Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2016, 2016:18
ISSN: 2052-2541

A STOCHASTIC TOXOPLASMOSIS SPREAD MODEL BETWEEN CAT AND OOCYST WITH JUMPS PROCESS

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Abstract. In this paper, Epidemic models are inevitably influenced by environmental white noise which is an important component in realism, using stochastic models can provide an additional degree of realism in comparison to their deterministic counterparts. Furthermore, it is possible for the population to confront emergency or sudden environmental changes such like chemical leak, abnormal weather, natural disaster and pestilence. In this paper, a toxoplasmosis spread model between cat and oocyst populations with independent stochastic perturbations and a jump process is proposed, the existence of global positive solution is derived. By the method of stochastic Lyapunov functions, we study its asymptotic behavior. When the perturbations about the the susceptible and infective cats are sufficiently small, as well as magnitude of the reproductive number is less than one, the infective cats, recovered cats and population oocysts decay to zero whilst the susceptible components converge to a class of explicit stationary distributions regardless of the perturbations on the recovered cats and population oocysts. When all the perturbations are small and the reproductive number is larger than one, we construct a new class of stochastic Lyapunov functions to show the positive recurrence, and our results reveal some cycling phenomena of recurrent diseases. These results mean that stochastic system has the similar property with the corresponding deterministic system when the white noise is small.

Keywords: stochastic model; Wiener process; jumps process; toxoplasmosis; asymptotically stable in the large.

2010 AMS Subject Classification: 91B70.

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Received November 24, 2015

1. Introduction

Variational inequalities, which include many important problems in nonlinear analysis and optimization such as the Nash equilibrium problem, complementarity problems, vector optimization problems, fixed point problems, saddle point problems and game theory, recently have been studied as an effective and powerful tool for studying many real world problems which arise in economics, finance, image reconstruction, ecology, transportation, and network; see [1-8] and the references therein.

Toxoplasma gondii, often referred to as T. gondii, is a parasite that is able to infect a wide range of hosts, including all mammals and birds[1]. Up to one third of the world human population are estimated to carry a Toxoplasma infection[2]. The increasing prevalence of infection in human population is probably due to the increase in the number of cats[3]. Cats are the key to control T. gondii due to the fact they shed, via feces[4], millions of oocysts, which after sporulation in the environment might infect warm-blooded animals including human beings.

Mathematical models are often used to research the transmission dynamics of diseases in population from an epidemiological point of view[5−8]. Abraham et al.[8] presented an epidemiological model to study the transmission dynamics of toxoplasmosis in a cat population under a continuous vaccination schedule. In this paper, authors assumed that the total number of cat population remains constant. But for many cases, taking into consideration the size of population varies is more reasonable. Furthermore, we assume that the vaccination rate of susceptible cats equals to zero. Then model formulated by Abraham et al. [8] is revised as follows:

$$\begin{cases} \dot{S}(t) = \Lambda - \beta S(t)O(t) - \mu S(t), \\ \dot{I}(t) = \beta S(t)O(t) - (\alpha + \mu)I(t), \\ \dot{R}(t) = \alpha I(t) - \mu R(t), \\ \dot{O}(t) = kI(t) - \mu_0 O(t). \end{cases} \tag{1.1}$$

In model (1.1), the total population of cats is divided into three disjoint subpopulations: cats who may become infected (Susceptible $S(t)$), cats infected by T. gondii (Infected $I(t)$), and cats who have immunity (Recovered $R(t)$). $O(t)$: number of oocysts in the environment. Λ:

the recruitment rate of susceptible cats. β: the rate of a susceptible cat transits to the infected subpopulation. μ: the natural death rate of cats. α: the rate of an infected cat transits to the recovered subpopulation. μ_0: the death rate of oocysts. k: the rate of appearance of new oocysts in the environment per infected cat.

The basic reproduction number

$$R_0 = k\beta\Lambda/(\mu\mu_0(\alpha+\mu)) \tag{1.2}$$

measures the average number of new infections generated by a single infected in a completely susceptible population. After a simple calculation, we find that the basic reproductive number R_0 controls completely the dynamics of the infection. In detail, system (1.1) has a disease-free equilibrium $E_0 = (\Lambda/\mu, 0, 0, 0)$, which is stable when $R_0 \leq 1$, whereas system (1.1) admits an epidemic equilibrium $E^* = (S^*, I^*, R^*, O^*)$, which is stable when $R_0 > 1$, where

$$S^* = (\alpha+\mu)\mu_0/\beta k,\ I^* = \mu_0 O^*/k,\ R^* = \alpha\mu_0 O^*/k\mu,\ O^* = \Lambda k/(\alpha+\mu)\mu_0 - \mu/\beta. \tag{1.3}$$

Thus the basic reproduction number R_0 is often considered as the threshold quantity that determines when an infection can invade and persist in a new host population. The disease-free equilibrium corresponds to maximal levels of susceptible, no infected and no recovered cats or oocysts. The epidemic equilibrium corresponds to positive levels of all four components including susceptible, infected, recovered cats as well as oocysts.

In fact epidemic models are inevitably influenced by environmental white noise which is an important component in realism, using stochastic models can provide an additional degree of realism in comparison to their deterministic counterparts. Many stochastic models for epidemic populations have been developed in Refs. [12–16]. Dalal et al.[12] previously used the technique of parameter perturbation to examine the effect of environmental stochasticity in a model of AIDS and condom use. Yu et al.[14] proved that the endemic equilibrium of the two-group SIR model with random perturbation is stochastic asymptotically stable. Meng [15] presented the stability conditions of the disease- free equilibrium of the SIR model without and with stochastic perturbation. Zhao et al.[16] investigated the extinction and persistence of the stochastic SIS epidemic model with vaccination. These results reveal the significant effect of the environmental noise on some epidemic models.

Furthermore, it is possible for the population to confront emergency or sudden environmental changes such like chemical leak, abnormal weather, natural disaster[19] and pestilence. For example, in 2000, the pesticides pollution problems led to the largest ever fish deaths in Rock Creek Park of Washington; in 2010, lightning or high-altitude hail resulted in about 3,000 birds suddenly falling to the ground and death in Arkansas; in 1999, Escherichia coli caused thousands of Americas Black Feather starlings being killed in northern Louisiana. But, a mathematical model only with stochastic extension cannot describe the situations above. In order to make our model more in line with the actual situation we need to introduce a jump process into underlying population dynamic model[20] .

However, to the best of our knowledge, few authors study the dynamics of a toxoplasmosis spread model between cat and oocyst populations, involving independent stochastic perturbations and a jump process. In this paper, first we explore the effect of randomly fluctuating environment on populations by a four dimensional Wiener process $B(t) = (B_1(t), B_2(t), B_3(t), B_4(t))$. Furthermore we use the jump diffusion to model the evolutions of population dynamics when they suffer emergency or sudden environmental shocks. Then the stochastic version corresponding to the deterministic model (1.1) takes the following form:

$$\begin{cases} dS(t) = (\Lambda - \beta S(t)O(t) - \mu S(t))dt + \sigma_1 S(t)dB_1(t) + \int_z C_1(z)S(t-)\tilde{N}(dt,dz), \\ dI(t) = (\beta S(t)O(t) - (\alpha+\mu)I(t))dt + \sigma_2 I(t)dB_2(t) + \int_z C_2(z)I(t-)\tilde{N}(dt,dz), \\ dR(t) = (\alpha I(t) - \mu R(t))dt + \sigma_3 R(t)dB_3(t) + \int_z C_3(z)R(t-)\tilde{N}(dt,dz), \\ dO(t) = (kI(t) - \mu_0 O(t))dt + \sigma_4 O(t)dB_4(t) + \int_z C_4(z)O(t-)\tilde{N}(dt,dz), \end{cases} \tag{1.3}$$

where the non-negative constants $\sigma_i (i = 1, \cdots, 4)$ denotes the intensity of the stochastic perturbations, respectively; $X(t-)$ means the left limit of $X(t)$; $\tilde{N}(dt,dz)$ is a Poisson counting measure with the stationary compensator $\pi(dz)dt$ and π is defined on a measurable subset Z of $[0, \infty)$ with $\pi(Z) < \infty$, $C_i(z) > -1 (i = 1, \cdots, 4)$.

Model (1.3) is the infectious diseases model with Wiener Process and jump perturbation. If $C_i(z) = 0$, model (1.3) degenerates into a stochastic model with only white noise. But when encountered with emergency situations like collective food poisoning, radiation and temperature plunge, such a disturbance may destroy the continuity of the solution. That is why we employ the stochastic differential equation with jump to imitate the dynamical behavior of the model. The main task of this paper is to study the effect of random disturbances and sudden fluctuations in the spread of toxoplasmosis in cat populations.

The section division of this article is as follows. In Section 2 the existence of global and nonnegative solutions for system (1.3) is proven. Since system (1.3) is constructed by adding stochastic perturbation in a deterministic system (1.1), it seems reasonable to investigate whether there are similar properties as in system (1.1). However seeing that there exists no disease-free or endemic equilibrium for system (1.3), it is essential to discuss the behavior of model (1.3) around E_0 and E^* to show the stability to some extent, which will be shown in Sections 3 and 4. We present numerical illustrations of the theoretical results in section 5. Finally, a concluding discussion is presented in Section 6.

2. Global positive solution

Before we discuss a stochastic model, firstly we should consider if there exists a solution for the system. Furthermore, a nonnegative solution is needed for a biological model. Jiang et al.[7] presented an SIR model only with white noise to study properties of the solution. It gives a method to prove the global nonnegativity of the solutions. However this model does not involve jump processes. In this part, through the Lyapunov analysis method[8], we shall show the jump processes can suppress the explosion and the solution of model (1.3) is positive and global.

For the jump diffusion coefficient we assume that for each $m > 0$ there exists $L_m > 0$ such that

(H1) $\int_Z |H_i(x,z) - H_i(y,z)|^2 \pi(dz) \le L_m |x-y|^2 (i = 1, \cdots, 4)$ where $H_1(x,z) = C_1(z)S(t-)$, $H_2(x,z) = C_2(z)I(t-), H_3(x,z) = C_3(z)R(t-)$ and $H_4(x,z) = C_4(z)O(t-)$ with $|x| \vee |y| \le m$.

(H2) $|\log(1+C_i(z))| \le K_1$, for $C_i(z) \ge -1 (i = 1, \cdots, 4)$, where K_1 is positive constant.

Theorem 2.1. Let the assumption $(H1)$ and $(H2)$ hold, for any given initial value $(S(0), I(0), R(0), O(0)) \in R^4_+$ there is a unique positive solution (S(t), I(t), R(t),O(t)) of model (1.3) on $t \ge 0$

and the solution will remain in R_+ with probability 1, namely (S(t), I(t),R(t),O(t))$\in R_+^4$ for all $t \geq 0$ almost surely.

Proof. By $(H1)$ and the drift and the diffusion are locally Lipschitz, there is a unique local solution $(S(t), I(t), R(t), O(t)) \in R_+^4$ on $t\in [0, \tau_e]$, for any given initial value $(S(0), I(0), R(0), O(0)) \in R_+^4$, where τ_e is the explosion time [20]. To prove this solution is global, it is necessary to show that $\tau_e = \infty$ a.s. At first, one confirms $S(t)$, $I(t)$ and $O(t)$ do not explode to infinity in a finite time. Set $k_0 > 0$ be sufficiently large such that $S(0) \in [1/k_0, k_0]$, $I(0) \in [1/k_0, k_0]$ and $O(0) \in [1/k_0, k_0]$. For each integer $k \geq k_0$, define the stopping time

$$\tau_k = \inf\{t \in [0, \tau_e] : S(t)\bar{\in}(1/k, k), I(t)\bar{\in}(1/k, k) \textit{ or } O(t)\bar{\in}(1/k, k)\}.$$

Here we set $\inf\emptyset = \infty$ ($\emptyset$ denotes the empty set). Obviously, τ_k is increasing as $k \to \infty$. Set $\tau_\infty = \lim_{k\to\infty} \tau_k$, therefore $\tau_\infty \leq \tau_e$ a.s. If $\tau_\infty = \infty$ a.s. is true, then $\tau_e = \infty$ a.s. and $(S(t), I(t)), O(t) \in R_+^3$ a.s. for $t \geq 0$. Hence, to complete the proof it is required to show that $\tau_\infty = \infty$ a.s. If this statement is false, then there exist a pair of constants $T > 0$ and $\varepsilon \in (0, 1)$ such that $P\{\tau_\infty \leq T\} > \varepsilon$. Thus there is an integer $k_1 \geq k_0$ such that $P\{\tau_\infty \leq T\} > \varepsilon$ for all $k \geq k_1$.

Let us define a C^2 function $V : R_+^2 \to R_+$ as follows

$$V(S, I) = S + I - \log I.$$

The nonnegativity of this function can be seen in view of $I - \log I \geq 0$ for $I > 0$. Applying It$\hat{o}$ formula, we obtain

$$\begin{aligned} dV(S,I) &= LV(S,I)dt + \sigma_1 S(t)dB_1(t) + (1 - 1/I)\sigma_2 I(t)dB_2(t) \\ &\quad + [\textstyle\int_z C_1(z)S(t-) + \int_z C_2(z)I(t-) - \log(1 + C_2(z))]\tilde{N}(dt, dz), \end{aligned} \tag{2.1}$$

where $LV : R_+^2 \longrightarrow R_+$ is defined by

$$\begin{aligned} LV(S,I) &= \Lambda - \beta S(t)O(t) - \mu S(t) + (1-1/I)(\beta S(t)O(t) - (\alpha+\mu)I(t)) + \tfrac{1}{2}\sigma_2^2 \\ &+ \int_z [C_2(z)I(t-) - \log(1+C_2(z))]\pi(dz) \\ &= \Lambda + \alpha + \mu + \sigma_2^2/2 - (\alpha+\mu)I - \mu S - \beta SO/I + \int_z [C_2(z)I(t-) - \log(1+C_2(z))]\pi(dz) \\ &\leq \Lambda + \alpha + \mu + \sigma_2^2/2 + K_2 = K, \end{aligned} \tag{2.2}$$

where $K_2 = \int_z [C_2(z)I(t-) - \log(1+C_2(z))]\pi(dz)$.

Therefore,

$$dV(S,I) \leq Kdt + [\sigma_1 S(t)dB_1(t) + (1-1/I)\sigma_2 I(t)dB_2(t)]. \tag{2.3}$$

Integrating the both sides of equation (2.3) from 0 to $\tau_k \wedge T$ and taking the expectation, yields

$$EV(S(\tau_k \wedge T), I(\tau_k \wedge T)) \leq EV(S(0), I(0)) + KE(\tau_k \wedge T).$$

As a result

$$EV(S(\tau_k \wedge T), I(\tau_k \wedge T)) \leq EV(S(0), I(0)) + KT. \tag{2.4}$$

Let $\Omega_k = \{\tau_k \leq T\}$ for $k \geq k_1$. By(2.1) $P(\Omega_k) \geq \varepsilon$. Note that for every $\omega \in \Omega_k$, we get $S(\tau_k, \omega)$ or $I(\tau_k, \omega)$ equals either k or $\frac{1}{k}$. Hence $V(S(\tau_k, \omega), I(\tau_k, \omega))$ is no less than either $k - \log k$ or $1/k - \log(1/k) = 1/k + \log k$. That is

$$V(S(\tau_k, \omega), I(\tau_k, \omega)) \geq [k - \log k] \wedge [1/k + \log k].$$

It then follows from (2.4) that

$$V(S(0), I(0)) + KT \geq E[1_{\Omega_k}(\omega)V(S(\tau_k, \omega)I(\tau_k, \omega))] \geq \varepsilon[k - \log k] \wedge [1/k + \log k],$$

where 1_{Ω_k} is the indicator function of Ω_k. Letting $k \to \infty$, it follows that

$$V(S(0), I(0)) + KT \geq \infty,$$

which is impossible, then we must have $\tau_\infty = \infty$. As a result $S(t)$, $I(t)$ and $O(t)$ will not explode in a finite time with probability one. At the same time, through the last two equations of system

(1.3), we can have solutions of $R(t)$ and $O(t)$ in the form of

$$R(t) = R(0)\phi(t) + \phi(t)^{-1} \int_0^t \alpha I(s)ds,$$

where

$$\begin{aligned}\phi(t) &= \exp \int_0^t [-(\mu+\sigma_3^2/2)t + \sigma_3 B_3(t) + \int_z (\log(1+C_3(z)) - C_3(z))\pi(dz)]ds \\ &+ \int_0^t \sigma_3 B_3(s) + \int_z (\log(1+C_3(z)))\pi(dz)ds.\end{aligned}$$

Since $I(t)$ has been proved to be global and positive, as a result $R(t)$ is also global and positive. This completes the proof.

3.Asymptotic behavior around the disease-free equilibrium of the deterministic model

As mentioned in the introduction, system (1.1) has a disease-free equilibrium $E_0 = (\Lambda/\mu, 0, 0, 0)$ and it is globally stable if $R_0 \leq 1$. While for the stochastic system (1.3), E_0 is no longer the disease-free equilibrium, and the stochastic solutions do not converge to E_0. In this section, we will study the asymptotic behavior around E_0.

Theorem 3.1. If $R_0 < 1$ and the following conditions are satisfied

$$\sigma_1^2 + \int_Z (2C_1^2(z) + C_1(z)C_2(z))\pi(dz) < 2\mu,\ \sigma_2^2 + \int_Z (2C_2^2(z) + C_1(z)C_2(z))\pi(dz) < 2(\alpha+\mu). \tag{3.1}$$

Then for any given initial value $(S(0), I(0), R(0), O(0)) \in R_+$, the solution of model (1.3) has the property

$$\limsup_{t\to\infty} 1/tE \int_0^t [(S(r)-\Lambda/\mu)^2 + I(r)^2 + R(r) + O(r)]dr \leq \Lambda^2/K_3\mu^2[2\sigma_1^2 + \int_Z 2C_1^2(z) + C_1(z)C_2(z)\pi(dz)]$$

where

$$\begin{aligned}K_3 &= \min\{(2\mu - \sigma_1^2) - \int_Z (2C_1^2(z) + C_1(z)C_2(z))\pi(dz), \\ &2(\alpha+\mu) - \sigma_2^2 - \int_Z (2C_2^2(z) + C_1(z)C_2(z))\pi(dz), e_2, e_3\}.\end{aligned}$$

Proof. First, change the variables $u = S - \Lambda/\mu$, $v = I$, $w = R$, $x = O$ then system (1.3) can be written as

$$\begin{cases} du(t) = & (-\mu u(t) - \beta u(t)x(t) - \beta\Lambda/\mu x(t))dt \\ & +\sigma_1(u(t)+\Lambda/\mu)dB_1(t) + \int_z C_1(z)(u(t-)+\Lambda/\mu)\tilde{N}(dt,dz), \\ dv(t) = & (\beta u(t)x(t) + \beta\Lambda/\mu x(t) - (\alpha+\mu)v(t))dt + \sigma_2 v dB_2(t) + \int_z C_2(z)v(t-)\tilde{N}(dt,dz), \\ dw(t) = & (\alpha v(t) - \mu w(t))dt + \sigma_3 w dB_3(t) + \int_z C_3(z)w(t-)\tilde{N}(dt,dz), \\ dx(t) = & (kv(t) - \mu_0 x(t))dt + \sigma_4 x dB_4(t) + \int_z C_4(z)x(t-)\tilde{N}(dt,dz). \end{cases} \tag{3.2}$$

and $u \in R, v > 0, w > 0, x > 0$. Define a function

$$V(u,v,w,x) = (u+v)^2 + e_1(u+v) + e_2 w + e_3 x,$$

where e_1, e_2, e_3 are three positive constants to be defined later. Applying Itô formula, we obtain

$$\begin{aligned} dV \;&= LVdt + [2(u+v)+e_1]\sigma_1(u+\Lambda/\mu)dB_1(t) \\ &+(2(u+v)+e_1)\sigma_2 v dB_2(t) + e_2\sigma_3 w dB_3(t) + e_3\sigma_4 x dB_4(t) \\ &+\int_Z\{[C_1(z)(u(t)+\Lambda/\mu)+C_2(z)v(t-)]^2 + e_1[C_1(z)(u(t)+\Lambda/\mu)+C_2(z)v(t-)] \\ &+2(u(t-)+v(t-))[C_1(z)(u(t)+\Lambda/\mu)+C_2(z)v(t-)] \\ &+e_2 C_3(z)w(t-) + e_3 C_4(z)x(t-)\}\tilde{N}(dt,dz), \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} LV = \;& (-2\mu+\sigma_1^2)u^2 - [2(\alpha+\mu)-\sigma_2^2]v^2 + (2\sigma_1^2\Lambda/\mu - e_1\mu)u + \sigma_1^2(\Lambda/\mu)^2 \\ & -e_2\mu w - e_3\mu_0 x + [e_2\alpha + e_3 k - e_1(\alpha+\mu)]v \\ & +u^2\int_Z C_1^2(z)\pi(dz) + 2u\Lambda/\mu\int_Z C_1^2(z)\pi(dz) + \tfrac{\Lambda^2}{\mu^2}\int_Z C_1^2(z)\pi(dz) \\ & +2uv\int_Z C_1(z)C_2(z)\pi(dz) + 2v\Lambda/\mu\int_Z C_1(z)C_2(z)\pi(dz) \\ & +v^2\int_Z C_2^2(z)\pi(dz). \end{aligned}$$

We choose e_1 such that $2\sigma_1^2\Lambda/\mu - e_1 = 0$, i.e $e_1 = 2\sigma_1^2\Lambda/\mu$, next we can find appropriate e_2, e_3 such that $[e_2\alpha + e_3 k - e_1(\alpha+\mu)]v \le 0$ and use the basic inequality $2ab \le a^2+b^2$, we can obtain

$$\begin{aligned} LV \quad &\leq -[(2\mu-\sigma_1^2)-\int_Z(2C_1^2(z)+C_1(z)C_2(z))\pi(dz)]u^2 \\ &-[2(\alpha+\mu)-\sigma_2^2-\int_Z(2C_2^2(z)+C_1(z)C_2(z))\pi(dz)]v^2 \\ &-e_2\mu w-e_3\mu_0 x \\ &+(\Lambda/\mu)^2[2\sigma_1^2+\int_Z 2C_1^2(z)+C_1(z)C_2(z)\pi(dz)], \end{aligned} \tag{3.4}$$

therefore

$$\begin{aligned} dV \leq \quad &(-2\mu-\sigma_1^2)u^2-[2(\alpha+\mu)-\sigma_2^2]v^2+\sigma_1^2(\Lambda/\mu)^2-e_2\mu w-e_3\mu_0 x \\ &+[2(u+v)+e_1]\sigma_1(u+\Lambda/\mu)dB_1(t)+(2(u+v)+e_1)\sigma_2 v dB_2(t) \\ &+e_2\sigma_3 w dB_3(t)+e_3\sigma_4 x dB_4(t). \end{aligned} \tag{3.5}$$

We can now integrate both sides of (3.5) from 0 to t and then take expectation. This yields

$$\begin{aligned} 0 \quad &\leq E[V(u(t),v(t),w(t),x(t))] \\ &\leq E[V(u(0),v(0),w(0),x(0))] \\ &+E\int_0^t\{-[(2\mu-\sigma_1^2)-\int_Z(2C_1^2(z)+C_1(z)C_2(z))\pi(dz)]u^2 \\ &-[2(\alpha+\mu)-\sigma_2^2-\int_Z(2C_2^2(z)+C_1(z)C_2(z))\pi(dz)]v^2 \\ &-e_2\mu w-e_3\mu_0 x\}ds \\ &+(\Lambda/\mu)^2[2\sigma_1^2+\int_Z 2C_1^2(z)+C_1(z)C_2(z)\pi(dz)], \end{aligned} \tag{3.6}$$

which implies

$$\begin{aligned} &E\int_0^t[(2\mu-\sigma_1^2)-\int_Z(2C_1^2(z)+C_1(z)C_2(z))\pi(dz)]u^2+[2(\alpha+\mu)-\sigma_2^2-\int_Z(2C_2^2(z) \\ &+C_1(z)C_2(z))\pi(dz)]v^2+e_2\mu w+e_3\mu_0 x \\ &\leq V(u(0),v(0),w(0),x(0))+(\Lambda/\mu)^2[2\sigma_1^2+\int_Z 2C_1^2(z)+C_1(z)C_2(z)\pi(dz)]t. \end{aligned} \tag{3.7}$$

Therefore

$$\begin{aligned} &\limsup_{t\to\infty}\frac{1}{t}E\int_0^t[(2\mu-\sigma_1^2)-\int_Z(2C_1^2(z)+C_1(z)C_2(z))\pi(dz)]u^2 \\ &+[2(\alpha+\mu)-\sigma_2^2-\int_Z(2C_2^2(z)+C_1(z)C_2(z))\pi(dz)]v^2+e_2\mu w+e_3\mu_0 x \\ &\leq (\Lambda/\mu)^2[2\sigma_1^2+\int_Z 2C_1^2(z)+C_1(z)C_2(z)\pi(dz)]. \end{aligned} \tag{3.8}$$

According to the conditions (3.1) we can guarantee that

$$(2\mu-\sigma_1^2)-\int_Z(2C_1^2(z)+C_1(z)C_2(z))\pi(dz)>0,\ 2(\alpha+\mu)-\sigma_2^2-\int_Z(2C_2^2(z)+C_1(z)C_2(z))\pi(dz)>0.$$

If letting

$$\begin{aligned} K_3 &= \min\{(2\mu-\sigma_1^2)-\textstyle\int_Z(2C_1^2(z)+C_1(z)C_2(z))\pi(dz),\\ &\quad 2(\alpha+\mu)-\sigma_2^2-\textstyle\int_Z(2C_2^2(z)+C_1(z)C_2(z))\pi(dz),e_2,e_3\}, \end{aligned}$$

then

$$\limsup_{t\to\infty}\frac{1}{t}E\int_0^t[(S(r)-\Lambda/\mu)^2+I(r)^2+R(r)+O(r)]dr\le\Lambda^2/K_3\mu^2[2\sigma_1^2+\int_Z 2C_1^2(z)+C_1(z)C_2(z)\pi(dz)].$$

This completes the proof.

4. Asymptotic behavior around the endemic equilibrium of the deterministic model

In this part, we assume $R_0>1$. This guarantees the existence of the endemic equilibrium E^* for model (1.1) but not the endemic equilibrium E^* for model (1.3), since this is no endemic equilibrium for model (1.3). As in section 3, we will investigate the asymptotic behavior around the endemic equilibrium of the deterministic model.

Theorem 4.1. *If $R_0>1$ and the following conditions are satisfied*

$$\sigma_1^2+\int_Z(2C_1^2(z)+C_1(z)C_2(z))\pi(dz)<2\mu,\sigma_3^2+\int_Z C_3^2(z)\pi(dz)<\mu,$$
$$\sigma_4^2+\int_Z C_4^2(z)\pi(dz)<\mu_0. \tag{4.1}$$

and appropriate positive constants p,q such that

$$\alpha+\mu-p\alpha^2/2\mu-qk^2/2\mu_0>\sigma_2^2/2+\int_Z C_2^2\pi(dz)$$

then for any given initial value $(S(0),I(0),R(0),O(0))\in R_+$, the solution of system (1.3) has the property

$$\begin{aligned} \limsup_{t\to\infty}1/tE\quad &\textstyle\int_0^t[S(r)-2\mu/A_1S^*]^2+[I(r)-(\alpha+\mu-p\alpha^2/2\mu+qk^2/2\mu_0)/A_2I^*]^2\\ &+[R(r)-\mu/2A_3R^*]^2+[O(r)-\mu_0/2A_4O^*]^2\}dr\le M/K_\sigma, \end{aligned}$$

where

$$K_\sigma = \min\{A_1/2, A_2A_3, A_4\}$$

$$A_1 = 2\mu - \sigma_1^2 - \int_Z (2C_1^2(z) + C_1(z)C_2(z))\pi(dz)$$

$$A_2 = \alpha + \mu - p\alpha^2/2\mu - qk^2/2\mu_0 - \sigma_2^2/2 - \int_Z C_2^2\pi(dz)$$

$$A_3 = p(\mu - \sigma_3^2 - \int_Z C_3^2(z))\pi(dz))/2$$

$$A_4 = q(\mu_0 - \sigma_4^2 - \int_Z C_4^2(z))\pi(dz))/2$$

$$\begin{aligned} M = \ & (\mu\sigma_1^2 + 2\mu \textstyle\int_Z C_1^2(z)\pi(dz))/A_1 S^{*2} \\ & + (\alpha + \mu - p\alpha^2/2\mu - qk^2/2\mu_0)(\sigma_2^2/2 + \textstyle\int_Z C_1^2(z)\pi(dz))/A_2 I^{*2} \\ & + p\mu[p\sigma_4^2 + p\textstyle\int_Z C_3^2(z)\pi(dz)]/(4A_3)R^{*2} + q\mu_0[q\sigma_4^2 + q\textstyle\int_Z C_4^2(z)\pi(dz)]/(4A_4)O^{*2} \end{aligned}$$

Proof. Define a C^2 function $V : R_+^4 \to R_+$ by

$$V(S,I,R,O) = (S - S^* + I - I^*)^2/2 + w_1(S+I) + p(R-R^*)^2/2 + q(O-O^*)^2/2, \qquad (4.2)$$

where $w_1 > 0, p > 0, q > 0$ are positive constants to be chosen later.

In order to make the proof more clear, we divide (4.2) into two parts:

$$V(x) = V_1(x) + V_2(x),$$

where

$$V_1(x) = (S - S^* + I - I^*)^2/2 + w_1(S+I), V_2(x) = p(R-R^*)^2/2 + q(O-O^*)^2/2.$$

Applying Itô formula, we obtain

$$\begin{aligned} dV_1(x) \ & = LV_1 dt + (S(t) - S^* + I(t) - I^* + w_1)(\sigma_1 S(t)dB_1(t) + \sigma_2 I(t)dB_2 t) \\ & + \textstyle\int_Z \{(S(t-) - S^* + I(t-) - I^* + w_1)(C_1(z)S(t-) + C_2(z)I(t-)) \\ & + (C_1(z)S(t-) + C_2(z)I(t-))^2/2\}\widetilde{N}(dt,dz), \end{aligned}$$

$$
\begin{aligned}
dV_2(x) &= LV_2dt + p(R(t)-R^*)RdB_3(t) + q(O(t)-O^*)OdB_4(t) \\
&\quad + \int_Z \{C_3^2(z)R^2(t-)/2 + pC_3(z)R(t-)(R(t-)-R^*) \\
&\quad + C_4^2(z)O^2(t-)/2 + qC_3(z)O(t-)(O(t-)-O^*)\}\widetilde{N}(dt,dz).
\end{aligned}
$$

In detail

$$
\begin{aligned}
LV_1 &= (S(t)-S^*+I(t)-I^*)[\Lambda - \mu S(t) - (\alpha+\mu)I(t)] \\
&\quad + w_1[\Lambda - \mu S(t) - (\alpha+\mu)I(t)] + (\sigma_1^2 S(t)^2 + \sigma_2^2 I(t)^2)/2 \\
&\quad + \int_Z ((C_1(z)S(t-) + C_2(z)I(t-))^2)/2\pi(dz) \\
&= -\mu(S(t)-S^*)^2 - (\alpha+\mu)(I(t)-I^*)^2 + w_1\Lambda + [(\alpha+\mu+\mu)I^* - w_1\mu]S(t) \\
&\quad + [(\alpha+\mu+\mu)S^* - (\alpha+\mu)w_1]I(t) + (\sigma_1^2 S(t)^2 + \sigma_2^2 I(t)^2)/2 \\
&\quad - (\alpha+\mu+\mu)(S(t)I(t) + S^*I^*) + \int_Z ((C_1(z)S(t-) + C_2(z)I(t-))^2)/2\pi(dz),
\end{aligned} \tag{4.3}
$$

and

$$
\begin{aligned}
LV_2 &= p(R-R^*)(\alpha I - \mu R) + p\sigma_3^2 R^2/2 + q(O-O^*)(kI - \mu_0 O) + q\sigma_4^2 O^2/2 \\
&\quad + p\int_Z C_3^2(z)R^2\pi(dz)/2 + q\int_Z C_4^2(z)O^2\pi(dz)/2 \\
&= p(R-R^*)[\alpha(I-I^*) - \mu(R-R^*)] + p\sigma_3^2 R^2/2 \\
&\quad + q(O-O^*)[K(I-I^*) - \mu_0(O-O^*)] + q\sigma_4^2 O^2/2 \\
&\quad + p\int_Z C_3^2(z)R^2\pi(dz)/2 + q\int_Z C_4^2(z)O^2\pi(dz)/2 \\
&\le p\alpha^2/2\mu(I-I^*)^2 + (\tfrac{p\mu}{2} - p\mu)(R-R^*)^2 + p\sigma_3^2 R^2/2 \\
&\quad + qk^2/2\mu_0(I-I^*)^2 + (\tfrac{q\mu_0}{2} - q\mu_0)(O-O^*)^2 + q\sigma_4^2 O^2/2 \\
&\quad + p\int_Z C_3^2(z)R^2\pi(dz)/2 + q\int_Z C_4^2(z)O^2\pi(dz)/2 \\
&= (p\alpha^2/2\mu + qk^2/2\mu_0)(I-I^*)^2 - \tfrac{p\mu}{2}(R-R^*)^2 - \tfrac{q\mu_0}{2}(O-O^*)^2 + p\sigma_3^2 R^2/2 + q\sigma_4^2 O^2/2 \\
&\quad + p\int_Z C_3^2(z)R^2\pi(dz)/2 + q\int_Z C_4^2(z)O^2\pi(dz)/2.
\end{aligned} \tag{4.4}
$$

Choose $w_1 = \max\{(\alpha+\mu+\mu)I^*/\mu, (\alpha+\mu+\mu)S^*/(\alpha+\mu)\}$. Hence $(\alpha+\mu+\mu)I^* - w_1\mu]S \le 0$ and $(\alpha+\mu+\mu)S^* - (\alpha+\mu)w_1 \le 0$. Next by using the basic inequality $2ab \le a^2+b^2$, we can obtain

$$\begin{aligned} LV_1 &= -\mu(S(t)-S^*)^2-(\alpha+\mu)(I(t)-I^*)^2+(\sigma_1^2S(t)^2+\sigma_2^2I(t)^2)/2 \\ &+S(t)^2\int_Z C_1^2(z)\pi(dz)+I(t)^2\int_Z C_2^2(z)\pi(dz). \end{aligned} \tag{4.5}$$

Taking (4.4) and (4.5) together, we get

$$\begin{aligned} LV=LV_1+LV_2 &\le -\mu(S-S^*)^2-(\alpha+\mu+p\alpha^2/2\mu+qk^2/2\mu_0)(I-I^*)^2 \\ &+(\sigma_1^2S^2+\sigma_2^2I^2+p\sigma_3^2R^2+\sigma_4^2O^2)/2-p\mu(R-R^*)^2/2-q\mu_0(O-O^*)^2/2 \\ &+S(t)^2\int_Z C_1^2(z)\pi(dz)+I(t)^2\int_Z C_2^2(z)\pi(dz) \\ &=-(\mu-\sigma_1^2/2)S^2+2\mu SS^*-\mu S^{*2} \\ &-(2\mu\mu_0(\alpha+\mu)-\mu_0p\alpha^2-\mu qk^2-\mu\mu_0\sigma_2^2)I^2 \\ &+(4\mu\mu_0(\alpha+\mu)-\mu_0p\alpha^2-\mu qk^2)II^* \\ &-(2\mu\mu_0(\alpha+\mu)-\mu_0p\alpha^2-\mu qk^2)I^{*2} \\ &-(p\mu/2-p\sigma_3^2/2)R^2+p\mu RR^*-p\mu R^{*2}/2 \\ &-(q\mu_0/2-q\sigma_4^2/2)R^2+q\mu_0OO^*-q\mu_0O^{*2}/2 \\ &+S(t)^2\int_Z C_1^2(z)\pi(dz)+I(t)^2\int_Z C_2^2(z)\pi(dz) \\ &=-A_1/2[S(r)-2\mu/A_1S^*]^2 \\ &-A_2[I(r)-(\alpha+\mu-p\alpha^2/2\mu+qk^2/2\mu_0)/A_2I^*]^2 \\ &-A_3[R(r)-\mu/2A_3R^*]^2-[O(r)-\mu_0/2A_4O^*]^2\}dr\le M/K_\sigma. \end{aligned} \tag{4.6}$$

Note that p,q are positive constants and satisfy

$$\alpha+\mu-p\alpha^2/2\mu-qk^2/2\mu_0>\sigma_2^2/2+\int_Z C_2^2\pi(dz).$$

Besides, the condition (4.1) implies implies $A1,A2,A3$, and $A4$ are positive constants.

Since

$$\begin{aligned} dV(x(t))= &\ LV(x)dt+(S-S^*+I-I^*+w_1)(\sigma_1S(t)dB_1(t)+\sigma_2I(t)dB_2t) \\ &+p(R-R^*)RdB_3(t)+q(O-O^*)OdB_4(t), \end{aligned} \tag{4.7}$$

we can now integrate both sides of (4.7) from 0 to t and take the expectation, next consider (4.6), one gets

$$\begin{aligned} 0 &\leq E[V(S(t),I(t),R(t),O(t))] \\ &\leq E[V(S(0),I(0),R(0),O(0))] \\ &+E\int_0^t\{-A_1/2[S(r)-2\mu/A_1S^*]^2-A_2[I(r)-(\alpha+\mu-p\alpha^2/2\mu+qk^2/2\mu_0)/A_2I^*]^2 \\ &-A_3[R(r)-\mu/2A_3R^*]^2-[O(r)-\mu_0/2A_4O^*]^2\}dr+Mt, \end{aligned} \tag{4.8}$$

where

$$\begin{aligned} M= &(\mu\sigma_1^2+2\mu\int_Z C_1^2(z)\pi(dz))/A_1S^{*2} \\ &+(\alpha+\mu-p\alpha^2/2\mu-qk^2/2\mu_0)(\sigma_2^2/2+\int_Z C_1^2(z)\pi(dz))/A_2I^{*2} \\ &+p\mu[p\sigma_4^2+p\int_Z C_3^2(z)\pi(dz)]/(4A_3)R^{*2}+q\mu_0[q\sigma_4^2+q\int_Z C_4^2(z)\pi(dz)]/(4A_4)O^{*2}, \end{aligned}$$

which implies that

$$\begin{aligned} &E\int_0^t\{A_1/2[S(r)-2\mu/A_1S^*]^2+A_2[I(r)-(\alpha+\mu-p\alpha^2/2\mu+qk^2/2\mu_0)/A_2I^*]^2 \\ &+A_3[R(r)-\mu/2A_3R^*]^2+[O(r)-\mu_0/2A_4O^*]^2\}dr \\ &\leq E[V(S(0),I(0),R(0),O(0))]+Mt. \end{aligned} \tag{4.9}$$

We can now divide both sides by t and let $t\to\infty$, yields

$$\begin{aligned} \limsup_{t\to\infty}\frac{1}{t} \quad &E\int_0^t\{A_1/2[S(r)-2\mu/A_1S^*]^2 \\ &+A_2[I(r)-(\alpha+\mu-p\alpha^2/2\mu+qk^2/2\mu_0)/A_2I^*]^2 \\ &+A_3[R(r)-\mu/2A_3R^*]^2+[O(r)-\mu_0/2A_4O^*]^2\}dr \\ &\leq M. \end{aligned} \tag{4.10}$$

Set

$$K_\sigma=\min\{A_1/2,A_2A_3,A_4\},$$

then it is easy to obtain

$$\limsup_{t\to\infty}\frac{1}{t}E\quad\int_0^t\{[S(r)-2\mu/A_1S^*]^2+[I(r)-(\alpha+\mu-p\alpha^2/2\mu+qk^2/2\mu_0)/A_2I^*]^2$$
$$+[R(r)-\mu/2A_3R^*]^2+[O(r)-\mu_0/2A_4O^*]^2\}dr\le M/K_\sigma.$$

The proof is complete.

5. Numerical Simulation

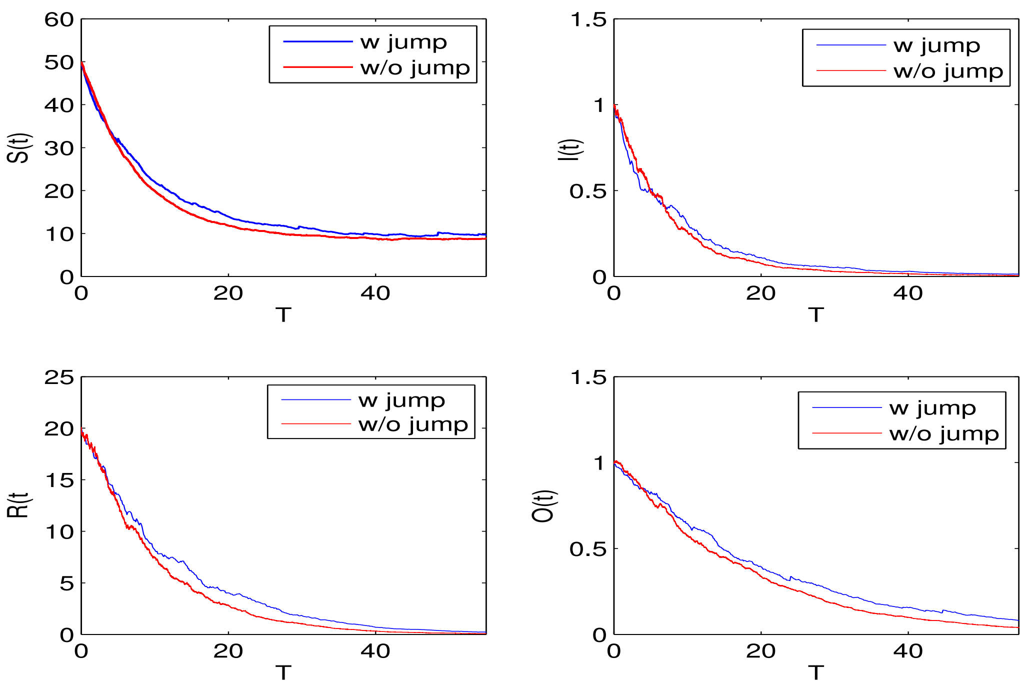

FIGURE 1. Solutions of system (1.3) with $R_0<1$. The parameter values are used as follows $\lambda=1,\ \beta=0.52/54,\ k=1/40,\ \mu=0.6/52,\ \mu_0=7/100,\ \alpha=0.5,\ \sigma_1=0.004,\ \sigma_2=0.003,\ \sigma_3=0.002,\ \sigma_4=0.001.$

In this section, we present numerical illustrations of the theoretical results. From Theorem 3.1, if the basic reproduction number is less than one, it is obtained that the numbers of all populations oscillate around the disease-free equilibrium. The smaller the values are, the weaker the fluctuation is. In other words, if the stochastic perturbations become small, the solution of Eq. (1.3) will be close to the disease-free equilibrium of Eq. (1.1). From the proof of Theorem

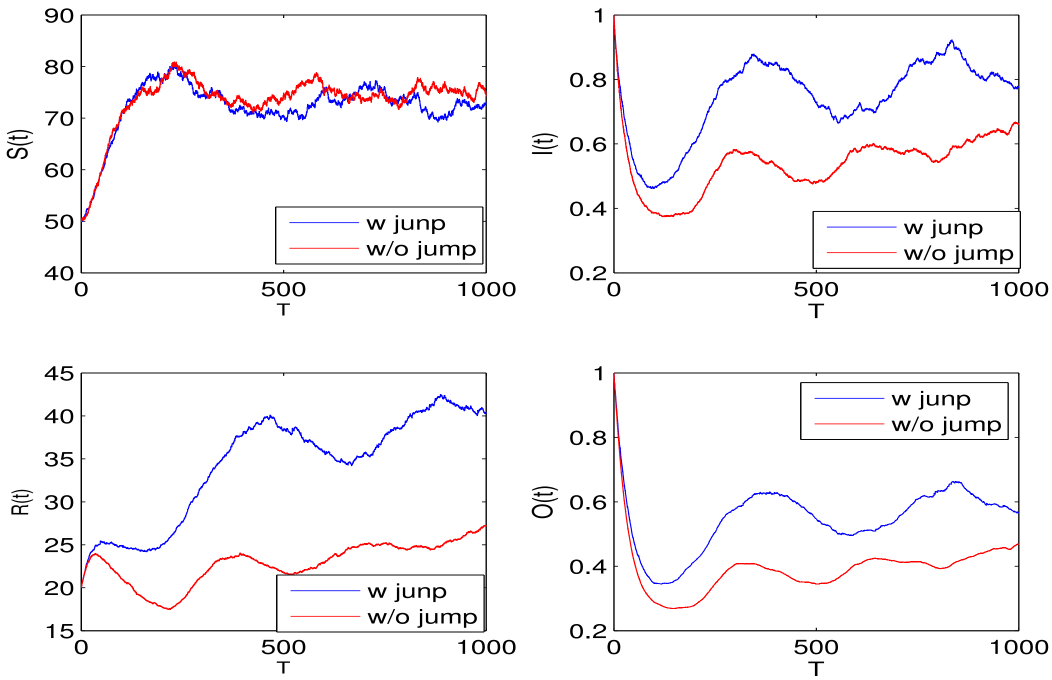

FIGURE 2. Solutions of system (1.3) with $R_0 > 1$. The parameter values are used as follows $\lambda = 1.154$, $\beta = 0.52/54$, $k = 1/20$, $\mu = 0.6/52$, $\mu_0 = 7/100$, $\alpha = 0.5$, $\sigma_1 = 0.004$, $\sigma_2 = 0.003$, $\sigma_3 = 0.002$, $\sigma_4 = 0.001$.

3.1, we can obtain

$$\begin{aligned} LV &\le -[2(\mu-\sigma_1^2) - \textstyle\int_Z (2C_1^2(z) + C_1(z)C_2(z))\pi(dz)]u^2 \\ &\quad -[2(\alpha+\mu) - \sigma_2^2 - \textstyle\int_Z (2C_2^2(z) + C_1(z)C_2(z))\pi(dz)]v^2 \\ &\quad -e_2\mu w - e_3\mu_0 x, \end{aligned}$$

which is negative-definite, therefore E_0 is stochastically asymptotically stable in the large which can be seen from figure 1). Theorem 4.1 shows that the solution of model (1.3) fluctuates around the certain level which is relevant to

$$P^* \left(2\mu/A_1S^*, (\alpha+\mu-p\alpha^2/2\mu+qk^2/2\mu_0)/A_2I^*, \mu/2A_3R^*, \mu_0/2A_4O^*\right)$$

and σ_i, for $i = 1,2,3$. With the value of σ_i decreasing, P^* will be closing to E^* (which is the epidemic equilibrium of system (1.1)) and the difference between the solution of system (1.3) and P^* also decreases(see figure 2).

6. Conclusion

Based on [8], we get a revised mathematical model to simulate the toxoplasmosis spread between cat and oocyst populations. To deterministic epidemic models we often investigate the behavior of equilibrium since by this mean we can better understand the stability of the system. However, as for stochastic differential equations there in no any equilibrium. As an alternative, we may discuss asymptotic behaviors around the equilibrium corresponding to its deterministic system. To make the discuss meaningful firstly the global existence and nonnegativity of the solutions are guaranteed. By the method of stochastic Lyapunov functions, we study its asymptotic behavior. When the reproductive number is less than one, the disease-free equilibrium is stochastically asymptotically stable in the large. When all the perturbations are small and the reproductive number is larger than one, we construct a new class of stochastic Lyapunov functions to show the positive recurrence, and our results reveal some cycling phenomena of recurrent diseases. These results mean that stochastic system has the similar property with the corresponding deterministic system when the white noise is small.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgements

This work is supported by the National Natural Science Foundation of China (11101305, 11471243).

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