Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2016, 2016:11
ISSN: 2052-2541

OPTIMAL HARVESTING CONTROL OF *N* SPECIES FOR A NONLINEAR POPULATION SYSTEM

JIANGBI LIU[1], ZHIXUE LUO[1,*], HONGWEI LUO[1,2]

[1]Department of Mathematics, Lanzhou Jiaotong University, Lanzhou 730070, China

[2]Department of Information Engineering, Gansu Forest College, Tianshui 741020, China

Abstract. In this paper, we investigate the optimal harvesting problem for a class of nonlinear population system with fertility and mortality depending on the population size. Fixed point theory is used to obtain the existence and uniqueness of nonnegative solution in terms of the controls. Optimality conditions are derived by means of normal cone technique. The existence of the optimal control is carefully verified via Ekeland's variation principle, some results in references are extended.

Keywords: optimal control; competition; age-dependent.

2010 AMS Subject Classification: 45K05, 92B05.

1. Introduction

The ecosystem problem has been paid more attention in recent years, especially in China, the ecological civilization has risen to national strategies, emphasizing sustainable development. So, population system as a main subsystem in ecosystem, it's investigation of the optimal control has very important practical significance. Many researchers in the world have made great

*Corresponding author

E-mail address: luozhixue2009@126.com

Received December 16, 2015

achievements; see [1-8] and the references therein. For single-species with age-structure, there are large numbers of research and the results is relatively perfect, the works of Brokate in [1], Anita in [3] and Barbu in [18] gave a detailed description, the methods of theirs will be a source of inspiration and can provide a reference to the follow-up related research. For periodic age-dependent population dynamics model, see [4]. Anita in [5] and Brauer in [15] investigated the impact of constant harvesting on a nonlinear age-dependent, but relatively less research on the continuous distribution multiple populations with age structure, for related work involving optimal control of interacting species, see [6] considered the well posedness and the optimal control of two competing species with age dependence. W.L.Chan in [7] and Z.-R. He in [8-10] analyzed optimal birth control of age-dependent competitive species II and III, the results were extended to N species by Zhixue Luo [11]. Recently, Zhixue Luo [12-13] first formulated a new age-dependent toxicant population model in an environment with small toxicant capacity, effectively bridge the research between age-structure and polluted environment. See Fister [14] for a two-stage age-dependent competitive system model. However, the birth and mortality rates of these model were not consider the total population size. Among the practical problems, it determines the real rate of the biological individual and the behavior of individual. In order to bridge this gap, this paper propose a more realistic nonlinear population models, which description of an optimal control of N species for a class of competitive system, the birth and mortality rates are here nonlinear functions of the population size.

2. The model and its well posedness

In this paper, the dynamics of the control problem can be described by the following equations:

$$\begin{cases} \frac{\partial p_i}{\partial t} + \frac{\partial p_i}{\partial a} = f_i(a,t) - u_i(a,t)p_i - \mu_i(a,t,\sum\limits_{i=1}^{n} P_i(t))p_i - \sum\limits_{k=1,k\neq i}^{n} \lambda_{ik}(a,t)P_k(t)p_i, \\ p_i(0,t) = \int_0^A \beta_i(a,t,\sum\limits_{i=1}^{n} P_i(t))p_i(a,t)da,\ i = 1,2,\dots,n, \\ p_i(a,0) = p_{i0}(a), \\ P_i(t) = \int_0^A p_i(a,t)da,\ (a,t) \in Q, \end{cases} \tag{2.1}$$

where $Q = (0,A) \times (0,T), p_i(a,t)$ are the density of ith population of age a at the moment t; μ_i and β_i represents the death and birth rates of ith population respectively; A is the maximal age of individuals in populations and T is a given finite horizon; $\lambda_{ik}(a,t)$ are the interaction coefficients $(i,k = 1,2,\ldots,n, k \neq i)$; $f_i(a,t)$ represents inputting rates of ith population respectively, such as migration, earthquakes and other natural disasters caused mortality.

The aim of this paper is to seek the maximum of the following functional, which gives the profit from harvesting less the cost of harvesting:

$$J(u) = \sum_{i=1}^{n} \int_0^A \int_0^T \big[K_i(a)u_i(a,t)p_i(a,t) - \frac{1}{2}B_i u_i^2(a,t)\big] dtda. \tag{2.2}$$

where $K_i(a)$ are selling price factors, positive constants B_i represents the cost factors of harvesting, $u = (u_1, u_2, \ldots, u_n)$ are the proportions of the populations to be harvested, and the state $p = (p_1, p_2, \ldots, p_n)$ is the solution of the system (2.1) corresponding to $(u_1, u_2, \ldots, u_n)$.

Definition 2.1. The control set is defined as

$$U_{ad} = \prod_{i=1}^{n} U_i,\ U_i = \big\{u_i(a,t) \in L^\infty(Q) |\ 0 \leq u_i(a,t) \leq N_i,\ a.e\ in\ Q\big\}.$$

Definition 2.2. We define our state solution space as

$$X = \big\{(p_1, p_2, \ldots, p_n) \in (L^\infty(Q))^n |\ 0 \leq \int_0^A p_i(a,t)da \leq M,\ a.e\ on\ Q\big\},$$

where $M = \|f\|_{L^1(Q)}(Ap^0 + 1)e^{\beta^0 T}$.

Throughout this paper, we always assume that:

(A_1) $\mu_i(a,t) \in L^1_{loc}(Q), 0 \leq \mu_i(a,t) \leq \mu^0, \mu^0$ is constant, $\int_0^A \mu_i(a,t)da = +\infty, a.e.\ t \in (0,T), i = 1,2,\ldots,n$;

(A_2) $\beta_i(a,t) \in L^1_{loc}(Q), 0 \leq \beta_i(a,t) \leq \beta^0, \beta^0$ is constant, $(a,t) \in Q$;

(A_3) $0 \leq \lambda_i(a,t) \leq \lambda^0, 0 \leq p_i(a,t) \leq M, 0 \leq p_{i0}(a) \leq p^0, f_i \in L^1(Q), f_i(a,t) \geq 0$;

(A_4) $\forall s \in R^+, |\beta_i(a,t,s_1) - \beta_i(a,t,s_2)| \leq L_{\beta_i}|s_1 - s_2|, |\mu_i(a,t,s_1) - \mu_i(a,t,s_2)| \leq L_{\mu_i}|s_1 - s_2|$.

Integrating (2.1) along almost every characteristic line $(a - t = k)$, draws the following proposition:

Proposition 2.1. *The solution of system (2.1) can be expressed as*

$$p_i(a,t) = \begin{cases} p_{i0}(a-t)\Pi(a,t,t;H_i) + \int_0^t f_i(a-s,t-s)\Pi(a,t,s;H_i)ds, & a \geq t \\ b(t-a;P_i)\Pi(a,t,a;H_i) + \int_0^a f_i(a-s,t-s)\Pi(a,t,s;H_i)ds, & a < t \end{cases} \tag{2.3}$$

where

$$H_i(a,t) = \sum_{k=1,k\neq i}^{n} \lambda_{ik}(a,t)P_k(t);$$

$$\Pi(a,t,s;H_i) = \exp\Big\{-\int_0^s \big[\mu_i(a-\tau,t-\tau,\sum_{i=1}^{n} P_i(t-\tau)) + u_i(a-\tau,t-\tau) + H_i(a-\tau,t-\tau)\big]d\tau\Big\},$$

$t \in [0,T],\ s \in \big(0,\min\{a,t\}\big),\ b(t;H_i) \in L^\infty(0,T)$ *is the solution of the Volterra integral equation*

$$b(t;H_i) = F(t,H_i) + \int_0^t K(t,s;H_i)b(t-s;H_i)ds. \tag{2.4}$$

Here we have set

$$K(t,a;H_i) = \beta_i(a,t,H_i(t))\Pi(a,t,a;H_i), \tag{2.5}$$

and

$$\begin{aligned} F(t;H_i) = &\int_0^\infty \beta_i(a+t,t,\sum_{i=1}^{n} P_i(t))p_{i0}(a)\Pi(a+t,t,t;H_i)da \\ &+ \int_0^\infty \beta_i(a,t,\sum_{i=1}^{n} P_i(t)) \int_0^{\min\{a,t\}} f_i(a-s,t-s)\Pi(a,t,s;H_i)dsda, \end{aligned} \tag{2.6}$$

where the functions p_0, β and Π are extended by zero outside their definition sets.

Notes that the assumptions imply that $K \in L^\infty(Q\times(0,\infty)),\ F \in L^\infty((0,T)\times(0,\infty))$,

$$0 \leq K(t,a;H_i) \leq \beta(a,t,0) \text{ a.e. in } Q\times(0,\infty),$$

$$0 \leq F(t;H_i) \leq F(t;0) \text{ a.e. in } (0,T)\times(0,\infty).$$

Before stating the proof of the existence and uniqueness of a solution to (2.1) we have to state some estimates.

Lemma 2.1. *There exist M_{1T}(constant depending on T), such that for any $P_i, P_k \in X (k \neq i)$, we have*

$$\begin{aligned} \big|F(t;H_i^1) - F(t;H_i^2)\big| \leq M_{1T}\Big(&\sum_{i=1}^{n}(|P_i^1(t) - P_i^2(t)| \\ &+ \int_0^t \big|P_i^1(s) - P_i^2(s)\big|ds\Big) + \sum_{k\neq i,k=1}^{n} \int_0^t \big|P_k^1(s) - P_k^2(s)\big|ds\Big). \end{aligned}$$

Proof. When $0 < t < A$, relation (2.6) imply that

$$\begin{aligned}
&\big|F(t;H_i^1) - F(t;H_i^2)\big| \\
&\leq \int_0^\infty \big|\beta_i(a+t,t,\sum_{i=1}^n P_i^1(t)) - \beta_i(a+t,t,\sum_{i=1}^n P_i^2(t))\big| p_{i0}(a)\Pi(a+t,t,t;P_i^1)da \\
&+ \int_0^\infty \beta_i(a+t,t,\sum_{i=1}^n P_i^2(t))p_{i0}(a)\big|\Pi(a+t,t,t;H_i^1) - \Pi(a+t,t,t;H_i^2)\big|da \\
&+ \int_0^\infty \big|\beta_i(a,t,\sum_{i=1}^n P_i^1(t)) - \beta_i(a,t,\sum_{i=1}^n P_i^2(t))\big| \int_0^\gamma \big|f_i(a-s,t-s)\Pi(a,t,s;H_i^1)\big|dsda \\
&+ \int_0^\infty \beta_i(a,t,\sum_{i=1}^n P_i^1(t)) \int_0^\gamma f_i(a-s,t-s)\big|\Pi(a,t,s;H_i^1) - \Pi(a,t,s;H_i^2)\big|dsda \\
&\leq \big(Ap^0 L_{\beta_i} + L_{\beta_i}\|f_i\|_{L^1(Q)}\big)\big(\sum_{i=1}^n \big|P_i^1(t) - P_i^2(t)\big|\big) \\
&+ \big(Ap^0\beta^0 L_{\mu_i} + \beta^0 L_{\mu_i}\|f_i\|_{L^1(Q)}\big)\big(\sum_{i=1}^n \int_0^t \big|P_i^1(s) - P_i^2(s)\big|ds\big) \\
&+ \big(Ap^0\beta^0\lambda^0 + A\beta^0\lambda^0\|f_i\|_{L^1(Q)}\big)\big(\sum_{k\neq i,k=1}^n \int_0^t \big|P_k^1(s) - P_k^2(s)\big|ds\big) \\
&\leq M_{1T}\big(\sum_{i=1}^n \big(|P_i^1(t) - P_i^2(t)| + \int_0^t \big|P_i^1(s) - P_i^2(s)\big|ds\big) + \sum_{k\neq i,k=1}^n \int_0^t \big|P_k^1(s) - P_k^2(s)\big|ds\big),
\end{aligned}$$

where

$$M_{1T} = \max\Big\{Ap_i^0 L_{\beta_i} + L_{\beta_i}\|f_i\|_{L^1(Q)}, Ap_i^0\beta^0 L_{\mu_i} + \beta^0 L_{\mu_i}\|f_i\|_{L^1(Q)}, Ap_i^0\beta^0\lambda^0 + A\beta^0\lambda^0\|f_i\|_{L^1(Q)}\Big\}.$$

Meanwhile, the same result we can get if $A < t < T$.

Lemma 2.2. *There exist M_{2T}(constant depending on T), such that for any $P_i, P_k \in X(k \neq i)$, we have*

$$\begin{aligned}
\big|b(t;H_i^1) - b(t;H_i^2)\big| \leq M_{2T}\big(&\sum_{i=1}^n (|P_i^1(t) - P_i^2(t)| \\
&+ \int_0^t \big|P_i^1(s) - P_i^2(s)\big|ds\big) + \sum_{k\neq i,k=1}^n \int_0^t \big|P_k^1(s) - P_k^2(s)\big|ds\big).
\end{aligned}$$

Proof. By (2.4),(2.5),(2.6) we get

$$\big|b(t;H_i)\big| \leq \beta^0(p^0 A + \|f_i\|_{L^1(Q)}) + \beta^0 \int_0^t \big|b(s;H_i)\big|ds,$$

then using Bellman's lemma we have

$$0 < b(t;H_i) < (A\beta^0 p^0 + \beta^0 \|f_i\|_{L^1(Q)})e^{T\beta^0} := M_T.$$

$$\begin{aligned}
&\big|b(t;H_i^1) - b(t;H_i^2)\big| \le \big|F(t;H_i^1) - F(t;H_i^2)\big| \\
&+ \int_0^t \big|K(t,t-s;H_i^1) - K(t,t-s;H_i^2)\big| b(s;P_i^1)ds + \int_0^t K(t,t-s;H_i^2)\big|b(s;H_i^1) - b(s;H_i^2)\big|ds \\
&\le M_{1T}\big(\sum_{i=1}^n (|P_i^1(t) - P_i^2(t)| + \int_0^t \big|P_i^1(s) - P_i^2(s)\big|ds) + \sum_{k\neq i,k=1}^n \int_0^t \big|P_k^1(s) - P_k^2(s)\big|ds\big) \\
&+ TM_T(L_{\beta_i} + \beta^0 L_{\mu_i} + \lambda^0)\big(\sum_{i=1}^n |P_i^1(t) - P_i^2(t)| + \sum_{i=1}^n \int_0^t \big|P_i^1(s) - P_i^2(s)\big|ds \\
&+ \sum_{k\neq i,k=1}^n \int_0^t \big|P_k^1(s) - P_k^2(s)\big|ds\big) + \beta^0 \int_0^t \big|b(s;H_i^1) - b(s;H_i^2)\big|ds.
\end{aligned}$$

Using Gronwall's lemma, we obtain

$$\begin{aligned}
\big|b(t;H_i^1) - b(t;H_i^2)\big| \le M_{2T}\big(&\sum_{i=1}^n (|P_i^1(t) - P_i^2(t)| \\
&+ \int_0^t \big|P_i^1(s) - P_i^2(s)\big|ds) + \sum_{k\neq i,k=1}^n \int_0^t \big|P_k^1(s) - P_k^2(s)\big|ds\big),
\end{aligned}$$

where $M' = \big(M_{1T} + TM_T(L_{\beta_i} + \beta^0 L_{\mu_i} + \lambda^0)\big)$, $M_{2T} = \max\big\{M_T, M'(2 + \beta^0 + T\beta^0)e^{\beta^0 T}\big\}$.

Theorem 2.1. *If T is small enough, then there are constants $K_i(t)$ with $\lim_{T\to 0} K_i(t) > 0$, $i = 1,2$, such that*

$$\sum_{i=1}^n \|p_i^1(\cdot,s) - p_i^2(\cdot,s)\|_{L^1(0,A)} \le K_1(T)T\big(\sum_{i=1}^n \|u_i(\cdot,s)^1 - u_i^2(\cdot,s)\|_{L^1(0,A)}\big) \tag{2.7}$$

$$\sum_{i=1}^n \|p_i^1(a,t) - p_i^2(a,t)\|_{L^\infty(Q)} \le K_2(T)T\big(\sum_{i=1}^n \|u_i^1(a,t) - u_i^2(a,t)\|_{L^\infty(Q)}\big). \tag{2.8}$$

Proof. For almost any $t \in (0,A)$

$$
\begin{aligned}
&\|p_i^1(a,t)-p_i^2(a,t)\|_{L^1(0,A)} = \int_0^t |p_i^1(a,t)-p_i^2(a,t)|da + \int_t^A |p_i^1(a,t)-p_i^2(a,t)|da \\
&\le \int_0^t |b(t-a;H_i^1)-b(t-a;H_i^2)|\Pi(a,t,a;H_i^1)da \\
&+ \int_0^t b(t-a;H_i^2)|\Pi(a,t,a;H_i^1)-\Pi(a,t,a;H_i^2)|da \\
&+ \int_0^t \int_0^a f_i(a-s,t-s)|\Pi(a,t,s;H_i^1)-\Pi(a,t,s;H_i^2)|dsda \\
&+ \int_t^A p_{i0}(a-t)|\Pi(a,t,t;H_i^1)-\Pi(a,t,t;H_i^2)|da \\
&+ \int_t^A \int_0^t f_i(a-s,t-s)|\Pi(a,t,s;H_i^1)-\Pi(a,t,s;H_i^2)|dsda.
\end{aligned}
$$

Using lemma 2.2 we may infer that

$$
\begin{aligned}
&\|p_i^1(a,t)-p_i^2(a,t)\|_{L^1(0,A)} \le M_{2T}\int_0^t \big(\sum_{i=1}^n |P_i^1(t-a)-P_i^2(t-a)| + \sum_{i=1}^n \int_0^{t-a} (|P_i^1(\tau)-P_i^2(\tau)| \\
&+ \sum_{k\neq i,k=1}^n |P_k^1(\tau)-P_k^2(\tau)|)d\tau\big)da + M_T L_{\mu_i}\int_0^t\int_0^a \sum_{i=1}^n |P_i^1(t-s)-P_i^2(t-s)|dsda \\
&+ \int_0^t\int_0^a |u_i^1-u_i^2|(a-\tau,t-\tau)dsda + M_T\lambda^0\int_0^t\int_0^a \sum_{k\neq i,k=1}^n |P_k^1(t-s)-P_k^2(t-s)|dsda \\
&+ L_{\mu_i}\|f_i\|_{L^1(Q)}\int_0^s \sum_{i=1}^n |P_i^1(t-\tau)-P_i^2(t-\tau)|ds + \|f_i\|_{L^1(Q)}\int_0^s |u_i^1-u_i^2|(a-\tau,t-\tau)ds \\
&+ \lambda^0\|f_i\|_{L^1(Q)}\int_0^s \sum_{k\neq i,k=1}^n |P_k^1(t-\tau)-P_k^2(t-\tau)|d\tau + p^0\int_t^A\int_0^t |u_i^1-u_i^2|(a-\tau,t-\tau)d\tau da \\
&+ p^0 L_{\mu_i}\int_t^A\int_0^t \sum_{i=1}^n |P_i^1(t-\tau)-P_i^2(t-\tau)|d\tau da + \|f_i\|_{L^1(Q)}\int_0^s |u_i^1-u_i^2|(a-\tau,t-\tau)d\tau \\
&+ p^0\lambda^0\int_t^A\int_0^t \sum_{k\neq i,k=1}^n |P_k^1(t-\tau)-P_k^2(t-\tau)|d\tau + L_{\mu_i}\|f_i\|_{L^1(Q)}\int_0^s \sum_{i=1}^n |P_i^1(t-\tau)-P_i^2(t-\tau)|d\tau \\
&+ \lambda^0\|f_i\|_{L^1(Q)}\int_0^s \sum_{k\neq i,k=1}^n |P_k^1(t-\tau)-P_k^2(t-\tau)|d\tau \\
&\le M_1\sum_{i=1}^n\int_0^t |P_i^1(s)-p_i^2(s)|ds + M_2\int_0^t \|u_i^1(\cdot,s)-u_i^2(\cdot,s)\|ds + M_3\sum_{k\neq i,k=1}^n\int_0^t |P_k^1(s)-P_k^2(s)|ds
\end{aligned}
$$

and consequently

$$\begin{aligned}\sum_{i=1}^{n}\|p_i^1(\cdot,t)-p_i^2(\cdot,t)\|_{L^1(0,A)} &\le n(M_1+M_3)\int_0^t\sum_{i=1}^{n}\|p_i^1(\cdot,s)-p_i^2(\cdot,s)\|_{L^1(0,A)}ds\\ &\quad+M_2\int_0^t\sum_{i=1}^{n}\|u_i^1(\cdot,s)-u_i^2(\cdot,s)\|_{L^1(0,A)}ds.\end{aligned}\tag{2.9}$$

By (2.9) and Gronwall's lemma we get

$$\sum_{i=1}^{n}\|p_i^1-p_i^2\|_{L^1(0,A)}\le K_1(T)T\big(\sum_{i=1}^{n}\|u_i^1-u_i^2\|_{L^1(0,A)}\big),$$

where $M_1=M_{2T}(1+T)+L_{\mu_i}TM_T+2L_{\mu_i}\|f_i\|_{L^1(Q)}, M_2=TM_T+Ap^0+2\|f_i\|_{L^1(Q)}, M_3=\lambda^0(M_{2T}+M_T+Ap^0+2\|f_i\|_{L^1(Q)})$.

In addition, (2.8) enable us to obtain the other estimate of the standard norm in L^∞ space if T is sufficiently small, thus, the proof is complete.

Theorem 2.2. *Under the hypothesis A_1-A_4, the system (2.1) has a unique nonnegative solution.*

Proof. Let $\mathscr{T}:X\to L^\infty(0,T;L^1(0,A))$, defined by

$$(\mathscr{T}q)(a,t)=p(a,t,Q),Q(t)=\int_0^A q(a,t)da,$$

absolutely, $\mathscr{T}q\in X$, for any $\lambda>M_4$, we can define the following equivalent norm on $L^\infty(0,T;L^1(0,A))$:

$$\|q\|=Ess\sup_{t\in(0,T)}e^{-\lambda t}\Big\{\sum_{i=1}^{n}\|q_i(a,t)\|_{L^1(0,A)}+\sum_{i=1}^{n}\|u_i(a,t)\|_{L^1(0,A)}\Big\},$$

we shall prove that $\mathscr{T}$ has a unique fixed point, $\forall q^1,q^2\in X$, the process of inequality is similar to theorem 2.1, then

$$\begin{aligned}&\|\mathscr{T}q^1-\mathscr{T}q^2\|=Ess\sup_{t\in(0,T)}e^{-\lambda t}\Big\{\sum_{i=1}^{n}\|(\mathscr{T}q_i^1)(a,t)-(\mathscr{T}q_i^2)(a,t)\|_{L^1(0,A)}\Big\}\\ &\le M_4\ Ess\sup_{t\in(0,T)}e^{-\lambda t}\int_0^t\Big\{\big(\sum_{i=1}^{n}\|q_i^1(a,s)-q_i^2(a,s)\|_{L^1(0,A)}+\sum_{i=1}^{n}\|u_i^1(a,s)-u_i^2(a,s)\|_{L^1(0,A)}\big)\Big\}ds\\ &\le M_4\ Ess\sup_{t\in(0,T)}e^{-\lambda t}\int_0^t e^{\lambda s}\Big\{e^{-\lambda s}\Big[\sum_{i=1}^{n}\big(\|q_i^1(a,s)-q_i^2(a,s)\|+\|u_i^1(a,s)-u_i^2(a,s)\|\big)_{L^1(0,A)}\big)\Big]\Big\}ds\\ &\le M_4\|q^1-q^2\|Ess\sup_{t\in(0,T)}\big\{e^{-\lambda t}\int_0^t e^{\lambda s}ds\big\}\le\frac{1}{\lambda}M_4\|q^1-q^2\|,\end{aligned}$$

where $M_4=\max\{n(M_1+M_3),M_2\}$. It means that $\mathscr{T}$ is a contraction on $(X,\|\cdot\|)$ and consequently has a unique fixed point, that is system (2.1) has a unique solution. The proof is complete.

3. Optimality conditions

Theorem 3.1. *If $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ is an optimal control and $p^* = (p_1^*, p_2^*, \ldots, p_n^*)$ is the corresponding optimal state, then*

$$u_i^*(a,t) = \mathscr{L}_i\Big(\frac{(K_i - q_i)p_i^*}{B_i}\Big) \tag{3.1}$$

where

$$\mathscr{L}_i(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le N_i \quad i = 1,2,\ldots,n. \\ N_i & x > N_i \end{cases}$$

and $q = (q_1, q_2, \ldots, q_n)$ is the solution of following adjoint system corresponding to $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$.

$$\begin{cases} \frac{\partial q_i}{\partial t} + \frac{\partial q_i}{\partial a} = [\mu_i(a,t,\sum\limits_{j=1}^n P_j^*(t)) + u_i^*]q_i - q_i(0,t)\beta_i(a,t,\sum\limits_{j=1}^n P_j^*(t)) \\ \quad + \sum\limits_{k\neq i,k=1}^n \lambda_{ik}P_k^*(t)q_i + \sum\limits_{k\neq i,k=1}^n \int_0^A (\lambda_{ki}p_k^*q_k)d\theta + K_iu_i^* \\ \quad + \sum\limits_{j=1}^n \int_0^A p_j(\theta,t)\Big[q_j(\theta,t)\dfrac{\partial \mu_j}{\partial x_i}(\theta,t,\sum\limits_{j=1}^n P_j^*(t)) - q_j(0,t)\dfrac{\partial \beta_j}{\partial x_i}(\theta,t,\sum\limits_{j=1}^n P_j^*(t))\Big]d\theta, \\ q_i(a,T) = 0, q_i(A,t) = 0. \end{cases} \tag{3.2}$$

Proof. Existence and uniqueness of the solution q to system (3.2) follows by theorem 2.2. Denote by $\mathscr{N}_{U_i}(u_i^*)$ the normal cone at U_i in u_i^*, $v = (v_1, v_2, \ldots, v_n)$, $\forall v \in \mathscr{N}_{U_{ad}}(u^*)$, as $\varepsilon > 0$ small enough, $u^* + \varepsilon v \in U_{ad}$, we get

$$J(u^* + \varepsilon v) \le J(u^*). \tag{3.3}$$

Substituting (2.2) into (3.3) gives that

$$\sum_{i=1}^n \int_0^T \int_0^A (K_iu_i^*z_i)(a,t) + [(K_ip_i^* - B_iu_i^*)v_i](a,t)dadt \le 0, \tag{3.4}$$

where

$$z_i = \lim_{\varepsilon\to 0^+} \frac{p_i^\varepsilon(a,t) - p_i^*(a,t)}{\varepsilon},$$

p_i^ε is the state corresponding to $u_i^*+\varepsilon v_i$, and $z=(z_1,z_2,\ldots,z_n)$ is the solution of

$$\begin{cases} \frac{\partial z_i}{\partial t}+\frac{\partial z_i}{\partial a}=-(\mu_i+u_i^*)z_i-\sum\limits_{j=1}^{n}P_j^*(t)\frac{\partial \mu_i}{\partial x_j}(a,t,\sum\limits_{j=1}^{n}P_j^*(t))z_i \\ \qquad -\sum\limits_{k\neq i,k=1}^{n}\lambda_{ik}[p_i^*Z_k(t)+P_k^*(t)z_i]-v_ip_i^*, \\ z_i(0,t)=\int_0^A\beta_i(a,t,\sum\limits_{j=1}^{n}P_j^*(t))z_i(a,t)da \\ \qquad +\int_0^A P_j^*(t)\sum\limits_{j=1}^{n}\frac{\partial \beta_i}{\partial x_j}(a,t,\sum\limits_{j=1}^{n}P_j^*(t))z_i(a,t)da, \\ z_i(a,0)=0, \\ P_i^*(t)=\int_0^A p_i^*(a,t)da, \\ Z_i(t)=\int_0^A z_i(a,t)da,\ i=1,2,\ldots,n, \end{cases} \tag{3.5}$$

where $\frac{\partial \mu_i}{\partial x_j}(a,t,\sum\limits_{j=1}^{n}P_j(t)),\frac{\partial \beta_i}{\partial x_j}(a,t,\sum\limits_{j=1}^{n}P_j(t))$ means the partial derivative of μ_i,β_i with respect to its third argument, multiplying the $(3.5)_i$ by q_i respectively, integrating on Q and since

$$\int_Q q_i(a,t)(\frac{\partial z_i}{\partial t}+\frac{\partial z_i}{\partial a})(a,t)dadt=-\int_Q z_i(a,t)\big[(\frac{\partial q_i}{\partial t}+\frac{\partial q_i}{\partial a})(a,t)$$
$$+\beta_i(a,t,\sum_{j=1}^{n}P_j(t))q_i(0,t)+\sum_{j=1}^{n}\int_0^A p_j^*(\theta,t)q_j(0,t)\frac{\partial \beta_i}{\partial x_j}(\theta,t,\sum_{j=1}^{n}P_j(t))d\theta\big]dadt,$$

now using (3.2) we obtain that

$$\sum_{i=1}^{n}\int_Q(K_iu_i^*z_i)(a,t)dadt=-\sum_{i=1}^{n}\int_Q(q_ip_i^*v_i)(a,t)dadt, \tag{3.6}$$

from (3.4) and (3.6) it follows that

$$\sum_{i=1}^{n}\int_Q v_i[(K_i-q_i)p_i^*-B_iu_i^*](a,t)dadt\le 0.$$

By using the concept of normal cone U_i at u_i^* [18], we get $(K_i-q_i)p_i^*-B_iu_i^*\in\mathscr{N}_{U_i}(u_i^*)$, the proof is complete by the characteristics properties of the normal vector [17].

4. Existence of optimal control

The characterization and uniqueness of the optimal control pair u^* is dependent on the use of Ekeland's principle [16]. To employ this principle, we embed our functional in the space $L^1(Q)$

by defining

$$\mathscr{J}(u) = \begin{cases} J(u) & (u) \in U_{ad} \\ -\infty & (u) \notin U_{ad} \end{cases}$$

Lemma 4.1. *$\mathscr{J}(u)$ is upper semi-continuous with respect to $L^1(Q)$ convergence.*

Proof. Let $u^n = (u_1^n, u_2^n, \dots, u_n^n) \to (u_1, u_2, \dots, u_n) = u$, as $n \to \infty$, by Riesz theorem there is a subsequence, denoted still by (u^n), such that

$$(u_i^n)^2 \to u_i^2,\ n \to \infty.$$

Thus, Lebesgue's dominated convergence theorem yields that

$$\lim_{n\to\infty} \int_Q (u_i^n)^2 dadt = \int_Q u_i^2 dadt.$$

On the other hand, it follows from (2.7) that

$$\begin{aligned} &|\int_Q K_i u_i^n(a,t) p_i^n(a,t) dadt - \int_Q K_i u_i(a,t) p_i(a,t) dadt| \\ &\le \int_Q K_i p_i^n(a,t) \|u_i^n - u_i\| dadt + \int_Q K_i u_i(a,t) \|p_i^n - p_i\| dadt \\ &\le (M + N_1 C_1 T) \|K_i\|_{L^1(Q)} \|u_i^n - u_i\|_{L^1(Q)}, \end{aligned}$$

using Fatou's lemma we conclude that on a subsequence, also denoted by (u^n) we have

$$\lim_{n\to\infty} \sup \int_Q K_i(a) u_i^n(a,t) p_i^n(a,t) dadt \le \int_Q K_i(a) u_i(a,t) p_i(a,t) dadt.$$

We may infer that $\mathscr{J}(u) \ge \limsup_{n\to\infty}(u^n)$.

Lemma 4.2. *For $u = (u_1, u_2, \dots, u_n) \in U$, the adjoint system (3.2) has a weak solution $q = (q_1, q_2, \dots, q_n)$ in $L^\infty(Q) \times L^\infty(Q)$ such that*

$$\sum_{i=1}^n \|q_i^1 - q_i^2\|_\infty \le CT(\sum_{i=1}^n (\|u_i^1 - u_i^2\|_\infty) \tag{4.1}$$

where adjoint solutions $(q_1^i, q_2^i, \dots, q_n^i)$ correspond to control pairs $(u_1^i, u_2^i, \dots, u_n^i), i = 1, 2$.

Theorem 4.1. *If $T \sum_{i=1}^n B_i^{-1}$ is sufficiently small, there exists one and only one optimal control pair $u^* = (u_1^*, u_2^*, \dots, u_n^*)$ in U_{ad} such that*

$$J(u^*) = \max_{u \in U_{ad}} J(u)$$

Proof. According to Ekeland's variational principle [16], for $\forall\varepsilon > 0$, there exists $u^\varepsilon = (u_1^\varepsilon, u_2^\varepsilon, \cdots, u_n^\varepsilon) \in [L^1(Q)]^n$ such that

(1) $\mathscr{J}(u^\varepsilon) > \sup\limits_{u\in U_{ad}} \mathscr{J}(u) - \varepsilon$,

(2) $\mathscr{J}(u^\varepsilon) > \mathscr{J}(u) - \sqrt{\varepsilon}\sum\limits_{i=1}^{n} \|u_i^\varepsilon - u_i\|_{L^1(Q)} \triangleq \mathscr{J}_\varepsilon(u)$.

Since u^ε is a maximum point for $\mathscr{J}_\varepsilon(u), \forall v = (v_1, v_2, \ldots, v_n) \in \mathscr{N}_U(u^\varepsilon)$, as δ small enough, $u^\varepsilon + \delta v = (u_1^\varepsilon + \delta v_1, u_2^\varepsilon + \delta v_2, \ldots, u_n^\varepsilon + \delta v_n) \in U_{ad}$, we have

$$J(u^\varepsilon) = J_\varepsilon(u^\varepsilon) \geq J_\varepsilon(u^\varepsilon + \delta v). \tag{4.2}$$

Substituting (2.2) into (4.3) and passing to the limit of both sides, as $\delta \to 0^+$gives that

$$\sum_{i=1}^{n}\int_0^T\int_0^A (K_i u_i^\varepsilon z_i)(a,t) + [(K_i p_i^\varepsilon - B_i u_i^\varepsilon)v_i](a,t)dadt + \sqrt{\varepsilon}\sum_{i=1}^{n}\|v_i\|_{L^1(Q)} \leq 0, \tag{4.3}$$

where $z_i = \lim\limits_{\varepsilon\to 0^+} \frac{p_i^\varepsilon - p_i(a,t)}{\varepsilon}$, p_i^ε is the state corresponding to $u_i + \varepsilon v_i$, and $z = (z_1, z_2, \ldots, z_n)$ is the solution of

$$\begin{cases} \frac{\partial z_i}{\partial t} + \frac{\partial z_i}{\partial a} = -(\mu_i + u_i^\varepsilon)z_i - \sum\limits_{j=1}^{n} P_j^\varepsilon(t)\frac{\partial \mu_i}{\partial x_j}(a,t,\sum\limits_{j=1}^{n} P_j^\varepsilon(t))z_i \\ \qquad\qquad - \sum\limits_{k\neq i,k=1}^{n} \lambda_{ik}[p_i^\varepsilon Z_k(t) + P_k^\varepsilon(t)z_i] - v_i p_i^\varepsilon, \\ z_i(0,t) = \displaystyle\int_0^A \beta_i(a,t,\sum_{j=1}^{n} P_j^\varepsilon(t))z_i(a,t)da + \int_0^A P_j^\varepsilon(t)\sum_{j=1}^{n}\frac{\partial \beta_i}{\partial x_j}(a,t,\sum_{j=1}^{n} P_j^\varepsilon(t))z_i(a,t)da, \\ z_i(a,0) = 0, \\ P_i^\varepsilon(t) = \displaystyle\int_0^A p_i^\varepsilon(a,t)da, \\ Z_i(t) = \displaystyle\int_0^A z_i(a,t)da, \ i = 1,2,\ldots,n. \end{cases}$$

Methods similar to theorem 3.1 can prove that

$$\sum_{i=1}^{n}\int_Q v_i[(K_i - q_i)p_i^\varepsilon - B_i u_i^\varepsilon - \sqrt{\varepsilon}v_i](a,t)dadt \leq 0, \tag{4.4}$$

a similar argument as that in theorem, there exists θ_i^ε (see [3]) gives

$$u^\varepsilon = \mathscr{L}(u^\varepsilon) = \Big(\mathscr{L}_1\big(\frac{(K_1 - q_1^\varepsilon)p_1^\varepsilon - \sqrt{\varepsilon}\theta_1^\varepsilon}{B_1}\big), \ldots, \mathscr{L}_n\big(\frac{(K_n - q_n^\varepsilon)p_n^\varepsilon - \sqrt{\varepsilon}\theta_n^\varepsilon}{B_n}\big)\Big)$$

where $\theta_i^\varepsilon \in L^\infty(Q)$, and with $|\theta_i^\varepsilon| \leq 1, \ i = 1,2,\cdots,n$.

Define $\mathscr{F}: U \to U, \mathscr{F}(u) = \big(\mathscr{L}_1(\frac{(K_1-q_1)p_1}{B_1}), \mathscr{L}_2(\frac{(K_2-q_2)p_2}{B_2}, \ldots, \mathscr{L}_n(\frac{(K_n-q_n)p_n}{B_n})\big)$, using fixed point, we prove uniqueness fist.

$$\begin{aligned}\|\mathscr{F}(u^1)-\mathscr{F}(u^2)\| &\equiv \sum_{i=1}^{n}\|\mathscr{L}_i\big(\frac{(K_i-q_i^1)p_i^1}{B_i}\big)-\mathscr{L}_i\big(\frac{(K_i-q_i^2)p_i^2}{B_i}\big)\|_\infty \\ &\leq (K_i+q_i^1)\sum_{i=1}^{n}B_i^{-1}\sum_{i=1}^{n}(\|p_i^1-p_i^2\|_\infty+p_i^2\|q_i^1-q_i^2\|_\infty) \\ &\leq C_2T\sum_{i=1}^{n}B_i^{-1}\sum_{i=1}^{n}\|u_i^1-u_i^2\|_\infty\end{aligned} \tag{4.5}$$

If T small enough, then the map $\mathscr{F}$ has a unique fixed point u^*, where $C_2 = (K+M)K_2(T)+CM$.

To prove this fixed point is an optimal control pair, we use the approximate maximizers u^ε from Ekeland's principle, for

$$\begin{aligned}\|\mathscr{F}(u^\varepsilon)-u^\varepsilon\|_\infty &= \sum_{i=1}^{n}\|\mathscr{L}\big(\frac{(K_i-q_i^\varepsilon)p_i^\varepsilon}{B_i}\big)-\mathscr{L}\big(\frac{(K_i-q_i^\varepsilon)p_i^\varepsilon-\sqrt{\varepsilon}\theta_i^\varepsilon}{B_i}\big)\|_\infty \\ &\leq \sum_{i=1}^{n}\|\frac{\sqrt{\varepsilon}\theta_i^\varepsilon}{B_i}\|_\infty \leq \sqrt{\varepsilon}\sum_{i=1}^{n}B_i^{-1}\end{aligned} \tag{4.6}$$

Next, using (4.5) and (4.6) to show that $u^\varepsilon \to u^*$ in $L^\infty(Q)$,

$$\begin{aligned}\|u^\varepsilon-u^*\|_\infty &\equiv \|u^\varepsilon-\mathscr{F}(u^\varepsilon)+\mathscr{F}(u^\varepsilon)-u^*\|_\infty \\ &\leq \|u^\varepsilon-\mathscr{F}(u^\varepsilon)\|_\infty+\|\mathscr{F}(u^\varepsilon)-u^*\|_\infty \\ &\leq \sqrt{\varepsilon}\sum_{i=1}^{n}B_i^{-1}+C_2T\sum_{i=1}^{n}B_i^{-1}\sum_{i=1}^{n}\|u_i^\varepsilon-u_i^*\|_\infty\end{aligned}$$

if $T\sum\limits_{i=1}^{n}B_i^{-1}$ is small enough, the following result holds:

$$\|u^\varepsilon-u^*\|_\infty \leq \frac{\sqrt{\varepsilon}\sum\limits_{i=1}^{n}B_i^{-1}}{1-C_2T\sum\limits_{i=1}^{n}B_i^{-1}}, \tag{4.7}$$

passing to the limit of both sides with $\varepsilon \to 0$, (4.7) imply that $u^\varepsilon \to u^*$ in $L^\infty(Q)$. Finally, using property (1) of Ekeland's principle, the inequality $\mathscr{J}(u^*) > \sup\limits_{u\in U_{ad}}\mathscr{J}(u)-\varepsilon$ implies $\mathscr{J}(u^*) \geq \sup\limits_{u\in U_{ad}}\mathscr{J}(u)$, but actually $\mathscr{J}(u^*) \leq \sup\limits_{u\in U_{ad}}\mathscr{J}(u)$, thus, $\mathscr{J}(u^*) = \sup\limits_{u\in U_{ad}}\mathscr{J}(u)$, the proof is complete.

Conflict of Interests

The authors declare that there is no conflict of interests.

Acknowledgements

The research has been supported by the Natural Science Foundation of China (11561041) and the Nature Science Foundation of Gansu Province of China (1506RJZA071).

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