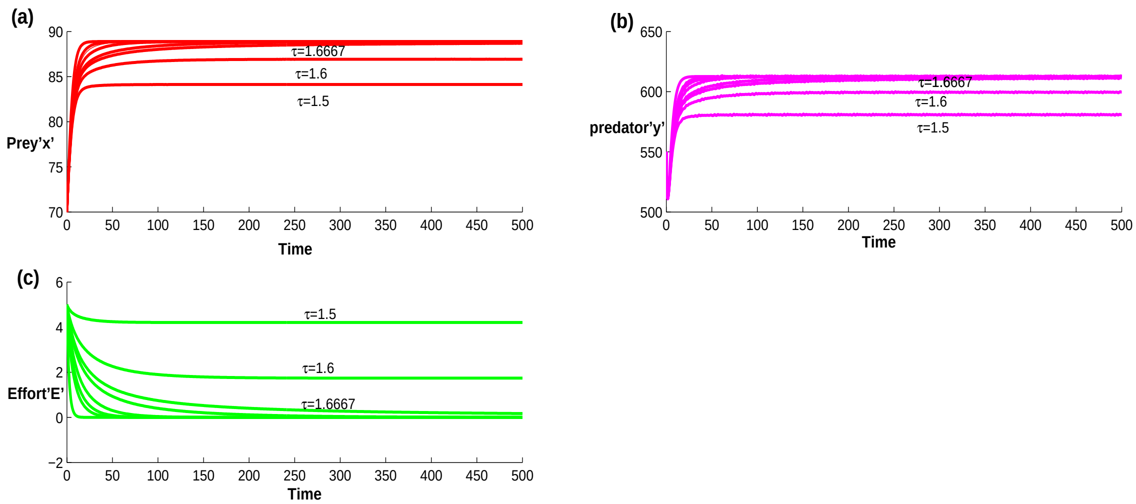

FIGURE 2. (a), (b) and (c) represent Solution curves of the prey population, predator population and Effort as a function of time for different high values of tax τ.

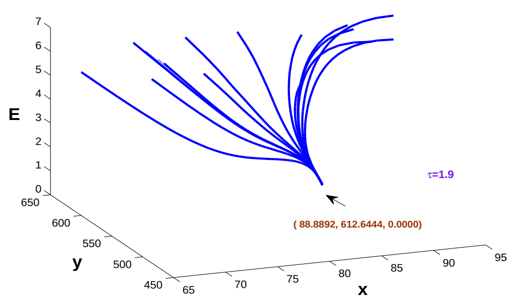

FIGURE 3. Phase plane trajectories of different biomasses with the different initial levels corresponding to $\tau = 1.9$ provided $\tau > 1.667$.

that the population densities for the prey 'x' and predator 'y' increase as the tax rates increase, where as the density of harvesting effort (E) decreases as the tax rates increase. After a time, a level of taxation i.e.,$\tau = 1.667$ is obtained where effort level will tends to zero.

The figure-3, represents phase plane trajectories of different biomasses with the different initial levels at the interior, which converge to the point $(88.8892, 612.6444, 0.0000)$ on the boundary plane i.e., x-y plane corresponding to $\tau = 1.9$, keeping other parameters fixed. Therefore, for the condition $\tau > 1.667$, it shows that, for the every different initial levels on the xyE space converge to a point on xy plane which means that if a threshold level of taxation i.e.,$\tau = 1.667$ is crossed, then it is not profitable to continue harvesting of prey species. In resultant, They have to stop harvesting of species.

7. Conclusion

This paper is concerned with the study of a Modified Leslie- Gower type predator in a predator- prey system with nonlinear harvesting of prey population. The harvesting effort is taken as a dynamic variable and taxation as a control instrument. The conditions for existence of steady states and their stability behavior have been examined by using Eigen Value Method, Routh- Hurwitz criteria and Lyapunov method. The existence of interior steady state strongly depends on range of tax. This range of tax may be useful for the regulatory agency for formulating a tax structure. The bionomical equilibrium of the system has been derived and it provides the range of harvesting co-efficient (or catch ability of harvest) that can be useful for a harvesting agency to get the profitable yields. The sufficient condition for global stability of unique interior equilibrium point provides a domain for global solution. The conditions of persistence for the system is derived. It is also investigated that the coexistence of prey and predator population depends upon the proper harvesting strategies such that the risk of extinction (or over exploitation) of the species can be avoided. The optimal taxation policy for the control problem has been studied by using Pontryagin's Maximum Principle. The optimum solution and optimum path has been derived. The impact of taxation on the system shows that the population densities for the prey(x) and predator(y) increased as the tax rates increased, where as the density of harvesting effort(E) decreased as the tax rates increased. It can be concluded that the equilibrium level of predator-prey system can be increased by increasing tax level. This observations gives the idea to obtain optimal level of taxation corresponding the optimal equilibrium level of prey, predator population and effort dynamics. Thus, the objective of this work includes, both ecological and economic aspects. The economic objective is to maximize the net economic revenue and ecologically, want to keep the prey and predator from extinction.

Conflict of Interests

The authors declare that there is no conflict of interests.

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