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A MODEL FOR HEPATITIS B DISEASE WITH AGE-DEPENDENT SUSCEPTIBILITY AND VERTICAL TRANSMISSION

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Abstract. In this paper, we construct a SEI model for hepatitis B where the susceptibility and transmission probabilities depend on the chronological age. We extend results of (Houpa *et al*, 2014) who studied it without vertical transmission. Moreover, we derive the basic reproductive rate R_0. Under given assumptions, we prove that the disease free equilibrium is globally asymptotically stable if $R_0 < 1$; the endemic equilibrium is globally asymptotically stable if $R_0 > 1$ and that the system is uniformly persistent. Numerical simulations are carried out to illustrate our results.

Keywords: Hepatitis B; basic reproductive rate; stability; epidemic equilibria.

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1. Introduction

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In this paper, we study a model of hepatitis B dynamic with age-dependent susceptibility. Infection with hepatitis B remains a serious global public health problem and a main cause of morbidity and mortality in high endemic areas such as Sub-Saharan Africa [1, 19, 3, 9]. According to Bonzi *et al* [2], Nokes *et al* [6], Zou *et al* [20], age factor is useful in the determination of the disease evolution. In his thesis [7], Fall highlighted vertical transmission of hepatitis B, in agreement with the results obtained by Diallo *et al* [5]. Moreover, Kouakep [12] showed that ignoring vertical transmission does not lead to a good approximation of reality to the virus infection hepatitis B in Cameroon.

We construct a SEI model for hepatitis B where the susceptibility and transmission probabilities depend on the chronological age. Moreover, we derive the basic reproductive rate R_0. Under given assumptions, we prove that the disease free equilibrium is globally asymptotically stable if $R_0 < 1$; the endemic equilibrium is globally asymptotically stable if $R_0 > 1$ and that the system is uniformly persistent for a transmission rate of chronic carriers $\beta_E = 0$.

We organise our work as follow. First of all, we study the mathematical model. The aim of this section is to present our model. Then, we give some main results relative to our model. Finally, we end by performing some simulations.

2. The Mathematical Model

2.1. Presentation

We consider the following age-dependent susceptibility model:

$$\begin{cases} (\partial_a + \partial_t)s(a,t) = -\mu(a)s(a,t) + \beta_I(a)I(t)s(a,t) \\ \frac{dE(t)}{dt} = \int_0^\infty q(a)\beta_I(a)I(t)s(a,t)da + (f_1(1-b_2-b_3) - \nu_E)E(t) \\ \frac{dI(t)}{dt} = \int_0^\infty p(a)\beta_I(a)I(t)s(a,t)da + (\varepsilon + f_1 b_3)E(t) + (f_1(1-b_1) - \nu_I)I(t) \end{cases} \tag{1}$$

with $a,t > 0$.

We denote by $s(a,t)$ the density of susceptible individuals to age a and time t; $I(t)$ and $E(t)$ denote respectively densities of acute infected and chronic carriers. We define a function

$p \in \mathrm{L}^{\infty}_{+}(0,\infty)$ such that $0 < p(a) \leq 1$ *a.e.* and $q(a) \equiv 1 - p(a)$; where functions p and q represent respectively the age-specific probabilities to develop an acute infection and to become a chronic carrier after being contaminated at the age a. μ_I and $\mu_E > 0$ denote the exit rates linked to each infected class (acute infected and chronic carrier). $\varepsilon > 0$ is the progression rate from chronic carrier to acute infected class. Here, we extend results of of Houpa *et al* [10, 11] who studied the problem (1) without vertical transmission. We define by function $f \in \mathrm{L}^{\infty}_{+}(0,\infty)$ the fertility rate of susceptible individuals and f_1 the fertility rate of infected. b_1 and b_2 are respectively the proportions of susceptible individuals born from acute infected and chronic carrier mothers. b_3 is the proportion of acute infected born from chronic carrier mothers.

The boundary and initial conditions associated to the problem (1) are:

$$s(0,t) = \Lambda + \int_0^{\infty} f(a')s(a',t)da' + f_1(b_1(I(t) + b_2E(t)) \tag{2}$$

and

$$\begin{cases} s(a,0) = s_0(a), \quad \forall a > 0 \\ E(0) = E_0 \\ I(0) = I_0 \end{cases} \tag{3}$$

The age-specific force of infection $\lambda(a,t)$ is given by:

$$\lambda(a,t) = \beta_E(a)E(t) + \beta_I(a)I(t).$$

Using the fact that asymptomatic carriers have a low infection rate (Fall *et al* [8], Bonzi *et al* [2], WHO [21]), we will assume that:

$$0 \simeq \beta_E(a) \ll \beta_I(a), \quad \forall a \in [0,\infty).$$

Then, we assume that:

$$\lambda(a,t) = \beta_I(a)I(t).$$

We perform simulations for the problem (1) by taking

$$q(a) = \exp(-0.645a^{0.455}), \quad \forall a > 0$$

given in Nokes *et al* [6].

2.1. Main results

Theorem 2.1.1. We assume that

$$\int_0^\infty f(a')e^{-\int_0^{a'}\mu(s)ds}da' < 1 \quad \text{and} \quad \frac{q(a)(\varepsilon+f_1b_3)+(\nu_E-f_1(1-b_2-b_3))p(a)}{(f_1(1-b_2-b_3)-\nu_E)(f_1(1-b_1)-\nu_I)} > 0.$$

The basic reproductive rate is given by:

$$R_0 = \frac{1}{(f_1(1-b_2-b_3)-\nu_E)(f_1(1-b_1)-\nu_I)}\int_0^\infty (q(a)(\varepsilon+f_1b_3)+(\nu_E-f_1(1-b_2-b_3))p(a))\beta_I(a)s_F(a)da \tag{4}$$

The disease free equilibrium (DFE) is given by:

$$(s_F(a),E_F,I_F) = \left(\frac{\Lambda}{1-\int_0^\infty f(a')e^{-\int_0^{a'}\mu(s)ds}da'}\left(e^{-\int_0^a\mu(s)ds}\right),0,0\right).$$

About the endemic equilibrium, we have its existence if and only if $R_0 > 1$. Then, we assume that

$$\int_0^\infty f(a')e^{-\int_0^{a'}(\mu(s)+\beta_I(s)I^*)ds}da' < 1$$

and we get

$$s^*(a) = \frac{\Lambda+f_1(b_1I^*+b_2E^*)}{1-\int_0^\infty f(a')e^{-\int_0^{a'}(\mu(s)+\beta_I(s)I^*)ds}da'}\left(e^{-\int_0^a(\mu(s)+\beta_I(s)I^*)ds}\right) \tag{5}$$

Equation 5 is linked to

$$\frac{1}{(f_1(1-b_2-b_3)-\nu_E)(f_1(1-b_1)-\nu_I)}\int_0^\infty (q(a)(\varepsilon+f_1b_3)+(\nu_E-f_1(1-b_2-b_3))p(a))\beta_I(a)s^*(a)da = 1 \tag{6}$$

Theorem 2.1.2.

Assumption 2.1. We assume that the function $a \mapsto \beta_I(a)$ is bounded and continuous from $[0,\infty)$ into itself.

We make the following assumptions in the order to prove the stability of equilibrium:

Assumption 2.2.

$$(\nu_E - f_1(1-b_2-b_3)) > (\varepsilon+f_1b_3).$$

Assumption 2.3.

(1)

$$H(a,t) = \left(\frac{s(a,t)}{s_F(a)} - 1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)} + \mu(a)\right)$$

has a constant sign on $[0,+\infty[\times[0,+\infty[$.

(2)

$$\forall\, t \geq 0, \quad \lim_{a\to 0} \frac{s(a,t)}{s_F(a)} = 1 \tag{7}$$

Let $G(x) = x - 1 - \ln x$ be a function which is defined, continuous and positive for all $x > 0$. Moreover, it has a global minimum 0 at $x = 1$.

We assume also that

Assumption 2.4.

(1)

$$H(a,t) = s^*(a)\left(\frac{s(a,t)}{s^*(a)} - 1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)} + \mu(a) + \beta_I(a)I^*\right)$$

has a constant sign on $[0,+\infty[\times[0,+\infty[$.

(2)

$$\forall\, t \geq 0, \quad \lim_{a\to 0} \frac{s(a,t)}{s^*(a)} = 1 \tag{8}$$

(3)

$$G\left(\frac{s(a,t)I(t)E^*}{s^*(a)I^*E(t)}\right) + G\left(\frac{I^*E(t)}{E^*I(t)}\right) - G\left(\frac{s(a,t)}{s^*(a)}\right) \geq 0.$$

The property of steady states is given by the following result.

If the assumptions 2.1., 2.2., 2.3. and 2.4. above are satisfied then:

- if $R_0 < 1$ then the disease free equilibrium is globally asymptotically stable;
- if $R_0 > 1$ then there exists an endemic equilibrium, strictly positive, which is globally asymptotically stable. Moreover, the problem (1) is uniformly persistent.

Proof. From Houpa *et al* [11], we have the existence and uniqueness of mild solution for the abstract Cauchy problem associated to problem (1) on $X_{0+} = X_0 \bigcap X_+$, where $X_0 = \mathrm{L}^1(0,\infty) \times \{0\} \times \mathbb{R} \times \mathbb{R}$ and $X_+ = \mathrm{L}^1_+(0,\infty) \times \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+$.

Let us prove the above theorem 2.1.2.

(1) Let us consider the Lyapunov function V defined on $]0,\infty[$ by:

$$U(s(a,t),E(t),I(t)) \equiv V(t) = \int_0^\infty A(a)G\left(\frac{s(a,t)}{s_F(a)}\right)da + BE(t) + CI(t)$$

$U(s_F(a),0,0) = 0$. Moreover, function V is defined and positive for all (s,E,I). $(s_F(a),0,0)$ is a global minimum of function V.

We have:

$$\frac{dV(t)}{dt} = \int_0^\infty A(a)\left(\frac{1}{s_F(a)} - \frac{1}{s(a,t)}\right)\partial_t s(a,t)da + B\frac{dE(t)}{dt} + C\frac{dI(t)}{dt}$$

where

$$A(a) = \left(q(a) + \frac{(\nu_E - f_1(1-b_2-b_3))}{(\varepsilon + f_1 b_3)}p(a)\right)s_F(a), \quad B = 1 \quad \text{and} \quad C = \frac{(\nu_E - f_1(1-b_2-b_3))}{(\varepsilon + f_1 b_3)}$$

From the equations of system (1), we get:

$$\frac{dV(t)}{dt} = -\int_0^\infty \left(q(a) + \frac{(\nu_E - f_1(1-b_2-b_3))}{(\varepsilon + f_1 b_3)}p(a)\right)s_F(a)\left(\frac{s(a,t)}{s_F(a)} - 1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)} + \mu(a)\right)da$$
$$+ \frac{(f_1(1-b_2-b_3) - \nu_E)(f_1(1-b_1) - \nu_I)}{(\varepsilon + f_1 b_3)}(R_0 - 1)I(t)$$

Since

$$1 \leq q(a) + \frac{(\nu_E - f_1(1-b_2-b_3))}{(\varepsilon + f_1 b_3)}p(a) \leq \frac{(\nu_E - f_1(1-b_2-b_3))}{(\varepsilon + f_1 b_3)}$$

and from assumption 2.4.-1, three cases occur.

(a) If $\left(\frac{s(a,t)}{s_F(a)} - 1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)} + \mu(a)\right) > 0$ then

$$\frac{dV(t)}{dt} \leq -\int_0^\infty s_F(a)\left(\frac{s(a,t)}{s_F(a)} - 1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)} + \mu(a)\right)da$$
$$+ \frac{(f_1(1-b_2-b_3) - \nu_E)(f_1(1-b_1) - \nu_i)}{(\varepsilon + f_1 b_3)}(R_0 - 1)I(t)$$

After integration by parts, and from assumption 2.4.-2, we have:

$$\frac{dV(t)}{dt} \leq -\left[s_F(a)G\left(\frac{s(a,t)}{s_F(a)}\right)\right]_{a=\infty} - \int_0^\infty \mu(a)s_F(a)G\left(\frac{s(a,t)}{s_F(a)}\right)$$
$$+ \frac{(f_1(1-b_2-b_3) - \nu_E)(f_1(1-b_1) - \nu_I)}{(\varepsilon + f_1 b_3)}(R_0 - 1)I(t)$$

Thus

$$\frac{dV(t)}{dt} < 0, \quad \text{since } R_0 < 1.$$

(b) If $\left(\frac{s(a,t)}{s_F(a)}-1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)}+\mu(a)\right)=0$ then

$$\begin{aligned}\frac{dV(t)}{dt} &= \frac{(f_1(1-b_2-b_3)-\nu_E)(f_1(1-b_1)-\nu_I)}{(\varepsilon+f_1b_3)}(R_0-1)I(t)\\ &< 0, \quad \text{since } R_0<1.\end{aligned}$$

(c) If $\left(\frac{s(a,t)}{s_F(a)}-1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)}+\mu(a)\right)<0$ then

$$\begin{aligned}\frac{dV(t)}{dt} \leq & -\frac{(\nu_E-f_1(1-b_2-b_3))}{(\varepsilon+f_1b_3)}\int_0^\infty s_F(a)\left(\frac{s(a,t)}{s_F(a)}-1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)}+\mu(a)\right)da\\ & +\frac{(f_1(1-b_2-b_3)-\nu_E)(f_1(1-b_1)-\nu_i)}{(\varepsilon+f_1b_3)}(R_0-1)I(t)\end{aligned}$$

From assumption 2.2 and using the result for the case $\left(\frac{s(a,t)}{s_F(a)}-1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)}+\mu(a)\right)>0$, we get $\frac{dV(t)}{dt}<0$, since $R_0<1$.

From the global stability Lyapunov-LaSalle theorem [16, 14, 15], the disease free equilibrium is globally asymptotically stable since the largest invariant set of orbits $(s(a,t),E(t),I(t))$ satisfying $\frac{dV(t)}{dt}=0$ is reduced to the disease free equilibrium.

(2) Any solution of the problem (1)-2 with positive initial condition remains positive indefinitely. Hence, the problem (1)-2 is uniformly persistent [16].

Let us consider the Lyapunov function V defined on $]0,\infty[$ by:

$$U(s(a,t),E(t),I(t))\equiv V(t)=\int_0^\infty A(a)G\left(\frac{s(a,t)}{s^*(a)}\right)da+BG\left(\frac{E(t)}{E^*}\right)+CG\left(\frac{I(t)}{I^*}\right)$$

$U(s^*(a),E^*,I^*)=0$. Moreover, function V is defined and positive for all (s,E,I). $(s^*(a),E^*,I^*)$ is a global minimum of function V.

We have:

$$\frac{dV(t)}{dt}=\int_0^\infty A(a)\left(\frac{1}{s^*(a)}-\frac{1}{s(a,t)}\right)\partial_t s(a,t)da+B\left(\frac{1}{E^*}-\frac{1}{E(t)}\right)\frac{dE(t)}{dt}+C\left(\frac{1}{I^*}-\frac{1}{I(t)}\right)\frac{dI(t)}{dt}$$

where

$$A(a)=\left(q(a)+\frac{(\nu_E-f_1(1-b_2-b_3))}{(\varepsilon+f_1b_3)}p(a)\right)s^*(a),\quad B=E^*\quad\text{and}\quad C=\frac{(\nu_E-f_1(1-b_2-b_3))}{(\varepsilon+f_1b_3)}I^*.$$

Thus

$$\begin{aligned}\frac{dV(t)}{dt} =& -\int_0^\infty \left(q(a)+\frac{(\nu_E - f_1(1-b_2-b_3))}{(\varepsilon+f_1 b_3)}p(a)\right)s^*(a)\left(\frac{s(a,t)}{s^*(a)}-1\right)\\ &\times\left(\frac{\partial_a s(a,t)}{s(a,t)}+\mu(a)+\beta_I(a)I^*\right)da+\int_0^\infty q(a)s^*(a)G\left(\frac{s(a,t)}{s^*(a)}\right)\beta_I(a)I^*da\\ &-\int_0^\infty q(a)\beta_I(a)I^*s^*(a)G\left(\frac{I^*E(t)}{E^*I(t)}\right)da-\int_0^\infty q(a)\beta_I(a)I^*s^*(a)G\left(\frac{s(a,t)I(t)E^*}{s^*(a)I^*E(t)}\right)da\end{aligned}$$

Since

$$1\leq q(a)+\frac{(\nu_E - f_1(1-b_2-b_3))}{(\varepsilon+f_1 b_3)}p(a)\leq\frac{(\nu_E - f_1(1-b_2-b_3))}{(\varepsilon+f_1 b_3)}$$

and from assumption 2.4.-1, three cases occur.

(a) If $s^*(a)\left(\frac{s(a,t)}{s^*(a)}-1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)}+\mu(a)+\beta_I(a)I^*\right)>0$ then

$$\begin{aligned}\frac{dV(t)}{dt}\leq& -\int_0^\infty s^*(a)\left(\frac{s(a,t)}{s^*(a)}-1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)}+\mu(a)+\beta_I(a)I^*\right)da\\ &+\int_0^\infty q(a)s^*(a)G\left(\frac{s(a,t)}{s^*(a)}\right)\beta_I(a)I^*da-\int_0^\infty q(a)\beta_I(a)I^*s^*(a)G\left(\frac{I^*E(t)}{E^*I(t)}\right)da\\ &-\int_0^\infty q(a)\beta_I(a)I^*s^*(a)G\left(\frac{s(a,t)I(t)E^*}{s^*(a)I^*E(t)}\right)da\end{aligned}$$

After integration by parts of the first term of the second member of the previous equality, and from assumption 2.4.-2, we have:

$$\begin{aligned}\frac{dV(t)}{dt}\leq& -\left[s^*(a)G\left(\frac{s(a,t)}{s^*(a)}\right)\right]_{a=\infty}-\int_0^\infty \mu(a)s^*(a)G\left(\frac{s(a,t)}{s^*(a)}\right)\\ &-\int_0^\infty p(a)s^*(a)G\left(\frac{s(a,t)}{s^*(a)}\right)\beta_I(a)I^*da-\int_0^\infty q(a)\beta_I(a)I^*s^*(a)G\left(\frac{I^*E(t)}{E^*I(t)}\right)da\\ &-\int_0^\infty q(a)\beta_I(a)I^*s^*(a)G\left(\frac{s(a,t)I(t)E^*}{s^*(a)I^*E(t)}\right)da\end{aligned}$$

Hence

$$\frac{dV(t)}{dt}<0.$$

(b) If $s^*(a)\left(\frac{s(a,t)}{s^*(a)}-1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)}+\mu(a)+\beta_I(a)I^*\right)=0$ then $s(a,t)=s^*(a)$ or $\partial_a s(a,t)=-\mu(a)s(a,t)-\beta_I I^*(a)s(a,t)$.

(i) If $s(a,t)=s^*(a)$ then

$$\frac{dV(t)}{dt}=-\int_0^\infty q(a)\beta_I(a)I^*s^*(a)G\left(\frac{I^*E(t)}{E^*I(t)}\right)da-\int_0^\infty q(a)\beta_I(a)I^*s^*(a)G\left(\frac{s(a,t)I(t)E^*}{s^*(a)I^*E(t)}\right)da$$

it means that

$$\frac{dV(t)}{dt}<0.$$

(ii) If $\partial_a s(a,t)=-\mu(a)s(a,t)-\beta_I I^*(a)s(a,t)$ and $s(a,t)\neq s^*(a)$ then

$$\frac{dV(t)}{dt}=\int_0^\infty q(a)s^*(a)G\left(\frac{s(a,t)}{s^*(a)}\right)\beta_I(a)I^*da-\int_0^\infty q(a)\beta_I(a)I^*s^*(a)G\left(\frac{I^*E(t)}{E^*I(t)}\right)da$$
$$-\int_0^\infty q(a)\beta_I(a)I^*s^*(a)G\left(\frac{s(a,t)I(t)E^*}{s^*(a)I^*E(t)}\right)da$$

Using assumption 2.4.-3, we show that

$$\frac{dV(t)}{dt}<0.$$

(c) If $s^*(a)\left(\frac{s(a,t)}{s^*(a)}-1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)}+\mu(a)+\beta_I(a)I^*\right)<0$ then

$$\frac{dV(t)}{dt}\leq-\frac{(\nu_E-f_1(1-b_2-b_3))}{(\varepsilon+f_1b_3)}\int_0^\infty s^*(a)\left(\frac{s(a,t)}{s^*(a)}-1\right)$$
$$\times\left(\frac{\partial_a s(a,t)}{s(a,t)}+\mu(a)+\beta_I(a)I^*\right)da+\int_0^\infty q(a)s^*(a)G\left(\frac{s(a,t)}{s^*(a)}\right)\beta_I(a)I^*da$$
$$-\int_0^\infty q(a)\beta_I(a)I^*s^*(a)G\left(\frac{I^*E(t)}{E^*I(t)}\right)da-\int_0^\infty q(a)\beta_I(a)I^*s^*(a)G\left(\frac{s(a,t)I(t)E^*}{s^*(a)I^*E(t)}\right)da$$

From assumption 2.2. and using the result for the case

$$s^*(a)\left(\frac{s(a,t)}{s^*(a)}-1\right)\left(\frac{\partial_a s(a,t)}{s(a,t)}+\mu(a)+\beta_I(a)I^*\right)>0,$$

we get

$$\frac{dV(t)}{dt}<0$$

.

From the global stability Lyapunov-LaSalle theorem [16, 14, 15], the endemic equilibrium is globally asymptotically stable since the largest invariant set of orbits $(s(a,t),E(t),I(t))$ satisfying $\frac{dV(t)}{dt} = 0$ is reduced to the endemic equilibrium.

This completes the proof.

3. Simulations

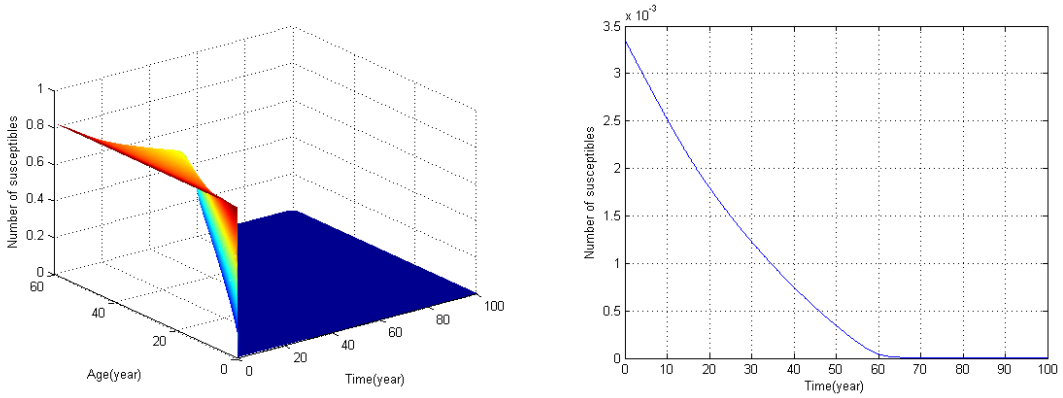
We perform our simulations by choosing the fertility rate f such as:

$$f(a) = \begin{cases} \frac{\pi}{10^4} \sin\left(\pi \frac{(a-15)}{30}\right), \text{if } 15 \leq a \leq 45 \\ 0 \text{ if not} \end{cases} \tag{9}$$

So, a woman with an age between 15-45 years old, will give birth to six children over a total population of 10^3 persons, during her life. Moreover, we perform three cases. The values of the parameters are giving by Table 1(for $R_0 < 1$) and Tables 2 and 3 (for $R_0 > 1$). Here, the recruitment rate Λ is constant.

- Case 1:

  The value of basic reproductive rate is $R_0 = 0.001277$. We have the following figures.



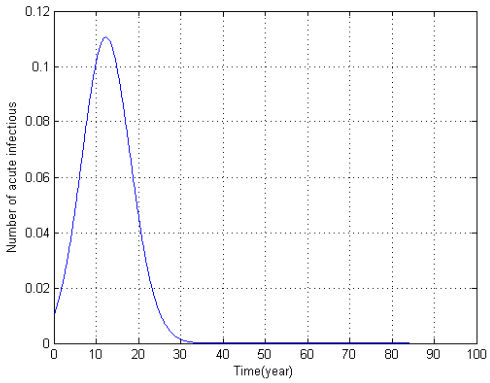
Function $s(a,t)$ Function $t \mapsto \int_0^\infty s(a,t)da$

FIGURE 1. Function $s(a,t)$ and Function $t \mapsto \int_0^\infty s(a,t)da$

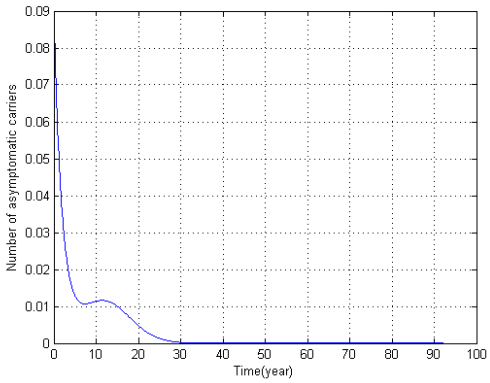
The corresponding value without vertical transmission is $R_0 = 0.001269$.

TABLE 1. Values of parameters used for numerical simulations.

Parameter/Variable	Value	Unit	Reference
Age	$[0,60]$	$year$	[13]
$\beta_I(a)$	3.10^{-2}	$year^{-1}$	Assumed
$q(a)$	$\exp\left(-0.645a^{0.455}\right)$		[6]
ε	10^{-2}		Assumed
ν_I	0.9818458	$year^{-1}$	Assumed
ν_E	0.5	$year^{-1}$	Assumed
$\mu(a)$	9.10^{-4}	$year^{-1}$	Assumed
Λ	10^{-3}		Assumed
f_1	0.002		Assumed
b_1	0.05		Assumed
b_2	0.1		[17, 18, 22]
b_3	0.15		[4]



Function $I(t)$



Function $E(t)$

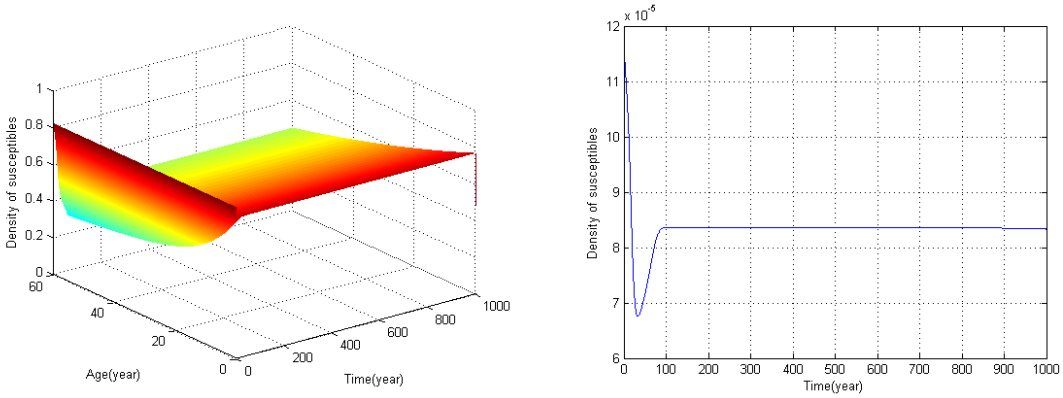
FIGURE 2. Function $I(t)$ and Function $E(t)$

- Case 2:

TABLE 2. Values of parameters used for numerical simulations.

Parameter/Variable	Value	Unit	Reference
Age	$[0,60]$	$year$	Kouakep
$\beta_I(a)$	31.10^{-3}	$year^{-1}$	Assumed
$q(a)$	$\exp\left(-0.645a^{0.455}\right)$		[6]
ε	10^{-3}		Assumed
ν_I	0.9818458	$year^{-1}$	Assumed
ν_E	0.9	$year^{-1}$	Assumed
$\mu(a)$	9.10^{-3}	$year^{-1}$	Assumed
Λ	0.76		Assumed
f_1	4.10^{-3}		Assumed
b_1	0.3		Assumed
b_2	0.3		[17, 18, 22]
b_3	0.14		[4]

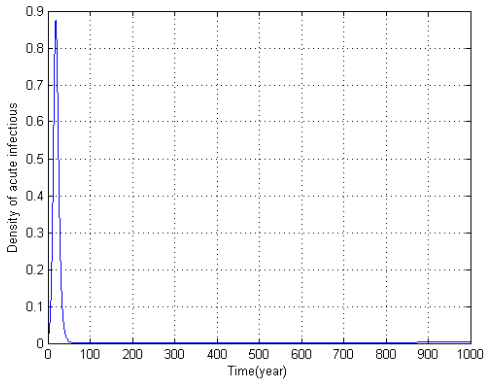
The value of basic reproductive rate is $R_0 = 1.002492$. We have the following figures.



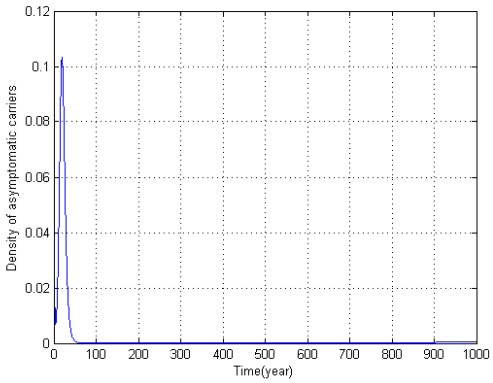
Function $s(a,t)$ Function $t \mapsto \int_0^\infty s(a,t)da$

FIGURE 3. Function $s(a,t)$ and Function $t \mapsto \int_0^\infty s(a,t)da$

The corresponding value without vertical transmission is $R_0 = 0.994943$.

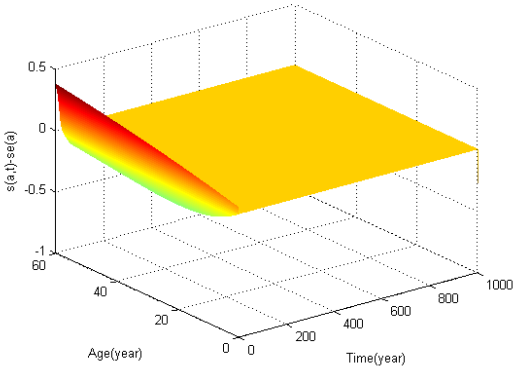


Function $I(t)$

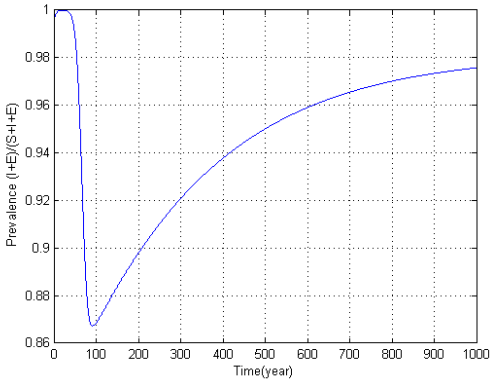


Function $E(t)$

FIGURE 4. Function $I(t)$ and Function $E(t)$



Difference $s(a,t)-s^*(a)$



Prevalence function

FIGURE 5. Difference $s(a,t)-s^*(a)$ and Prevalence function

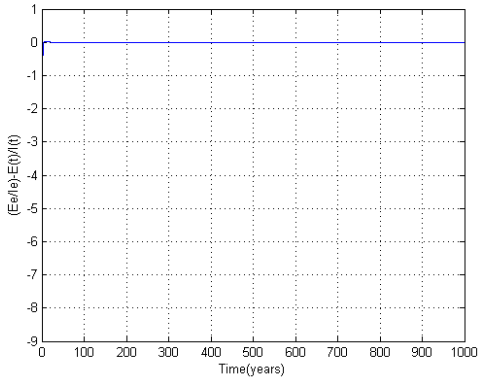


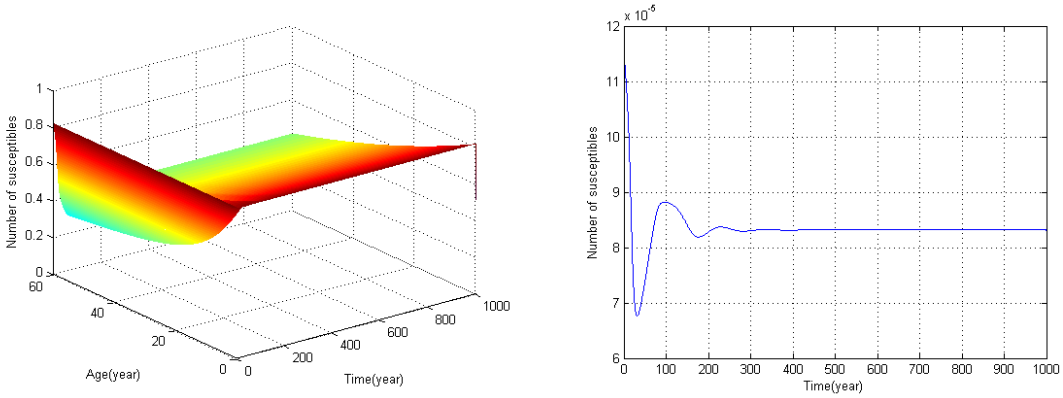
FIGURE 6. Difference $\frac{E_e}{I_e}-\frac{E(t)}{I(t)}$

- Case 3:

TABLE 3. Values of parameters used for numerical simulations.

Parameter/Variable	Value	Unit	Reference
Age	$[0,60]$	$year$	[13]
$\beta_I(a)$	31.10^{-3}	$year^{-1}$	Assumed
$q(a)$	$\exp\left(-0.645a^{0.455}\right)$		[6]
ε	10^{-2}		Assumed
ν_I	0.9818458	$year^{-1}$	Assumed
ν_E	0.7	$year^{-1}$	Assumed
$\mu(a)$	9.10^{-3}	$year^{-1}$	Assumed
Λ	0.81		Assumed
f_1	0.002		Assumed
b_1	0.05		Assumed
b_2	0.1		[17, 18, 22]
b_3	0.15		[4]

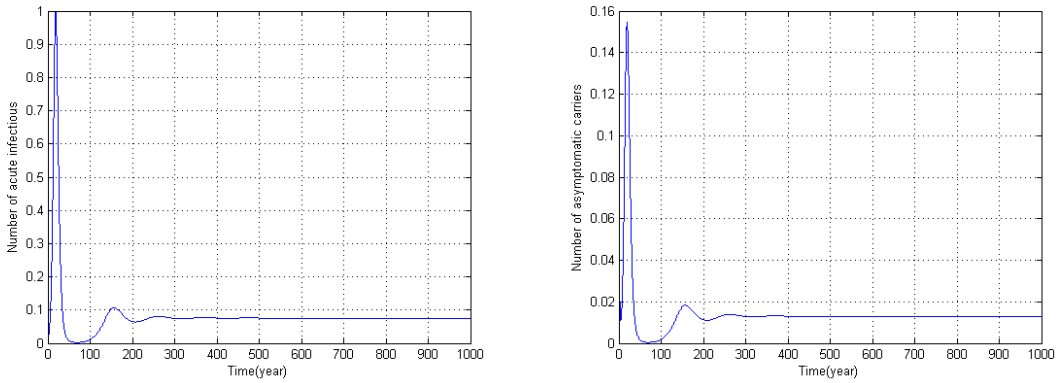
The value of basic reproductive rate is $R_0 = 1.069098$. We have the following figures.



Function $s(a,t)$ Function $t \mapsto \int_0^\infty s(a,t)da$

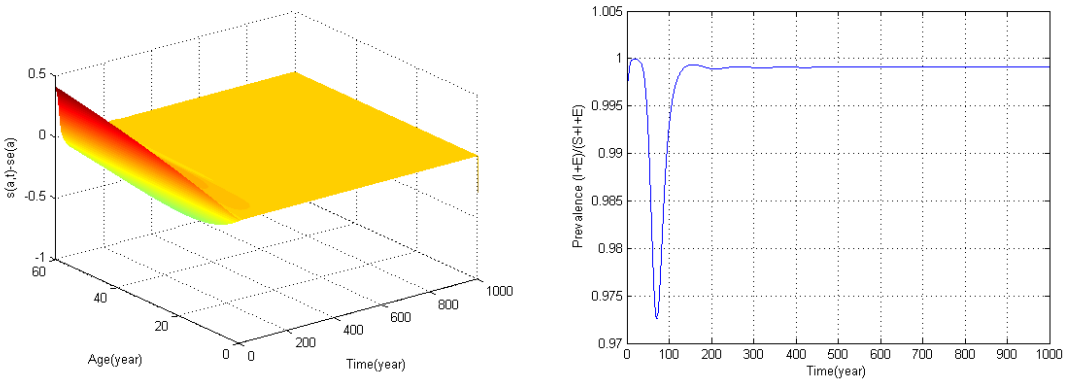
FIGURE 7. Function $s(a,t)$ and Function $t \mapsto \int_0^\infty s(a,t)da$

The corresponding value without vertical transmission is $R_0 = 1.062043$.



Function $I(t)$ Function $E(t)$

FIGURE 8. Function $I(t)$ and Function $E(t)$



Difference $s(a,t)-s^*(a)$ Prevalence function

FIGURE 9. Difference $s(a,t)-s^*(a)$ and Prevalence function

From figure 2, we observe that densities of acute infected and chronic carriers vanish over time, evidence of the disparition of the disease in population.

In figures 4 and 8, we observe that densities of acute infected and chronic carriers don't vanish over time, evidence of the persistence of the disease in population.

From our simulations, we realize that the value of the basic reproduction rate of the model (1) is higher than the one without vertical transmission. Moreover, this basic reproductive rate increases when the transmission rate $\beta_I(a)$ increases. In all cases, stabilization occurs after a period of severe manifestation of disease.

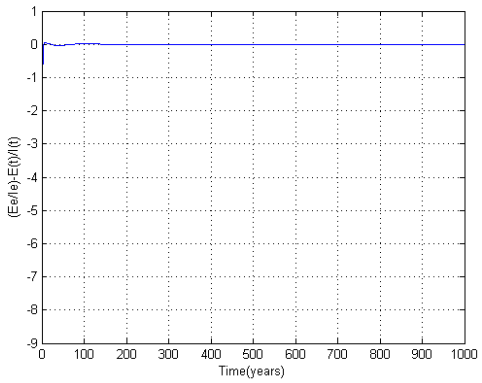


FIGURE 10. Difference $\frac{E_e}{I_e} - \frac{E(t)}{I(t)}$

4. Discussion

In this paper, we considered a mathematical model of hepatitis B dynamic with age-dependent susceptibility, taking into account vertical transmission. We have derived the basic reproductive rate and proved that the disease free equilibrium is globally asymptotically stable if $R_0 < 1$; the endemic equilibrium is globally asymptotically stable if $R_0 > 1$ and that the system is uniformly persistent under given some assumptions. Moreover, our theoretical study for stability of steady states was proved by simulations. In the future, it would be judicious to incorporate into the modeling of transmission Hepatitis B immunization, in order to assess their impact in the process of transmission control of disease. Furthermore, migration of the population (recruitment of patients) should also be taken into account. Following the demographic boom that known worldwide because of scientific advances, nowadays birth policies have been adopted to limit births. Therefore, we must consider that fertility depends of the age and time. In addition, the mortality of different compartments will depend of the age and time.

Conflict of Interests

The authors declare that there is no conflict of interests.

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