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DIRECTION AND STABILITY OF HOPF BIFURCATION

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Abstract. The dynamics of the spatial competition mathematical model for the invasion, removal of Kappaphycus Algae (KA) in Gulf of Mannar (GoM) with propagation delays is investigated by applying the normal form theory and the center manifold theorem.

Keywords: delay differential equation; Hopf bifurcation.

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1. Introduction

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In recent years we have witnessed an increasing interest in dynamical systems with time delays, especially in applied mathematics. Stability and direction of the Hopf bifurcation for the predator-prey system have been discussed by using normal form theory and center manifold theory [5,6,10,13,15,16,17]. Direction and stability of the equilibrium for a neural network model with two delays have been investigated [7,12]. Bifurcation analysis of the predtor-prey model has been detailed [8]. Direction and stability of the equilibrium involving various fields have been discussed [4,9,11,14]. We reported the shifting of algal dominated reef ecosystem due to the invasion of KA in Gulf of Mannar [1]. Subsequently, the dominance of KA over NA and corals in competing for space has also been reported. KA sexual reproduction by spores in the Gulf of Mannar Marine Biosphere Reserve (GoM) in future, when environmental conditions unanimously favor this alga has been deliberated [2]. To simulate the three way competition among corals, KA and NA, we proposed the following system of non-linear ODE's [3].

$$\begin{aligned}\frac{dx}{dt} &= rx - rx^2 - rxy - rxz - a_1xy - a_2xz + dy \\ \frac{dy}{dt} &= a_1yx + a_3yz + \nu y(t-\tau_1) - \nu yx - \nu y^2 - \nu yz - dy \\ \frac{dz}{dt} &= a_2zx + hz(t-\tau_2) - hxz - hzy - hz^2 - a_3zy\end{aligned} \tag{1.1}$$

2. Direction and Stability of Hopf bifurcation

We assume that the system undergoes a Hobf bifurcation at the positive equilibrium $E(0,y^*,0)$ for $\tau_1 = \tau_1^*$ and then $\pm i\omega$ denotes the corresponding purely imaginary roots of the characteristic equation at the positive equilibrium $E(0,y^*,0)$.

Without loss of generality, we assume that $\tau_2^* < \tau_1^*$ where $\tau_2^* \in (0, \tau_{2_0}^*)$ and $\tau_1 = \tau_1^* + \mu$. Let $x_{11} = x - x^*$, $x_{21} = y_1 - y_1^*$, $x_{31} = y_2 - y_2^*$, $\bar{x_{i1}} = \mu_i(\tau t)$, i=1,2,3... Here $\mu = 0$ is the bifurcation parameter and dropping the bars, the system becomes a functional differential equation in $C = C([-1,0], R^3)$ as

$$\frac{dX}{dt} = L_\mu(X_t) + f(\mu, X_t) \tag{2.1}$$

where $x(t)=(x_{11},x_{21},x_{31})\in R^3$ and $L_\mu : C\to R^3$, $f:R\times C\to R^3$ are respectively given by

$$L_\mu(\phi)=(\tau_1^*+\mu)B\begin{pmatrix}\phi_1(0)\\ \phi_2(0)\\ \phi_3(0)\end{pmatrix}+(\tau_1^*+\mu)C\begin{pmatrix}\phi_1(\frac{-\tau_2^*}{\tau_1})\\ \phi_2(\frac{-\tau_2^*}{\tau_1})\\ \phi_3(\frac{-\tau_2^*}{\tau_1})\end{pmatrix}+(\tau_1^*+\mu)D\begin{pmatrix}\phi_1(-1)\\ \phi_2(-1)\\ \phi_3(-1)\end{pmatrix} \tag{2.2}$$

and

$$f(\mu,\phi)=(\tau_1^*+\mu)Q \tag{2.3}$$

where Q= $\begin{pmatrix}(r-r\phi_2(0)-a_1\phi_2(0))\phi_1(0)+d\phi_2(0)\\ (a_1\phi_2(0)-\nu\phi_2(0))\phi_1(0)-2\nu\phi_2^2(0)-d\phi_2(0)+(a_3\phi_2(0)-\nu\phi_2(0))\phi_3(0)+\nu e^{-\lambda\tau_1}\phi_2(-1)\\ (-h\phi_2(0)-a_3\phi_2(0))\phi_3(0)+he^{-\lambda\tau_2}\phi_3(-1)\end{pmatrix}$

respectively where $\phi(\theta)=(\phi_1(\theta),\phi_2(\theta),\phi_3(\theta))^T\in C$,

$$B=\begin{pmatrix}r-ry^*-a_1y^* & d & 0\\ a_1y^*-\nu y^* & -2\nu y^*-d & a_3y^*-\nu y^*\\ 0 & 0 & -hy^*-a_3y^*\end{pmatrix},$$

$$C=\begin{pmatrix}0&0&0\\0&0&0\\0&0&he^{-\lambda\tau_2}\end{pmatrix},\ D=\begin{pmatrix}0&0&0\\0&\nu e^{-\lambda\tau_1}&0\\0&0&0\end{pmatrix}.$$

By the Riesz representation theorem, we claim about the existence of a function $\eta(\theta,\mu)$ of bounded variation for $\theta\in[-1,0)$ such that

$$L_\mu(\phi)=\int_{-1}^{0}d\eta(\theta,\mu)\phi(\theta)\ for\ \phi\in C \tag{2.4}$$

Now let us choose ,

$$\eta(\theta,\mu)=\begin{cases}(\tau_1^*+\mu)(B+C+D), & \theta=0\\ (\tau_1^*+\mu)(C+D), & \theta\in[\frac{-\tau_2^*}{\tau_1},0)\\ (\tau_1^*+\mu)(D), & \theta\in(-1,\frac{-\tau_2^*}{\tau_1})\\ 0, & \theta=-1.\end{cases}$$

For$\phi \in C([-1,0],R^3)$, we define

$$A(\mu)\phi = \begin{cases} \frac{d\phi(\theta)}{d\theta}, & \theta \in [-1,0) \\ \\ \int_{-1}^{0} d\eta(s,\mu)\phi(s), & \theta = 0 \end{cases}$$

and

$$R(\mu)\phi = \begin{cases} 0, & \theta \in [-1,0) \\ \\ f(\mu,\phi), & \theta = 0 \end{cases}$$

Then the system is equivalent to

$$\frac{dX}{dt} = A(\mu)X_t + R(\mu)X_t, \tag{2.5}$$

where $X_t(\theta) = X(t+\theta)$ for $\theta \in [-1,0]$.

Now for $\psi \in C([-1,0],(R^3)^*)$, we define

$$A^*\psi(s) = \begin{cases} \frac{-d\psi(s)}{ds}, & s \in (0,1] \\ \\ \int_{-1}^{0} d\eta^T(t,0)\psi(-t), & s = 0 \end{cases}$$

Further we define a bilinear inner product

$$< \psi(s),\phi(0) >= \bar{\psi}(0)\phi(0) - \int_{-1}^{0}\int_{\zeta=0}^{\theta} \bar{\psi}(\zeta-\theta)d\eta(\theta)\phi(\zeta)d\zeta. \tag{2.6}$$

where $\eta(\theta) = \eta(\theta,0)$. Clearly here A and A^* are adjoint operators and $\pm i\omega^*\tau_0^*$ are eigen values of $A(0)$ and so they are also eigen values A^*. Let $q(\theta) = (1\ \alpha\ \beta)^T e^{i\omega^*\tau_0^*\theta}$ be the eigen vector of $A(0)$ corresponding to $i\omega^*\tau_0^*$ where

$\alpha = \frac{-[r-ry^*-a_1y^*-iw]}{d}$, $\beta = \frac{(r-ry^*-a_1y^*-iw)(\nu e^{-iw_0}-2\nu y^*-d-iw)-d(a_1y^*-\nu y^*)}{d(a_3y^*-\nu y^*)}$

Similarly if $q^*(s) = M(1\ \alpha^*\ \beta^*)e^{i\omega^*\tau_0^*s}$ be the eigen vector of A^* where

$\alpha^* = \frac{-(r-ry^*-a_1y^*-iw)}{a_1y^*-\nu y^*}$,

$\beta^* = \frac{(a_3y^*-\nu y^*)(r-ry^*-a_1y^*-iw)}{(a_1y^*-\nu y^*)(he^{-iw\tau_0}-hy^*-a_3y^*-iw)}$

Then we have to determine M from $< q^*(s), q(\theta) >= 1$.

Thus we can take

$$\bar{M} = \frac{1}{1 + \alpha\bar{\alpha^*} + \beta\bar{\beta^*} + \tau_1^* e^{i\omega_0^*\tau_0^*}(\alpha\alpha^*\nu + \beta\beta^* h)} \tag{2.7}$$

We first compute the coordinate to describe the center manifold C_0 at $\mu = 0$. Let X_t be the solution of the system (2.5)when $\mu = 0$. Define $z(t) =< q^*, X_t >$

$$W(t,\theta) = X_t(\theta) - 2Rez(t)q(\theta) \tag{2.8}$$

On the center manifold C_0, we have
$W(t,\theta) = W(z(t),\bar{z}(t),\theta)$ where

$$W(z,\bar{z},\theta) = W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + \dots. \tag{2.9}$$

and z and $\bar{z}$ are local coordinates for center manifold C_0 in the direction of q^* and $\bar{q}^*$.
Note that W is real if X_t is real. We consider only real solutions. For solution $X_t \in C_0$ of Eq. (2.1), since $\mu = 0$ we have

$$\begin{aligned} \dot{z}(t) &= i\omega^*\tau_0^* z + < \bar{q^*}(0) f(0, W(z,\bar{z},0) + 2Rezq(\theta)) > \\ &\cong i\omega^*\tau_0^* z + \bar{q^*}(0) f_0(z,\bar{z}) \\ &= i\omega^*\tau_0^* z + g(z,\bar{z}) \end{aligned} \tag{2.10}$$

where

$$\begin{aligned} g(z,\bar{z}) &= \bar{q^*}(0) f_0(z,\bar{z}) \\ &= g_{20}\frac{z^2}{2} + g_{11}z\bar{z} + g_{02}\frac{z^2}{2} + g_{21}\frac{z^2\bar{z}}{2} + \dots. \end{aligned} \tag{2.11}$$

From (2.8) and (2.9), we get

$$\begin{aligned} X_t(\theta) &= W(t,\theta) + 2Rez(t)q(\theta) \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + zq + \bar{z}\bar{q} + \dots. \\ &= W_{20}(\theta)\frac{z^2}{2} + W_{11}(\theta)z\bar{z} + W_{02}(\theta)\frac{\bar{z}^2}{2} + (1\ \alpha\ \beta)^T e^{i\omega^*\tau_0^*} + (1\ \bar{\alpha}\ \bar{\beta})^T e^{i\omega^*\tau_0^*}\bar{z} + \dots. \end{aligned} \tag{2.12}$$

Hence we have

$$\begin{aligned} g(z,\bar{z}) &= \bar{q^*}(0) f_0(z,\bar{z}) \\ &= \bar{q^*}(0) f(0,X_t) \\ &= \tau_0^* \bar{M}(1\ \bar{\alpha^*}\ \bar{\beta^*}) T \\ &= \tau_0^* \bar{M}(p_1 z^2 + 2p_2 z\bar{z} + p_3 \bar{z}^2 + p_4 z^2 \bar{z}) + H.O.T \end{aligned} \tag{2.13}$$

where $T = \begin{pmatrix} (r - rx_{2t}(0) - a_1 x_{2t}(0)) x_{1t}(0) + d x_{2t}(0) \\ (a_1 x_{2t}(0) - \nu x_{2t}(0)) x_{1t}(0) - 2\nu x_{2t}^2(0) - d x_{2t} + (a_3 x_{2t} - \nu x_{2t}) x_{3t}(0) + \nu e^{-\lambda \tau_1} x_{2t}(-1) \\ (-h x_{2t}(0) - a_3 x_{2t}(0)) x_{3t}(0) + h e^{-\lambda \tau_2} x_{3t}(-1) \end{pmatrix}$

p_1, p_2, p_3 and p_4 values can be calculaed by using the formula.

Comparing (2.11) and (2.13)

$g_{20} = 2\tau_0^* \bar{M} p_1$

$g_{11} = 2\tau_0^* \bar{M} p_2$

$g_{02} = 2\tau_0^* \bar{M} p_3$

$g_{21} = 2\tau_0^* \bar{M} p_4$

For unknown $W_{20}^{(i)}(\theta)$, $W_{11}^{(i)}(\theta)$, i=1,2 in g_{21}, we still have to compute them. From (2.5) and (2.8)

$$\begin{aligned} \dot{W} &= \dot{X}_t - \dot{z}q - \dot{\bar{z}}_{\bar{q}} \\ &= \begin{cases} AW - 2Re\{\bar{q^*}(0) f_0 q(\theta)\}, & -1 \le \theta \le 0, \\ AW - 2Re\{\bar{q^*}(0) f_0 q(\theta)\} + f_0, & \theta = 0, \end{cases} \\ \dot{W} &= AW + H(z,\bar{z},\theta) \end{aligned} \tag{2.14}$$

where

$$H(z,\bar{z},\theta) = H_{20}(\theta)\frac{z^2}{2} + H_{11}(\theta) z\bar{z} + H_{02}(\theta)\frac{\bar{z}^2}{2} + ... \tag{2.15}$$

From (2.14) and (2.15)

$$[A(0) - 2i\omega^* \tau_0^* I] W_{20}(\theta) = -H_{20}(\theta)$$

$$A(0) W_{11}(\theta) = -H_{11}(\theta)$$

From (2.14) we have for $\theta \in [-1,0)$

$$H(z,\bar{z},\theta) = -g(z,\bar{z})q(\theta) - \bar{g}(z,\bar{z})\bar{q}(\theta) \tag{2.18}$$

Comparing (2.15) and (2.18)

$$H_{20}(\theta) = -g_{20}q(\theta) - \bar{g}_{02}\bar{q}(\theta) \tag{2.19}$$

$$H_{11}(\theta) = -g_{11}q(\theta) - \bar{g}_{11}\bar{q}(\theta) \tag{2.20}$$

By the definition of $A(\theta)$ and from the above equations

$$W_{20}(\theta) = \frac{ig_{20}}{\omega^*\tau_0^*}q(0)e^{i\omega^*\tau_0^*\theta} + \frac{i\bar{g}_{02}}{3\omega^*\tau_0^*}\bar{q}(0)e^{-i\omega^*\tau_0^*\theta} + E_1 e^{2i\omega^*\tau_0^*\theta}. \tag{2.21}$$

and

$$W_{11}(\theta) = \frac{-ig_{11}}{\omega^*\tau_0^*}q(0)e^{i\omega^*\tau_0^*\theta} + \frac{i\bar{g}_{11}}{\omega^*\tau_0^*}\bar{q}(0)e^{-i\omega^*\tau_0^*\theta} + E_2. \tag{2.22}$$

where $q(\theta) = (1\ \alpha\ \beta)^T e^{i\omega^*\tau_0^*\theta}$, $E_1 = (E_1^{(1)}, E_1^{(2)}, E_1^{(3)}) \in R^3$ and $E_2 = (E_2^{(1)}, E_2^{(2)}, E_2^{(3)}) \in R^3$ are constant vectors. From (2.14) and (2.15)

$$H_{20}(0) = -g_{20}q(0) - \bar{g}_{02}\bar{q}(0) + 2\tau_0^*(c_1\ c_2\ c_3)^T$$

$$H_{11}(0) = -g_{11}q(0) - \bar{g}_{11}\bar{q}(0) + 2\tau_0^*(d_1\ d_2\ d_3)^T \tag{2.23}$$

where $(c_1\ c_2\ c_3)^T = C_1$, $(d_1\ d_2\ d_3)^T = D_1$ are respective coefficients of z^2 and $z\bar{z}$ of $f_0(z\,\bar{z})$ and they are

$$C_1 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} -(a_1+r)\alpha \\ (a_1-\nu)\alpha - 2\nu\alpha^2 - \frac{dW_{20}^{(2)}(0)}{2} + (a_3-\nu)\alpha\beta + \frac{\nu e^{-\lambda\tau_1}}{2}W_{20}^{(2)}(-1) \\ (-h-a_3)\alpha\beta + \frac{he^{-\lambda\tau_2}}{2}W_{20}^{(3)}(-1) \end{pmatrix} \text{ and}$$

$$D_1 = \begin{pmatrix} d_1 \\ d_2 \\ d_3 \end{pmatrix} = 2\begin{pmatrix} \frac{rW_{11}^{(1)}(0)}{2} - rRe(\alpha) - a_1Re(\alpha) + \frac{dW_{11}^{(2)}(0)}{2} \\ (a_1-\nu)Re(\alpha) - 4\nu\alpha - \frac{dW_{11}^{(2)}(0)}{2} + (a_3-\nu)Re(\bar{\alpha}\beta) + \frac{\nu e^{-\lambda\tau_1}W_{11}^{(2)}(-1)}{2} \\ (-h-a_3)Re(\bar{\alpha}\beta) + he^{-\lambda\tau_2}\frac{W_{11}^{(3)}(-1)}{2} \end{pmatrix}$$

Finally we have $(2i\omega^*\tau_0^* I - \int_{-1}^{0} e^{2i\omega^*\tau_0^*\theta} d\eta(\theta))E_1 = 2\tau_0^* C_1$ or $C^* E_1 = 2C_1$ where

$$C^* = \begin{vmatrix} 2rx+ry+rz+a_1y+a_2z-r+2i\omega & rx+a_1x-d & rx+a_2x \\ vy-a_1y & vx+2vy+vz+d-a_1x-a_3z-ve^{-2i\omega^*\tau_1}+2i\omega & vy-a_3y \\ hz-a_2z & hz+a_3z & -a_2x-he^{-2i\omega\tau_2+hx+hy+2hz+a_3y+2i\omega} \end{vmatrix}, \quad (2.24)$$

Thus $E_1^i = \frac{2\Delta i}{\Delta}$ where $\Delta = Det(C^*)$ and Δ_i be the value of the determinant U_i, where U_i formed by replacing i^{th} column vector of C^* by another column vector $(c_1\ c_2\ c_3)^T$, i =1, 2, 3. Similarly $D^* E_2 = 2D_1$, where

$$D^* = \begin{vmatrix} 2rx+ry+rz+a_1y+a_2z-r & rx+a_1x-d & rx+a_2x \\ vy-a_1y & vx+2vy+vz+d-a_1x-a_3z-v & vy-a_3y \\ hz-a_2z & hz+a_3z & -a_2x-h+hx+hy+2hz+a_3y \end{vmatrix}, \quad (2.25)$$

Thus $E_2^i = \frac{2\bar{\Delta} i}{\bar{\Delta}}$ where $\bar{\Delta} = Det(D^*)$ and $\bar{\Delta}_i$ be the value of the determinant V_i, where V_i formed by replacing i^{th} column vector of D^* by another column vector $(d_1\ d_2\ d_3)^T$, i =1,2, 3. Thus we can determine $W_{20}(\theta)$ and $W_{11}(\theta)$ from (2.12) and (2.13). Furthermore using them we can compute g_{21} and derive the following values.

$$C_1(0) = \frac{i}{2\omega^*\tau_0^*}(g_{20}g_{11} - 2|g_{11}|^2 - \frac{|g_{02}|^2}{3}) + \frac{g_{21}}{2}$$

$$\mu_2 = \frac{-Re\{C_1(0)\}}{Re\left\{\frac{d\lambda(\tau_0^*)}{d\tau}\right\}}$$

$$\beta_2 = 2Re\{C_1(0)\}$$

$$T_2 = \frac{-Im\left\{C_1(0) + \mu_2 Im\left\{\frac{d\lambda(\tau_0^*)}{d\tau}\right\}\right\}}{\omega^*\tau_0^*}$$

These formulae give a description of the Hopf bifurcation periodic solutions of system (1.1) at $\tau = \tau_0^*$ on the center manifold. Hence we have the following result.

Theorem 2.1. *The periodic solutions is supercritical (resp.subcritical) if* $\mu_2 > 0$ *(resp.* $\mu_2 < 0$*). The bifurcating periodic solutions are orbitally asymptotically stable with an asymptotical* *phase (resp.unstable) if $\beta_2 < 0$ (resp. $\beta_2 > 0$). The period of bifurcating periodic solutions increases(resp.decreases) if $T_2 > 0$ (resp.$T_2 < 0$).*

4. Conclusion

We have derived the bifurcating periodic solutions are orbitally asymptotically stable with an asymptotical phase if $\beta_2 < 0$ and unstable if $\beta_2 > 0$ and the period of bifurcating periodic solutions increases if $T_2 > 0$ and decreases if $T_2 < 0$.

Conflict of Interests

The authors declare that there is no conflict of interests.

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