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# OPTIMAL CONTROL OF A COMPUTER VIRUS MODEL WITH NETWORK ATTACKS

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**Abstract.** This paper addresses the issue of how to promote the spread of computer virus, which is regarded as an intelligence weapon, by means of the optimal control method. First, a state-delay computer virus spread model is established. Second, an objective functional for optimizing the control cost and effect is described. Third, an algorithm for computing the gradient of the optimal control problem is developed. Finally, several numerical examples is shown.

**Keywords:** state delay; computer virus; optimal control; network attacks.

**2010 AMS Subject Classification:** 49N90.

## 1. Introduction

Computer virus is a malicious mobile code that are intended to spread among computers and perform detrimental operations including virus, worm, logic bomb and so on[1]. From their first appearance in 1986, computer virus has become a great threat to our work and daily life. Especially, in recent years, this threat has become more and more serious because of the development

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of hardware and software technology and the popularity of computer networks. So computer viruses are an important risk to computational systems endangering either corporations of all sizes or personal computers used for domestic applications, then we can use computer viruses to cripple a computational system, which will be serious damaged. Computer virus is considered as one of the most important weapon in the internet, and their emergence and spread may have great effect on the computer world[2]. For example, the U.S. military has been using the primary computer virus techniques against the Iraqi command heart when the outbreak of the Gulf War in the 1991. This is the first time in the world of battle with a computer virus and uncover the prelude of computer virus in actual combat. Different codes have different ways to spread in the internet. Virus mainly attack the file system and worm uses system vulnerability to search and attack the computer. And for trojan horses, they camouflage themselves and thus induce the users to download them[3].

Since Kephart and White [4,5] took the first step towards computer virus spread model, much effort has been done in the area of developing a mathematical model for the computer virus propagation[1,2,3,6-10]. But we noticed that the above-mentioned work ignores the fact that computer viruses can be regarded as not only a threat to our information system, but also a intelligence weapon to attack computer systems of the enemies. Some scientists asserted: in the future, the greatest threat of mankind is not a nuclear war any more, but computer virus which is more direct and more dangerous. So the computer virus attack has become a new and important topic in modern information warfare.

The classic epidemic SIR model have been borrowed to depict the spread of a computer virus[7,11,12]. In [13], Zou et al. investigated how the spread of red worms is affected by the worm characteristics based on the SIR model. But this paper present the issue of how to let the damage of computer virus for computational systems as serious as possible. First, a delayed computer virus spread model with control is established by introducing an appropriate control variable into a delayed model presented in [3]. Second, an optimal control problem is described by suggesting an objective functional for optimizing the control cost and control effect. Third, we develop a numerical algorithm for computing the gradient of the optimal control problem by the lights of optimal state-delay control problem and Pontryagin-type minimum (maximum) principle. Finally, numerical examples demonstrate that the spread of computer virus can be maximized effectively by use of an optimal control strategy, while the budget of the spread can be minimized.

## 2. Model formulation

Our work is based on the delayed computer virus model on internet.

$$\begin{cases} \frac{dS}{dt} = (1-p)b - \mu S - \beta S(t-\tau_1)I(t-\tau_1) + \nu R(t-\tau_2), \\ \frac{dI}{dt} = \beta S(t-\tau_1)I(t-\tau_1) - (\mu+\gamma+\alpha)I \\ \frac{dR}{dt} = pb + \gamma I - \mu R - \nu R(t-\tau_2). \end{cases} \tag{2.1}$$

It is assumed that each node is denoted as one computer and the state of it can be the healthy computers who are susceptible ($S(t)$) to infection, the already infected computers ($I(t)$) who can transmit the virus to the healthy ones or the recovered ($R(t)$) who cannot get the virus or transmit it[3,14,15]. Here $b$ is the number of newly coming nodes, $p$ is the immune rate of a newly coming node, $\beta$ is the infection rate of an infected node, $\mu$ is the natural leaving rate of a node, $\nu$ is the loss rate of immunity of a recovered node, $\gamma$ is the recovery rate of an infected node, $\alpha$ is the removing rate of an infected node due to the action of the virus, $\tau_1$ is the latent period of an infected node, and $\tau_2$ is the immune period of a recovered node [3].

Next, we will introduce network attacks into model (2.1). Let $\tau_3$ be the latent period of an offensive node, $\eta$ be the budget for attacking susceptible computers. Then the corresponding model with the network attack of computer virus is employed as follows:

$$\begin{cases} \frac{dS}{dt} = (1-p)b - \mu S - \beta S(t-\tau_1)I(t-\tau_1) + \nu R(t-\tau_2) - \eta S(t-\tau_3), \\ \frac{dI}{dt} = \beta S(t-\tau_1)I(t-\tau_1) - (\mu+\gamma+\alpha)I + \eta S(t-\tau_3) \\ \frac{dR}{dt} = pb + \gamma I - \mu R - \nu R(t-\tau_2), \end{cases} \tag{2.2}$$

The initial conditions for model (2.2) are

$$(\phi_1(\theta), \phi_2(\theta), \phi_3(\theta)) \in C_+ = C([-\tau, 0], R^3_+),\ \phi_i(0) > 0,\ i = 1,2,3, \tag{2.3}$$

where $\tau = \max\{\tau_1, \tau_2, \tau_3\}, R^3_+ = \{(S,I,R) \in R^3, S \geq 0, I \geq 0, R \geq 0\}$. Therefore, all the standard results on existence, uniqueness and continuous dependence on initial condition of solutions are evidently satisfied.

## 3. Optimal control problems with state-delay controls

To formulate an optimal problem on the network attack, we introduce some preliminary definitions[16]. Let $\tau = (\tau_1, \tau_3)$ be a vector of delays and $\eta$ be the budget for attacking susceptible computers. We impose the following bound constraints on the delays and budget:

$$\tilde{\tau}_i \leq \tau_i \leq \bar{\tau}_i, \ i = 1,3, \tag{3.1}$$

and

$$\tilde{\eta} \leq \eta \leq \bar{\eta}, \tag{3.2}$$

where $\tilde{\tau}_i$, $\bar{\tau}_i$, $\tilde{\eta}_j$ and $\bar{\eta}_j$ are given constants such that $0 \leq \tilde{\tau}_i < \bar{\tau}_i$ and $\tilde{\eta}_j < \bar{\eta}_j$.

Any vector $\boldsymbol{\tau} = (\tau_1, \tau_3)^T$ satisfying (3.1) is called an admissible state-delay vector. Let $\Gamma$ denote the set of all such admissible state-delay vectors. The vector $\eta$ satisfying (3.2) is called an admissible budget vector. Let $Z$ denote the set of all such admissible parameter vectors. Any combined pair $(\boldsymbol{\tau}, \eta) \in \Gamma \times Z$ is called an admissible control pair for system (2.2). Let

$$\boldsymbol{X}(t) = \begin{bmatrix} S(t) & I(t) & R(t) \end{bmatrix}^T.$$

For a given terminal time $t_f$ and given time points $t_i$, $i = 1,2,...,p$ satisfying

$$0 < t_1 \leq t_2 \ldots \leq t_n \leq t_p, < t_f \tag{3.3}$$

our aim is to find an admissible control pair $(\boldsymbol{\tau}, \eta) \in \Gamma \times Z$ that minimizes the following cost function

$$\begin{aligned} J = \min\{&\Phi(S(t_i \mid \boldsymbol{\tau}, \eta), I(t_i \mid \boldsymbol{\tau}, \eta), R(t_i \mid \boldsymbol{\tau}, \eta)) \\ &+ \int_0^{t_p} L(\boldsymbol{X}(t \mid \boldsymbol{\tau}, \eta), \boldsymbol{X}(t - \tau_1 \mid \boldsymbol{\tau}, \eta), \boldsymbol{X}(t - \tau_2 \mid \boldsymbol{\tau}, \eta), \eta) dt \Big\}, \end{aligned} \tag{3.4}$$

where

$$\Phi = -\sum_{i=1}^{p} I(t_i \mid \tau_1, \tau_3, \eta)$$

$$L=\int_0^{t_p}\frac{\eta^2}{2}dt$$

In (3.4), our objective aim is to find a control to maximize the accumulated number of infected nodes at the characteristic times and meaning while to minimize the total budget for network attack during the time period $[0,t_p]$. In order to achieve those goals, we will focus on the following algorithms.

## 4. Gradient computation

For convenient calculation, define the right side of (2.2) for $t\in[0,t_p]$ as:

$$\boldsymbol{f}(\boldsymbol{X}(t|\boldsymbol{\tau},\eta),\boldsymbol{X}(t-\tau_1|\boldsymbol{\tau},\eta),\boldsymbol{X}(t-\tau_3|\boldsymbol{\tau},\eta))=\begin{bmatrix}\frac{dS}{dt} & \frac{dI}{dt} & \frac{dR}{dt}\end{bmatrix}^T,$$

and initial conditions for t $\leq 0$ as

$$\boldsymbol{X}(t)=\begin{bmatrix}S(t) & I(t) & R(t)\end{bmatrix}^T=\phi(t,\eta).$$

Furthermore, in order to find an optimal solution, define

$$\psi(t\mid\boldsymbol{\tau},\eta)=\begin{cases}\frac{\partial\phi(t,\eta)}{\partial t}, & \text{if } t<0,\\ \boldsymbol{f}(\boldsymbol{X}(t|\boldsymbol{\tau},\eta),\boldsymbol{X}(t-\tau_1|\boldsymbol{\tau},\eta),\boldsymbol{X}(t-\tau_3|\boldsymbol{\tau},\eta)), & \text{if } t\in[0,t_f],\end{cases}$$

and for $i=1,3$

$$\frac{\partial\bar{\boldsymbol{f}}(t\mid\boldsymbol{\tau},\eta)}{\partial\boldsymbol{X}}=\frac{\partial\boldsymbol{f}(\boldsymbol{X}(t\mid\boldsymbol{\tau},\eta),\boldsymbol{X}(t-\tau_1\mid\boldsymbol{\tau},\boldsymbol{E}),\boldsymbol{X}(t-\tau_3\mid\boldsymbol{\tau},\eta),\eta)}{\partial\boldsymbol{X}},$$

$$\frac{\partial\bar{\boldsymbol{f}}(t\mid\boldsymbol{\tau},\eta)}{\partial\tilde{\boldsymbol{x}}^i}=\frac{\partial\boldsymbol{f}(\boldsymbol{X}(t\mid\boldsymbol{\tau},\eta),\boldsymbol{X}(t-\tau_1\mid\boldsymbol{\tau},\eta),\boldsymbol{X}(t-\tau_3\mid\boldsymbol{\tau},\eta),\eta)}{\partial\boldsymbol{X}(t-\tau_i)},$$

$$\frac{\partial\bar{L}(t\mid\boldsymbol{\tau},\eta)}{\partial\tilde{\boldsymbol{x}}^i}=\frac{\partial L(\boldsymbol{X}(t\mid\boldsymbol{\tau},\eta),\boldsymbol{X}(t-\tau_1\mid\boldsymbol{\tau},\eta),\boldsymbol{X}(t-\tau_3\mid\boldsymbol{\tau},\eta),\eta)}{\partial\boldsymbol{X}(t-\tau_i)}.$$

According to literature [16] and model (2.2), we deduce the auxiliary impulsive system corresponding to this problem as follows:

$$\dot{\boldsymbol{\lambda}}(t)=-\left[\frac{\partial\bar{\boldsymbol{f}}(t\mid\boldsymbol{\tau},\eta)}{\partial\boldsymbol{X}(t)}\right]^T\boldsymbol{\lambda}(t)-\sum_{l=1,3}\left\{\left[\frac{\partial\bar{\boldsymbol{f}}(t+\tau_l\mid\boldsymbol{\tau},\eta)}{\partial\tilde{\boldsymbol{x}}^l}\right]^T\boldsymbol{\lambda}(t+\tau_l)+\left[\frac{\partial\bar{L}(t+\tau_l\mid\boldsymbol{\tau},\eta)}{\partial\tilde{\boldsymbol{x}}^l}\right]^T\right\}.$$

Then for the system(2.2), given the admissible control pair $(\boldsymbol{\tau},\eta)\in\Gamma\times Z$ and the corresponding solution of state system (2.2), there exist adjoint variables $\lambda_i$ (i=1, 2, 3) satisfying

$$\begin{cases} \dot{\lambda}_1(t)=\mu\lambda_1(t)+\beta I\lambda_1(t+\tau_1)-\beta I\lambda_2(t+\tau_1), \\ \dot{\lambda}_2(t)=(\mu+\gamma+\alpha)\lambda_2(t)-\gamma\lambda_3(t)+\beta S\lambda_1(t+\tau_1)-\beta S\lambda_2(t-\tau_1)+\eta\lambda_2(t+\tau_3), \\ \dot{\lambda}_2(t)=\mu\lambda_3(t). \end{cases} \tag{4.1}$$

By jump conditions,

$$\boldsymbol{\lambda}(t_k^-)=\boldsymbol{\lambda}(t_k^+)+\left[\frac{\partial\Phi(\boldsymbol{X}(t_1\mid\boldsymbol{\tau},\eta),\ldots,\boldsymbol{X}(t_p\mid\boldsymbol{\tau},\eta),\eta)}{\partial\boldsymbol{X}(t_k)}\right]^T, k=1,2,\ldots,p \tag{4.2}$$

and

$$\frac{\partial\Phi(\boldsymbol{X}(t_1\mid\boldsymbol{\tau},\eta),\ldots,\boldsymbol{X}(t_p\mid\boldsymbol{\tau},\eta),\eta)}{\partial\boldsymbol{X}(t_k)}=\begin{pmatrix}\frac{\partial\boldsymbol{J}}{\partial S} & \frac{\partial\boldsymbol{J}}{\partial I} & \frac{\partial\boldsymbol{J}}{\partial R}\end{pmatrix}=\begin{pmatrix}0 & -1 & 0\end{pmatrix}, \tag{4.3}$$

we get jump conditions at points $t_k$ $(k=1,2,3,...,p)$ :

$$\begin{cases} \lambda_1(t_k^-)=\lambda_1(t_k^+) \\ \lambda_2(t_k^-)=\lambda_2(t_k^+)-1 \\ \lambda_3(t_k^-)=\lambda_3(t_k^+), \end{cases}$$

with boundary conditions $\lambda_i(t_f)=0$ $(i=1,2,3)$. So, from the above derivation, we can get following conclusion.

**Theorem 4.1** *For the solution of system (2.2) corresponding to the admissible control pair $(\boldsymbol{\tau},\eta)\in\Gamma\times Z$, the auxiliary impulsive system (4.1)-(4.3) is established.*

Next, we devote to gradient with respect to state-delays $\tau_1$and $\tau_3$:

$$\frac{\partial J(\boldsymbol{\tau},\eta)}{\partial\tau_i}=\frac{\partial\Phi}{\partial\tau_i}+\int_0^{t_p}\left\{\frac{\partial\bar{L}}{\partial\tau_i}-\left(\frac{\partial\bar{L}(t\mid\boldsymbol{\tau},\eta)}{\partial\tilde{\boldsymbol{x}}^i}+\boldsymbol{\lambda}^T(t\mid\boldsymbol{\tau},\eta)\frac{\partial\bar{\boldsymbol{f}}(t\mid\boldsymbol{\tau},\eta)}{\partial\tilde{\boldsymbol{x}}^i}\right)\psi(t-\tau_i\mid\boldsymbol{\tau},\eta)\right\}dt,$$

here $i=1,3$. After simple computer, one gets

$$\frac{\partial\Phi}{\partial\tau_i}=\frac{\partial I}{\partial\tau_i}=0,\ \frac{\partial\bar{L}}{\partial\tau_i}=\frac{\partial\frac{\eta^2}{2}}{\partial\tau_i}=0,\ \frac{\partial\bar{L}(t\mid\boldsymbol{\tau},\zeta)}{\partial\tilde{\boldsymbol{x}}^i}=\frac{\partial\frac{\eta^2}{2}}{\partial\tilde{\boldsymbol{x}}^i}=0.$$

We also get

$$\frac{\partial \bar{\boldsymbol{f}}(t \mid \boldsymbol{\tau},\eta)}{\partial \tilde{\boldsymbol{x}}^i} = \begin{bmatrix} \frac{\partial \dot{S}(t)}{\partial S(t-\tau_i)} & \frac{\partial \dot{S}(t)}{\partial I(t-\tau_i)} & \frac{\partial \dot{S}(t)}{\partial R(t-\tau_i)} \\ \frac{\partial \dot{I}(t)}{\partial S(t-\tau_i)} & \frac{\partial \dot{I}(t)}{\partial I(t-\tau_i)} & \frac{\partial \dot{I}(t)}{\partial R(t-\tau_i)} \\ \frac{\partial \dot{R}(t)}{\partial S(t-\tau_i)} & \frac{\partial \dot{R}(t)}{\partial I(t-\tau_i)} & \frac{\partial \dot{R}(t)}{\partial R(t-\tau_i)} \end{bmatrix}, \psi(t-\tau_i \mid \boldsymbol{\tau},\eta) = \begin{bmatrix} \dot{S}(t-\tau_i) \\ \dot{I}(t-\tau_i) \\ \dot{R}(t-\tau_i) \end{bmatrix},$$

and

$$\frac{\partial \bar{\boldsymbol{f}}(t \mid \boldsymbol{\tau},\boldsymbol{q})}{\partial \boldsymbol{X}(t-\tau_1)} = \begin{bmatrix} -\beta I(t-\tau_1) & -\beta I(t-\tau_1) & 0 \\ \beta I(t-\tau_1) & \beta I(t-\tau_1) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \frac{\partial \bar{\boldsymbol{f}}(t \mid \boldsymbol{\tau},\boldsymbol{q})}{\partial \boldsymbol{X}(t-\tau_3)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In conclusion, we obtain gradient with respect to state-delays $\tau_1$and $\tau_3$:

$$\begin{aligned} \frac{\partial J(\boldsymbol{\tau},\eta)}{\partial \tau_1} &= -\int_0^{t_p} \begin{bmatrix} \lambda_1(t) & \lambda_2(t) & \lambda_3(t) \end{bmatrix} \begin{bmatrix} -\beta I(t-\tau_1) & -\beta I(t-\tau_1) & 0 \\ \beta I(t-\tau_1) & \beta I(t-\tau_1) & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{S}(t-\tau_1) \\ \dot{I}(t-\tau_1) \\ \dot{R}(t-\tau_1) \end{bmatrix} dt \\ &= \int_{\tau_1}^{t_p} [-\lambda_1(t)+\lambda_2(t)][\beta I(t-\tau_1)\dot{S}(t-\tau_1)+\beta S(t-\tau_1)\dot{I}(t-\tau_1)]dt, \end{aligned} \tag{4.4}$$

$$\begin{aligned} \frac{\partial J(\boldsymbol{\tau},\eta)}{\partial \tau_3} &= -\int_0^{t_p} \begin{bmatrix} \lambda_1(t) & \lambda_2(t) & \lambda_3(t) \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{S}(t-\tau_3) \\ \dot{I}(t-\tau_3) \\ \dot{R}(t-\tau_3) \end{bmatrix} dt \\ &= \int_{\tau_3}^{t_p} \lambda_2(t)\dot{I}(t-\tau_3)dt, \end{aligned} \tag{4.5}$$

where $\dot{S}(t),\dot{I}(t),\dot{R}(t)$ are given by the right hand side of (2.2).

**Theorem 4.2** *For each admissible control pair $(\boldsymbol{\tau},\eta) \in \Gamma \times Z$, we obtain gradients (4.4) and (4.5) of (2.2) with respect to delays $\tau_1$ and $\tau_3$ .*

Analogously, for $\eta$ according to

$$\begin{aligned}\frac{\partial J(\boldsymbol{\tau},\boldsymbol{\eta})}{\partial\eta} =&\frac{\partial\Phi(\boldsymbol{x}(t_1\mid\boldsymbol{\tau},\boldsymbol{\eta}),\ldots,\boldsymbol{x}(t_p\mid\boldsymbol{\tau},\boldsymbol{\eta}),\boldsymbol{\tau},\boldsymbol{\eta})}{\partial\eta}+\boldsymbol{\lambda}^T(0^+\mid\boldsymbol{\tau},\boldsymbol{\eta})\frac{\partial\phi(0,\boldsymbol{\eta})}{\partial\eta}\\ &+\int_0^{t_p}\left\{\frac{\partial\bar{L}(t\mid\boldsymbol{\tau},\boldsymbol{\eta})}{\partial\eta}+\boldsymbol{\lambda}^T(t\mid\boldsymbol{\tau},\boldsymbol{\eta})\frac{\partial\bar{\boldsymbol{f}}(t\mid\boldsymbol{\tau},\boldsymbol{\eta})}{\partial\eta}\right\}dt\\ &+\sum_{l=1}^{m}\int_{-\tau_l}^{0}\left\{\frac{\partial\bar{L}(t+\tau_l\mid\boldsymbol{\tau},\boldsymbol{\eta})}{\partial\tilde{\boldsymbol{x}}^l}+\boldsymbol{\lambda}^T(t+\tau_l\mid\boldsymbol{\tau},\boldsymbol{\eta})\frac{\partial\bar{\boldsymbol{f}}(t+\tau_l\mid\boldsymbol{\tau},\boldsymbol{\eta})}{\partial\tilde{\boldsymbol{x}}^l}\right\}\frac{\partial\phi(t,\boldsymbol{\zeta})}{\partial\eta}dt,\end{aligned}\tag{4.6}$$

where

$$\frac{\partial\Phi(\boldsymbol{x}(t_1\mid\boldsymbol{\tau},\boldsymbol{\zeta}),\ldots,\boldsymbol{x}(t_p\mid\boldsymbol{\tau},\boldsymbol{\zeta}),\boldsymbol{\tau},\boldsymbol{\zeta})}{\partial\zeta_j}=\frac{\partial I}{\partial\eta}=0,$$

$$\frac{\partial\bar{L}(t\mid\boldsymbol{\tau},\boldsymbol{\zeta})}{\partial\zeta_j}=\frac{\partial\frac{\eta^2}{2}}{\partial\eta}=1.$$

Combined with $\varphi(t,q)$ which is a constant in $[-\tau,0]$, together with cost function $J$ which is a explicit function of $\eta$, it is obtain gradient with respect to the budget $\eta$:

$$\begin{aligned}\frac{\partial J(\boldsymbol{\tau},\boldsymbol{\eta})}{\partial\eta}&=\int_0^{t_p}\{\begin{bmatrix}\lambda_1(t) & \lambda_2(t) & \lambda_3(t)\end{bmatrix}\begin{bmatrix}\frac{\partial\dot{S}(t)}{\partial\eta}\\ \frac{\partial\dot{I}(t)}{\partial\eta}\\ \frac{\partial\dot{R}(t)}{\partial\eta}\end{bmatrix}+1\}dt\\ &=\int_0^{t_p}\{\begin{bmatrix}\lambda_1(t) & \lambda_2(t) & \lambda_3(t)\end{bmatrix}\begin{bmatrix}0\\ I(t-\tau_3)\\ 0\end{bmatrix}+1\}dt\\ &=-\int_0^{t_p}\{\lambda_2(t)I(t-\tau_3)+1\}dt\end{aligned}\tag{4.7}$$

**Theorem 4.3** *For each admissible control pair $(\boldsymbol{\tau},\boldsymbol{\eta})\in\Gamma\times Z$, we obtain gradients (4.7) with respect to budget $\eta$.*

## 5. Optimal attack strategies

As in [3], we shall take the parameter values given in Table 1. Furthermore, we shall assume that S(0) = 8, I(0) = 1, and R(0) = 1[17].

TABLE 1. Values of the parameters.

| Parameters | p | b | $\beta$ | $\mu$ | $\nu$ | $\gamma$ |
|---|---|---|---|---|---|---|
| Values | 0.9 | 1 | 0.2 | 0.32 | 0.7 | 0.35 |

### 5.1. Optimal attacking problem with delay and without delay network attacks

The subject of attacking in computer systems is considered as a multidisciplinary area of research which includes economists, network technology and resource management. And what role does the time delay select play in optimal attack?

**Simulation 1.**

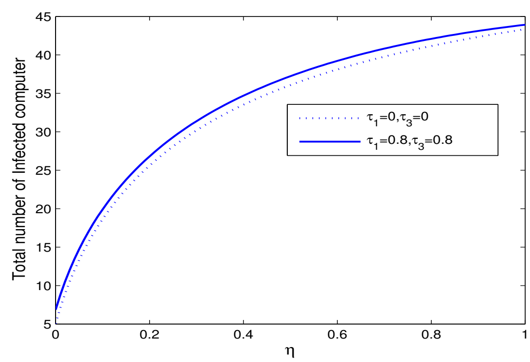


FIGURE 1. The total number of infected computer with respect to budget effort $\eta$ in interval (0, 1] for without and with delays attacking.

In this subsection, we assume that the terminal time is $t_p = 50$ and the immune period of a recovered computer $\tau_2 = 0.2$, and we will firstly probe into optimal attacking problem based on non-selective attack. That is, for fixed $\tau_1 = \tau_3 = 0$, maximum attack is studied when $\eta$ is regarded as a control parameter. Then the optimal harvest strategy with selective catch along with $\tau_1 = 1, \tau_3 = 1$ are researched.

Simulation 1 reveal several significant points. The total number of infected computer fluctuate between selective and non-selective attacking, and the number become higher with the growth of budget for both of selective and non-selective attacking. It declare that selective attacking is more appropriate. The growth of the number not really obvious with the budget grow.

### 5.2. Composite optimization problem (COP) based on the time delay attack

Problem (3.4) is a composite optimization problem related to the resources exploitation and computer virus attacking from the perspective of economics and computer science when $\phi \neq 0$ and $L \neq 0$.

**Simulation 2.**

When the initial of budget and ages are chosen, optimal budget and ages are computed by algorithmic design. Furthermore the corresponding number of Infected computer is obtained in Table 2. The terminal time is $t_p = 50$.

**Table 2: Optimal delays and budget with respect to COP**

| set | initial of budget and ages $\eta_0$ $\tau_{10}$ $\tau_{30}$ | initial of the total number of infected computer $I_0$ | optimal budget and ages $\eta$ $\tau_1$ $\tau_3$ | minimum of the total number of infected computer $I$ |
|---|---|---|---|---|
| 1 | 0.4 0.3 1.5 | 35.07 | 0.55 0.31 1.45 | 38.70 |
| 2 | 0.1 1 1 | 20.19 | 0.55 0.86 0.86 | 38.30 |

### 5.3. Evolution of number of computers with or without man-made attack.

**Simulation 3.**

In this section, we shall assume that S(0)=8, I(0) = 1, and R(0) = 1. Let $\tau_1 = 0.1$, $\tau_3 = 0.2$ and $\eta = 0.5$. Figure. 2 plots the evolution of number of susceptible nodes both with and without man-made attack, Figure. 3 depicts the evolution of number of infected nodes both with and without man-made attack, and Figure. 4 shows the evolution of number of recovered nodes both with and without man-made attack. From these figures one can see that a system with optimal man-made attack is dramatically superior to one without man-made attack.

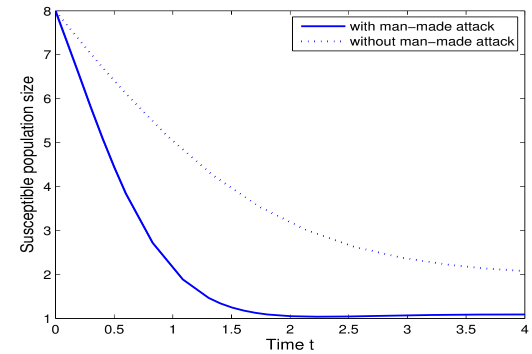


FIGURE 2. Evolution of number of susceptible computers with or without man-made attack.

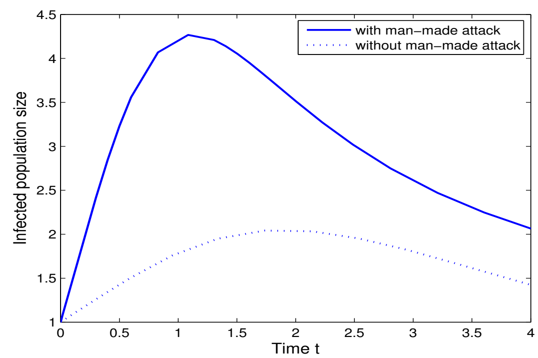


FIGURE 3. Evolution of number of infected computers with or without man-made attack.

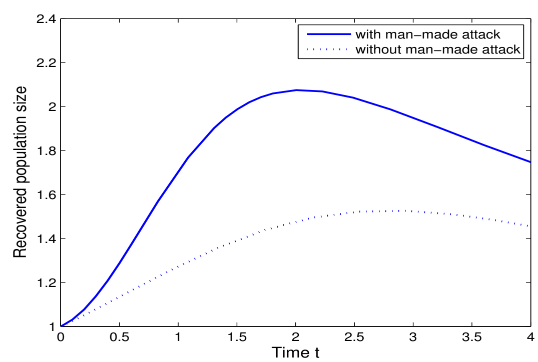


FIGURE 4. Evolution of number of recovered computers with or without man-made attack.

## 6. Conclusion

his paper aims to promote the spread of computer virus and the cost savings effectively by means of the optimal control method. First, a controlled delayed computer virus spread model has been established. Second, an optimal control problem has been proposed by maximizing the number of infected nodes and minimizing the budget. Third, develop a numerical algorithm for computing the gradient of the optimal control problem by the lights of optimal state-delay control problem and Pontryagin-type minimum (maximum) principle. Finally, some numerical examples have been provided, which show that the spread of computer virus can be promoted effectively by taking optimal control measures.

**Conflict of Interests**

The authors declare that there is no conflict of interests.

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