



Available online at <http://scik.org>

Commun. Math. Biol. Neurosci. 2017, 2017:19

ISSN: 2052-2541

PERSISTENCE AND GLOBAL STABILITY OF A LESLIE-GOWER PREDATOR-PREY REFUGE SYSTEM WITH A COMPETITOR FOR THE PREY

CHANDAN MAJI^{1,2}, DEBASIS MUKHERJEE^{2,*}, DIPAK KESH¹

¹Centre for Mathematical Biology and Ecology, Department of Mathematics, Jadavpur University,
Kolkata-700032, India

²Department of Mathematics, Vivekananda College, Thakurpukur, Kolkata-63, India

Communicated by F. Chen

Copyright © 2017 Maji, Mukherjee and Kesh. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Abstract. This paper analyzes the effect of refuge on the dynamics of a Leslie-Gower predator-prey model in which one predator feeds on one of two competing species. Existence conditions for equilibrium points are discussed. By using differential inequality argument, we developed persistence criterion. Sufficient condition for global stability of the unique positive equilibrium point is derived. Different type of local bifurcation near the equilibrium points has been investigated. The role of refuges have been shown on equilibrium densities of prey, competitor for prey and predator respectively. The results establish the fact that the effects of refuges used by prey increase the equilibrium density of prey population under certain restrictions, whereas opposite hold for competitor of prey population. However equilibrium density of predator may decrease or increase by increasing the amount of prey refuge. Some numerical simulations are performed to validate the results obtained.

Keywords: Leslie-Gower model; refuge; global stability; persistence.

2010 AMS Subject Classification: 47H17, 47H05, 39A30, 37N25, 93A30, 93C10.

*Corresponding author

E-mail address: mukherjee1961@gmail.com

Received May 26, 2017

1. Introduction

There has been a growing interest in the study of refuges in predator-prey system. González-Olivares and Ramos-Jiliberto [6] studied a predator-prey system with Holling type-II functional response and a prey refuge. They showed that there is a trend from limit cycles through non-zero stable points up to predator extinction and prey stabilizing at high densities. Kar [9] investigated a Lotka-Volterra type predator-prey system incorporating a constant proportion of prey refuges with Holling type-II response function. He remarked that it is possible to break the cyclic behaviour of the system if harvesting effects as controls. Chen et al. [4] analysed the uniqueness of limit cycles and global stability of the unique positive equilibrium of predator-prey system with Holling type-II functional response and a constant number of refuges. Chen et al. [3], and Yue [22] studied Leslie-Gower predator-prey system incorporating a constant proportion prey refuge and showed the global stability at the interior equilibrium point. More results on the effects of a prey refuge can be found in [2, 5, 7, 10, 12, 14, 15, 16, 17, 19]. Previous studies on Leslie-Gower predator-prey system are mainly confined into constant proportion of refuge which acts on the system as an external decreasing of the uptake rate and half saturation constant, does not change the dynamical behaviour of the prey-predator model. Thus our main object in this work to modify the refuge term. Recently, Mukherjee [14] studied the effect of immigration and refuge on the dynamics of three species system. He discussed about the persistence of the system and global stability. Model considered by him is of Lotka-Volterra type. In another paper [16] Mukherjee investigated same type of situation without immigration and predation process follows Holling-type II response function. Both of the papers, he did not address what will be dynamical consequence if Leslie-Gower form is taken. Further we are interested to know the dynamics consequence of the predator-prey system in presence of a competitor for the prey in a Leslie-Gower model.

This paper is structured as follows. In Section 2, we propose our model. Some preliminaries (positivity and boundedness) is discussed in Section 3. In Section 4, we analyze our model with regard to equilibria and stabilities. Section 5, deals with the local bifurcation analysis around the equilibria. Persistence criterion is established in Section 6. Influence of prey refuge is given

in Section 7. Numerical simulations is presented in Section 8. A brief discussion is presented in Section 9.

2. Mathematical Model

In Leslie-Gower prey-predator model, predator equation is taken logistic growth with carrying capacity proportional to the prey density. This type of situation are applicable in ecology [11, 13, 20] because the direct conversion of prey density into offspring is inappropriate for a small mammalian predator that uses most of its energy intake on generating heat and because model of Leslie's type assume interferences of predators which is justifiable for territorial predators [20]. In this paper we introduce a predator-prey model with Leslie-Gower functional response incorporating a positive constant prey refuge with the presence of a competitor for the prey :

$$\begin{aligned}\frac{dx}{dt} &= x(r_1 - b_1x) - \alpha xy - a_1(x - m)z \\ \frac{dy}{dt} &= y(r_2 - b_2y) - \beta xy \\ \frac{dz}{dt} &= z\left(r_3 - \frac{a_2z}{k + x - m}\right)\end{aligned}\tag{1.1}$$

with initial conditions $x(0) > m, y(0) \geq 0, z(0) \geq 0$.

Here x, y, z denotes the density of the prey, competitor for the prey and predator respectively. r_1 is the intrinsic growth rate of the prey species and r_2 is the intrinsic growth rate of the competitor for the prey species. b_1 is the intraspecific competition coefficient of the prey. α denotes the interspecific competition coefficient of the competitor for the prey. b_2 represents the intraspecific competition coefficient of the competitor for the prey. β corresponds to the intraspecific competition coefficient of the competitor for the prey. r_3 describes the growth rate of predator. a_1 is the per capita predator consumption rate. a_2 is the efficiency with which predators convert consumed prey. m is the constant number of prey using refuge. k is the half saturation constant.

Specific example illustrates the above model: Consider two species of aphid (*Acyrtosiphon pisum* and *Megoura viciae*) competing for the same food plant and a species of a specialist parasitoid (*Aphidius ervi*) that attacks only one of the aphids (*A. pisum*). From experimental

studies van Veen et al. [21] showed that (i) when the two aphid species compete for resources in the absence of parasitoid, *A. pisum* excludes *M. viciae*. (ii) When the aphid species and the parasitoid are all present, all three species can coexist.

3. Preliminaries

1.1. Positivity

Lemma 1.1 All solution of system (1.1) with positive initial conditions are positive i.e $x(t) > 0, y(t) > 0, z(t) > 0$ for all $t \geq 0$ in the interval $[0, \infty)$.

Proof. Since the right hand side of system (1.1) is continuous and locally Lipschitzian on C , the solution $(x(t), y(t), z(t))$ of system (1.1) with initial conditions exists and is unique on $[0, \phi)$, where $0 < \phi \leq \infty$ [8].

From system (1.1), we have

$$\begin{aligned} x(t) &\geq x(0) \exp \left\{ \int_0^t (r_1 - b_1 x(\xi) - \alpha y(\xi) - a_1 z(\xi)) d\xi \right\} \geq 0 \\ y(t) &= y(0) \exp \left\{ \int_0^t (r_2 - b_2 y(\xi) - \beta x(\xi)) d\xi \right\} \geq 0 \\ z(t) &= z(0) \exp \left\{ \int_0^t (r_3 - \frac{a_2 z(\xi)}{k+x(\xi)-m}) d\xi \right\} \geq 0. \end{aligned}$$

Thus any trajectory starting in R_+^3 cannot cross the co-ordinate axes. This completes the proof.

1.2. Boundedness

Lemma 1.2. The set $B = \{(x, y, z) \in R_+^3 : 0 < W(t) = x + y + z \leq \frac{M}{\zeta}\}$ is a region of attraction for all solutions initiating in R_+^3 with positive initial conditions, where $M = \frac{(r_1 + \lambda)^2}{4b_1} + \frac{(r_2 + \lambda)^2}{4b_2} + \frac{\zeta(r_3 + \lambda)^2}{4a_2}$, $\zeta = \frac{b_1}{(r_1 + b_1(k - m))}$ provided $k > m$.

Proof. Let us define $W(t) = x + y + z$ and $\lambda > 0$ be a constant. Then

$$\begin{aligned}
 \frac{dW}{dt} + \lambda W &= x(r_1 - b_1x) - \alpha xy - a_1(x - m)z + \lambda x + y(r_2 - b_2y - \beta x + \lambda) + z(r_3 + \lambda - \frac{a_2z}{k+x-m}) \\
 &\leq x(r_1 - b_1x + \lambda) + y(r_2 - b_2y + \lambda) + z(r_3 + \lambda - \frac{a_2z}{k+x-m}) \\
 &\leq \frac{(r_1+\lambda)^2}{4b_1} + \frac{(r_2+\lambda)^2}{4b_2} + \frac{\zeta(r_3+\lambda)^2}{4a_2} = M, \text{ where } \zeta = \frac{b_1}{(r_1+b_1(k-m))}.
 \end{aligned}$$

By using differential inequality [1] we obtain,

$$0 < W(x(t), y(t), z(t)) \leq \frac{M(1-e^{-\zeta t})}{\zeta} + (x(0), y(0), z(0))e^{-\zeta t}$$

Taking limit $t \rightarrow \infty$, we have $0 < W(t) \leq \frac{M}{\zeta}$.

This proves the Lemma.

4. Equilibria

Evidently, system (1.1) has at most five equilibrium points: the trivial equilibrium point $E_0 = (0, 0, 0)$ which does not belongs to B . The axial equilibrium point $E_1 = (\frac{r_1}{b_1}, 0, 0)$. Planner equilibrium point $E_{12} = (\bar{x}, \bar{y}, 0)$ where $\bar{x} = \frac{r_1 b_2 - r_2 \alpha}{b_1 b_2 - \alpha \beta}$, $\bar{y} = \frac{r_2 b_1 - r_1 \beta}{b_1 b_2 - \alpha \beta}$. E_{12} is feasible if $b_1 b_2 > \alpha \beta$ and $r_1 b_2 > r_2 \alpha, r_2 b_1 > r_1 \beta$ or $b_1 b_2 < \alpha \beta$ and $r_1 b_2 < r_2 \alpha, r_2 b_1 < r_1 \beta$. Another planner equilibrium point $E_{13} = (\hat{x}, 0, \hat{z})$ where \hat{x} is the positive root of the equation

$$(a_2 b_1 + a_1 r_3)x^2 + (a_1 r_3 k - 2m a_1 r_3 - a_2 r_1)x + a_1 r_3 m(m - k) = 0. \quad (4.1)$$

and $\hat{z} = \frac{r_3(k+\hat{x}-m)}{a_2}$. The interior equilibrium point is given by $E^* = (x^*, y^*, z^*)$ where $y^* = \frac{r_2 - \beta x^*}{b_2}$, $z^* = \frac{r_3(k+x^*-m)}{a_2}$ and x^* is the positive root of the equation

$$(b_1 a_2 b_2 - \alpha \beta a_2 + r_3 b_2 a_1)x^2 - \{r_1 a_2 b_2 - \alpha r_2 a_2 - r_3 b_2 a_1(k - 2m)\}x - r_3 b_2 a_1 m(k - m) = 0. \quad (4.2)$$

E^* is feasible if $r_2 > \beta x^*, k + x^* > m$.

Theorem 1.1 (i) Equilibrium points E_1 and E_{12} are always unstable. (ii) E_{13} is locally asymptotically stable if $r_2 < \beta\hat{x}$.

Proof. Proof follows immediately by linearising around the equilibria.

Theorem 1.2. The interior equilibrium point E^* of system (1) is locally asymptotically stable if $(\frac{a_1 m z^*}{x^*} + b_1 x^*)b_2 \geq \alpha \beta x^*$.

Proof. The Jacobian matrix of system (1) at the equilibrium point E^* is given by

$$J(E^*) = \begin{pmatrix} -\frac{a_1 m z^*}{x^*} - b_1 x^* & -\alpha x^* & -a_1(x^* - m) \\ -\beta_2 y^* & -b_2 y^* & 0 \\ \frac{r_3^2}{a_2} & 0 & -r_3 \end{pmatrix}$$

The characteristic equation about E^* is given by

$$\lambda^3 + A_1 \lambda^2 + A_2 \lambda + A_3 = 0 \quad (4.3)$$

$$\text{where } \begin{cases} A_1 = \frac{a_1 m z^*}{x^*} + b_1 x^* + b_2 y^* + r_3, \\ A_2 = (\frac{a_1 m z^*}{x^*} + b_1 x^*)(b_2 y^* + r_3) + b_2 r_3 y^* - \alpha \beta x^* y^* + a_1(x^* - m)\frac{r_3^2}{a_2}, \\ A_3 = (\frac{a_1 m z^*}{x^*} + b_1 x^*)b_2 y^* r_3 - \alpha \beta x^* y^* r_3 + a_1(x^* - m)\frac{r_3^2}{a_2} b_2 y^* \end{cases}$$

Now $A_1 > 0, A_3 > 0$ follows from the assumption of the Theorem 1.2. Also $A_1 A_2 > A_3$. Therefore by Routh-Hurwitz criterion the result follows.

5. Local bifurcation analysis

In this section, we use the application of Sotomayor's theorem [18] to investigate the local bifurcation around the equilibrium points of system (1.1). As the existence of non-hyperbolic

equilibrium point is a necessary but not sufficient condition for bifurcation to occur therefore we choose a parameter which gives zero eigenvalues to the Jacobian at the equilibria. Now rewrite system (1.1) in the form :

$\frac{dX}{dt} = F(X)$ where $X = (x, y, z)^t$ and $F = (F_1, F_2, F_3)$ where $F_1 = x(r_1 - b_1x) - \alpha xy - a_1(x - m)z$, $F_2 = y(r_2 - b_2y) - \beta xy$ and $F_3 = z(r_3 - \frac{a_2z}{k+x-m})$. The local bifurcation near the equilibrium points is investigated in the following theorems:

Theorem 1.3. System (1.1) undergoes a transcritical bifurcation at the axial equilibrium point E_1 but no saddle node bifurcation can occur when the parameter β crosses the critical value $\beta^* = \frac{b_1r_2}{r_1}$.

Proof. One of the eigenvalues of the Jacobian matrix $J(E_1)$ will be zero if $\beta = \beta^* = \frac{b_1r_2}{r_1}$. Now the Jacobian matrix of system (1.1) at E_1 with zero eigenvalue is given by

$$J(E_1) = \begin{pmatrix} -r_1 & -\frac{\alpha r_1}{b_1} & -a_1(\frac{r_1}{b_1} - m) \\ 0 & 0 & 0 \\ 0 & 0 & r_3 \end{pmatrix}$$

Let $V = (v_1, v_2, v_3)^t$ be the eigenvector corresponding to eigenvalue $\lambda = 0$. Thus $V = (v_1, -\frac{v_1 b_1}{\alpha}, 0)^t$ where v_1 be any non zero real number. Also, let $W = (w_1, w_2, w_3)^t$ represents the corresponding eigenvector of $J(E_1)^t$ to the eigenvalues of $\lambda = 0$. Hence $J(E_1)^t W = 0$ gives that $W = (0, w_2, 0)^t$ where w_2 be any non zero real number. Now $F_\beta(E_1, \beta^*) = (0, 0, 0)^t$, here $F_\beta(E_1, \beta)$ represents the derivative of $F = (F_1, F_2, F_3)^t$ with respect to β . Then we get $W^t [F_\beta(E_1, \beta^*)] = 0$.

Thus according to Sotomayor's theorem system (1.1) has no saddle-node bifurcation at $\beta = \beta^*$.

Again

$$DF_\beta(E_1, \beta^*) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\frac{r_1}{b_1} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then, $W^t[DF_\beta(E_1, \beta^*)V] = -\frac{r_1 v_2 w_2}{b_1} \neq 0$.

Now

$$D^2F(E_1, \beta^*)(V, V) = \begin{pmatrix} (-2b_1 - \alpha - a_1)v_1^2 - \alpha v_1 v_2 - a_1 v_1 v_3 \\ -\beta v_1 v_2 - (\beta + 2b_2)v_2^2 \\ -\frac{2a_2}{k+x-m}v_3^2 \end{pmatrix}$$

Therefore, $W^t[D^2F(E_1, \beta^*)(V, V)] = \frac{b_1 k^2}{\alpha} [\beta - (\beta + 2b_2)b_1] \neq 0$.

Thus according to Sotomayor's theorem system (1.1) has a transcritical bifurcation at E_1 when the parameter β crosses the critical value β^* . Furthermore, as the Jacobian matrix of E_1 has three linear factors, so no Hopf bifurcation can occurs.

5.1. Numerical example for trnscritical bifurcation

Choose $r_1 = 12, b_1 = 10, \alpha = 2, a_1 = 2, m = 0.5, r_2 = 6, \beta = 5, a_2 = 2, b_2 = 1, r_3 = 1, k = 1.5$ then system (1.1) admits a transcritical bifurcation at $E_1(1.2, 0, 0)$ with respect to β .(see Fig. 1.)

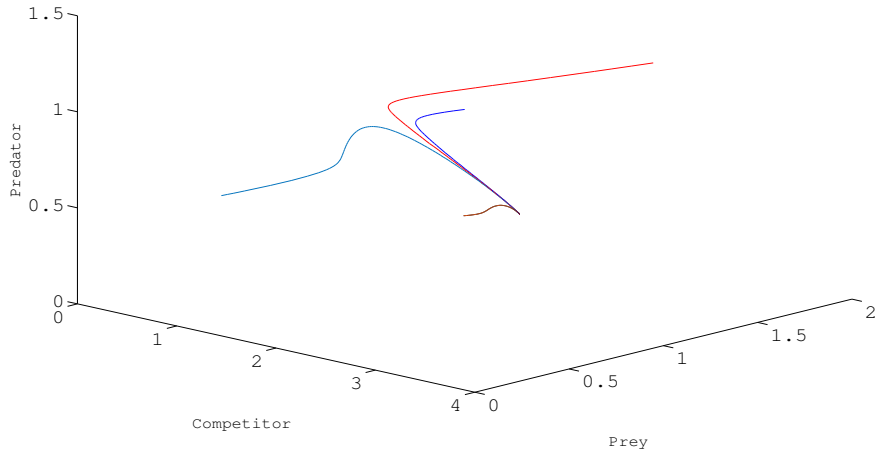


Fig. 1. Transcritical bifurcation near E_1

Remark 1: System (1.1) does not admits any local bifurcation (saddle-node, transcritical or

Hopf-bifurcation) at E_{12} as the Jacobian matrix $J(E_{12})$ has no zero eigenvalues due to the existence of E_{12} .

Theorem 1.4. System (1.1) admits a transcritical bifurcation but no saddle-node bifurcation around the equilibrium point E_{13} when r_2 crosses the critical value $r_2^* = \beta\hat{x}$.

Proof. Proof is similar to the proof of Theorem 1.3.

Remark 2: System (1) does not undergoes any Hopf-bifurcation around the interior equilibrium point E^* as in equation (4), $A_1 > 0$ and $A_1A_2 - A_3$ cannot be equal to zero.

Theorem 1.5. Suppose that $\frac{4b_1b_2a_2}{k+x^*-m} > \left\{ \frac{(\alpha+\beta)^2a_2}{k+x^*-m} + b_2 \left(\frac{(x^*-m)a_1}{x^*} + \frac{r_3^2}{k+x^*-m} \right) \right\}$ and $r_2 > \beta x^*$. Then the interior equilibrium point E^* is globally asymptotically stable.

Proof. First note that, E_{13} is unstable as $r_2 > \beta x^*$ and other boundary equilibrium points are always unstable whenever they exist.

Consider the following positive definite function about E^*

$$V(t) = (x - x^* - x^* \ln \frac{x}{x^*}) + (y - y^* - y^* \ln \frac{y}{y^*}) + (z - z^* - z^* \ln \frac{z}{z^*})$$

Differentiating V with respect to t along the solution of system (1.1), we get

$$\begin{aligned} \frac{dV}{dt} &= (x - x^*) \left\{ r_1 - b_1x - \alpha y - \frac{a_1(x-m)}{x} \right\} + (y - y^*) \left\{ r_2 - b_2y - \beta x \right\} + (z - z^*) \left\{ r_3 - \frac{a_2z}{k+x-m} \right\} \\ &= (x - x^*) \left\{ -b_1(x - x^*) - \alpha(y - y^*) + \frac{a_1(x^*-m)}{x^*} - \frac{a_1(x-m)}{x} \right\} + (y - y^*) \left\{ -b_2(y - y^*) - \beta(x - x^*) \right\} + \\ &\quad (z - z^*) \left\{ \frac{a_2z^*}{k+x^*-m} - \frac{a_2z}{k+x-m} \right\} \\ &\leq -b_1(x - x^*)^2 + (\alpha + \beta) |(x - x^*)| |(y - y^*)| - b_2(y - y^*)^2 - \frac{a_2(z - z^*)^2}{k+x^*-m} + \left\{ \frac{(x^*-m)a_1}{x^*} + \frac{r_3}{k+x^*-m} \right\} |x - x^*| |z - z^*| \end{aligned}$$

We note that \dot{V} is negative definite if

$$\frac{b_1 b_2 a_2}{k + x^* - m} > \frac{1}{4} \left[\frac{(\alpha + \beta)^2 a_2}{k + x^* - m} + b_2 \left\{ \frac{(x^* - m) a_1}{x^*} + \frac{r_3^2}{k + x^* - m} \right\} \right]$$

Thus the condition of Theorem 1.5 implies that V is a Lyapunov function and hence the theorem follows.

6. Persistence

Biologically persistence means the long time survival of all population in a future time whatever may be the initial populations. By differential inequality argument we state some result guaranteeing the persistence of all the populations of system (1.1).

Theorem 1.6. (i) If $x(t) > m$ then $\lim_{t \rightarrow \infty} \sup x(t) \leq \frac{r_1}{b_1}$ (ii) If $x(t) \leq m$ and $r_1 b_2 > \alpha r_2$ then $\lim_{t \rightarrow \infty} \inf x(t) \geq \frac{r_1 b_2 - \alpha r_2}{b_1 b_2}$

Proof.(i) When $x(t) > m$, $\frac{dx}{dt} \leq (r_1 - b_1 x)x \implies \lim_{t \rightarrow \infty} \sup x(t) \leq \frac{r_1}{b_1}$

(ii) When $x(t) \leq m$, $\frac{dx}{dt} \geq (r_1 - b_1 x)x - \alpha x \frac{r_2}{b_2} = (r_1 - \frac{\alpha r_2}{b_2} - b_1 x)x \implies \lim_{t \rightarrow \infty} \inf x(t) \geq \frac{r_1 b_2 - \alpha r_2}{b_1 b_2}$

Theorem 1.7. (i) If $x(t) > m$ and $r_2 > \frac{\beta r_1}{b_1}$ then $\lim_{t \rightarrow \infty} \inf y(t) \geq \frac{b_1 r_2 - \beta r_1}{b_1 b_2}$ (ii) If $x(t) \leq m$ and $r_2 > \beta m$ then $\lim_{t \rightarrow \infty} \inf y(t) \geq \frac{r_2 - \beta m}{b_2}$

Proof. when $x(t) > m$, $\lim_{t \rightarrow \infty} \sup x(t) \leq \frac{r_1}{b_1}$ then from the second equation of (1) we have $\frac{dy}{dt} \geq (r_2 - \frac{\beta r_1}{b_1} - b_2 y)y \implies \lim_{t \rightarrow \infty} \inf y(t) \geq \frac{b_1 r_2 - \beta r_1}{b_1 b_2}$.

(ii) If $x(t) \leq m$, then $\frac{dy}{dt} \geq (r_2 - \beta m - b_2 y)y$.

As, $r_2 > \beta m$, this implies that $\lim_{t \rightarrow \infty} \inf y(t) \geq \frac{r_2 - \beta m}{b_2}$

Theorem 1.8. If $k > m$ then $\lim_{t \rightarrow \infty} \inf z(t) \geq \frac{r_3(k-m)}{a_2}$.

Proof. Since $k > m$ then $k + x - m > k - m$ and hence $-\frac{1}{k+x-m} > -\frac{1}{k-m}$.

From third equation of (1.1), we have $\frac{dz}{dt} \geq z(r_3 - \frac{a_2 z}{k-m}) \implies \lim_{t \rightarrow \infty} \inf z(t) \geq \frac{r_3(k-m)}{a_2}$.

7. Influence of prey refuge

In the following we shall discuss the influence of prey refuge on each population when the coexistence equilibrium point E^* is exists and is stable. It is easy to see that x^*, y^*, z^* are all continuous differential functions of parameter m .

Now let α be any positive root of equation (4.2).

Then $\alpha = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$ where

$$A = b_1 a_2 b_2 - \alpha \beta a_2 + r_3 b_2 a_1, B = -\{r_1 a_2 b_2 - \alpha r_2 a_2 - r_3 b_2 a_1 (k - 2m)\}, C = -r_3 b_2 a_1 m (k - m).$$

Now $\frac{d\alpha}{dm} = -\frac{dB}{dm} + \frac{1}{2} \frac{2B \frac{dB}{dm} - 4A \frac{dC}{dm}}{\sqrt{B^2 - 4AC}} > 0$ provided,

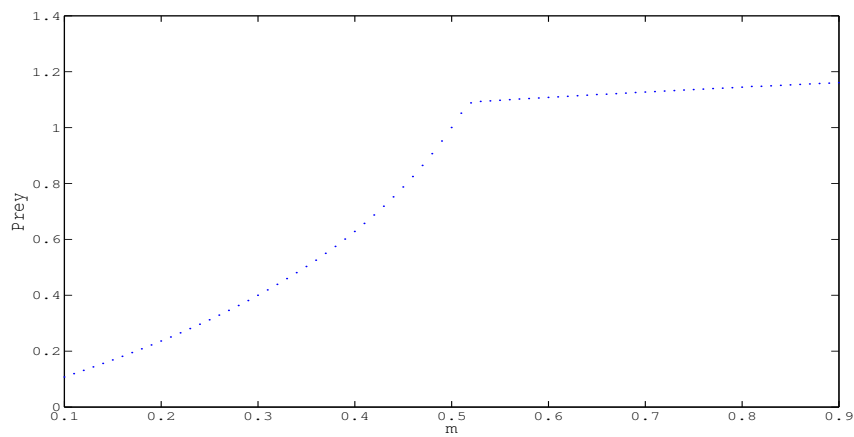
$$\frac{a_2}{a_1} > \frac{r_3}{b_1} \text{ and } \min \left\{ \frac{\alpha \beta a_2}{a_1 r_3}, \frac{r_2 a_2}{r_1 a_1} \right\} < b_2 < \frac{\alpha \beta}{b_1}. \quad (7.1)$$

Again $\frac{dy^*}{dm} = -\frac{\beta}{b_2} \frac{dx^*}{dm} < 0$ and $\frac{dz^*}{dm} = \frac{r_3}{a_2} \left(\frac{dx^*}{dm} - 1 \right)$.

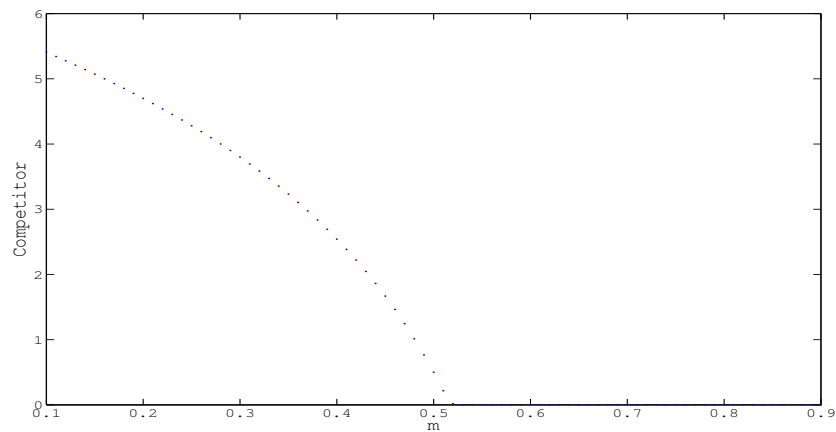
Clearly x^* is strictly increasing function of parameter m whenever (7.1) holds and increasing the amount of prey refuge leads to the increasing density of the prey species. y^* is strictly decreasing function of parameter m and increasing the amount of prey refuge leads to the decreasing density of the competitor prey species. The presence of negative term in the third equation indicates that increasing the amount of prey refuge may decrease the predator density as long as $\frac{dx^*}{dm} < 1$.

7.1 Numerical example for influence of prey refuge

Here we choose a set of parameters $r_1 = 12, b_1 = 10, \alpha = 2, a_1 = 2, m = 0.5, r_2 = 6, \beta = 5.5, a_2 = 2, b_2 = 1, r_3 = 1, k = 1.5$ and in this case interior equilibrium point E^* is locally asymptotically stable. Influence of prey refuge on susceptible and infected prey population is given in Fig. 2. and Fig. 3.



(a)



(b)

Fig. 2. Influence of prey refuge on (a) susceptible prey population and (b) infected prey population

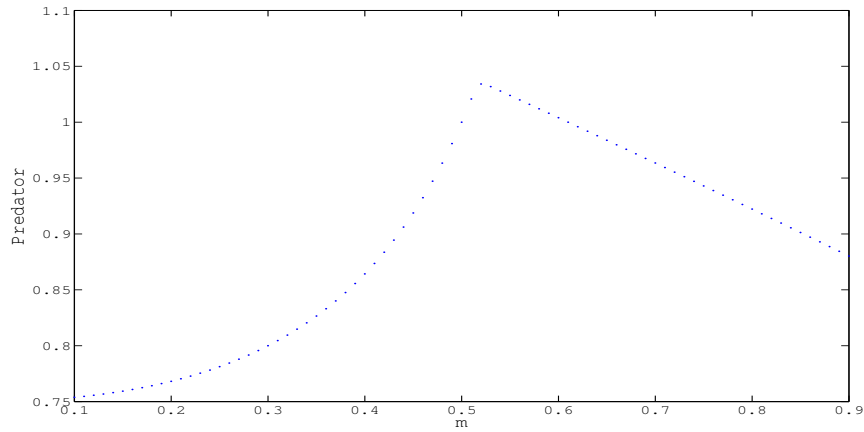


Fig. 3. Influence of prey refuge on predator population

8. Numerical Simulation

Dynamical behaviour of Leslie-Gower predator-prey model is not affected by refuge. If interspecific competition is allowed into the system, oscillation can emerge. Keeping this in mind, we have taken the rate of interspecific competition low and high. We select $r_1 = 12, b_1 = 10, \alpha = 2, a_1 = 2, m = 0.5, r_2 = 6, \beta = 5.5, a_2 = 2, b_2 = 1, r_3 = 1, k = 1.5$. Our numerical result shows that system (1) converges to this point $E^*(1, 0.5, 1)$ (see Fig. 4.).

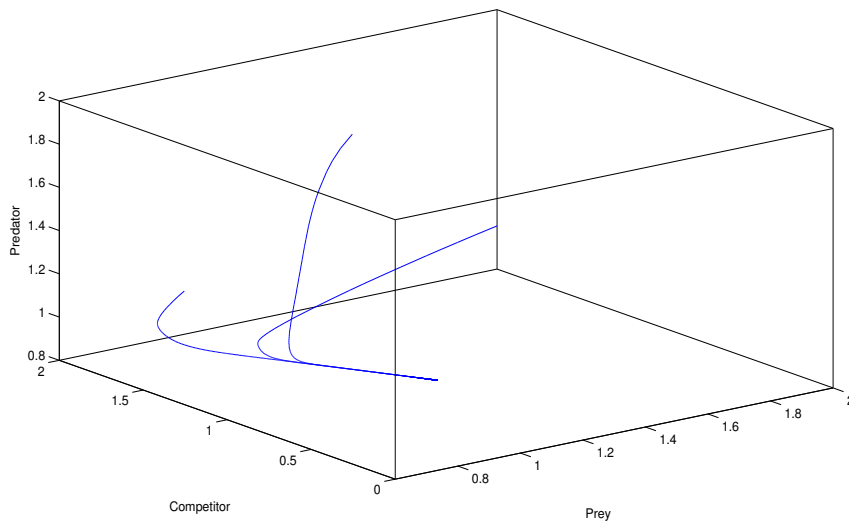


Fig. 4. The figure shows that system (1.1) is locally stable

Further we considered $r_1 = 4, b_1 = 1, \alpha = 1.2, a_1 = 2, m = 0.5, r_2 = 6, \beta = 5, a_2 = 1.5, b_2 = 1, r_3 = 2, k = 0.25$, the system oscillates near this equilibrium point $E^*(\frac{5}{7}, \frac{17}{7}, \frac{13}{21})$. Also some chaotic type oscillation is observed (see Fig. 5.).

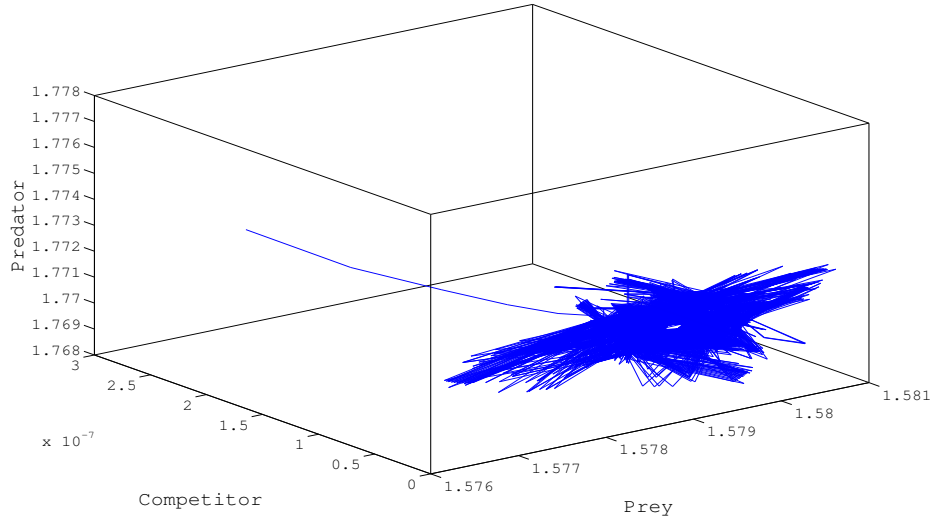


Fig. 5. The system (1.1) is unstable and chaotic type oscillation observed

In the first case $\alpha\beta = 11$ and $b_1b_2 = 10$ that indicates interspecific competition is low. In the second case $\alpha\beta = 6$ and $b_1b_2 = 1$ that implies interspecific competition is high i.e all the equilibrium points are unstable in nature. That causes chaotic type motion.

9. Discussion

In this paper we have considered a prey-predator system where prey has a competitor. Due to predation pressure prey population uses refuge mechanism. The model is formulated according to Leslie-Gower. This type of model usually exhibits stable behaviour with or without refuge. But the dynamics of the model may be changed if the interspecific competition is allowed. We have thoroughly investigated bifurcation analysis around the equilibria. We note that certain parameters of the system are very sensitive to give transcritical bifurcation. We further observe that Hopf-bifurcation cannot occur around the equilibria which is either interior or boundary.

Thus our system is either stable or unstable around the coexistence equilibrium point. We have found five possible equilibria, namely trivial equilibrium point E_0 , axial equilibrium point E_1 , predator free equilibrium point E_{12} , competition free equilibrium point E_{13} and coexistence equilibrium point E^* . Here the boundary equilibrium points E_0, E_1, E_{12} are always unstable in nature where as E_{13} may be stable when the intrinsic growth rate of competitor remains below certain threshold value ($r_2 < \beta \hat{x}$). Local stability at the coexistence equilibrium point can be checked from the condition of Theorem 1.2. We observed that from the numerical simulation that chaotic motion can arise if the condition of Theorem 1.2 is violated. Further we have derived a sufficient condition for global stability condition of the coexistence equilibrium point. By using differential inequality argument we found persistence condition of the population. The novelty of our paper is the occurrence of transcritical bifurcation but no Hopf-bifurcation around the equilibria. Though Mukherjee [14, 16] showed Hopf-bifurcation in his system and did not carried out local bifurcation analysis.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

- [1] G. Birkhoff, C.G. Rota, Ordinary Differential Equation, Ginn and Co., Boston (1982).
- [2] R. Cressman, J. Garay, A predator-prey refuge system: Evolutionary stability in ecological systems, *Theor. Popula. Biol.* 76 (2009), 248-257.
- [3] F.D. Chen, L.I. Chen, X.I. Xie, On a Leslie-Gower predator-prey model incorporating a prey refuge. *Nonlinear Anal., Real World Appl.* 10 (2009), 2905-2908.
- [4] L.S. Chen, F.D. Chen, L.J. Chen, Qualitative analysis of a predator-prey model with Holling type II functional response incorporating a constant prey refuge. *Nonlinear Anal., Real World Appl.* 11 (2010), 246-252.
- [5] L.S. Chen, F.D. Chen, Global analysis of a harvested predator-prey model incorporating a constant prey refuge. *Int. J. Biomath.* 3 (2010), 205-223.
- [6] E. González-Olivares, R. Ramos-Jiliberto, Dynamic consequence of prey refuges in a simple model system: More prey, fewer predators and enhanced stability. *Ecol. Mod.* 166 (2003), 135-146.
- [7] L.L. Ji, C.Q. Wu, Qualitative analysis of a predator-prey model with constant-rate prey harvesting incorporating a constant prey refuge. *Nonlinear Anal., Real World Appl.* 11 (2010), 2285-2295.
- [8] J.K. Hale, Theory of functional differential equations. Springer, New York (1977).

- [9] T.K. Kar, Stability analysis of a prey-predator model incorporating a prey refuge. *Commun. Nonlinear Sci. Numer. Simul.* 10 (2005), 681-691.
- [10] T.K. Kar, Modelling and analysis of a harvested prey-predator system incorporating a prey refuge. *J. Comput. Appl. Math.* 185 (2006), 19-33.
- [11] P.H. Leslie, J.C. Gower, The properties of a stochastic model for the predator-prey type of interaction between two species, *Biometrika* 47 (1960), 219-234.
- [12] X. Liu, M.A. Han, Chaos and Hopf bifurcation analysis for a two species predator-prey system with prey refuge and diffusion. *Nonlinear Anal., Real World Appl.* 12 (2011), 1047-1061.
- [13] R.M. May, J.R. Beddington, C.W. Clark, S.J. Holt, R.M. Laws, Management of multi-species fisheries, *Science* 205 (1979), 267-277.
- [14] D. Mukherjee, The effect of refuge and immigration in a predator-prey system in the presence of a competitor for the prey. *Nonlinear Anal., Real World Appl.* 31 (2016), 277-287.
- [15] D. Mukherjee, The effect of prey refuges on a three species food chain model. *Differ. Eqns. Dyn. Sys.* 22 (4) (2014), 413-426.
- [16] D. Mukherjee, Study of refuge use on a predator-prey system with a competitor for the prey. *Int. J. Biomath.*, (10)2 (2017), Article ID 1750023.
- [17] D. Mukherjee, Bifurcation analysis of a Holling type II predator-prey model with refuge. *Int. J. Bifurcation Chaos* (2017), in press.
- [18] J. Sotomayor, Generic bifurcations of dynamical systems. In : Peixoto, M.M.(eds.) *Dyn. Sys.*, Academic Press, New York (1973), 549-560. .
- [19] Y.D. Tao, X. Wang, X.Y. Song, Effect of prey refuge on a harvested predator-prey model with generalised functional response. *Commun. Nonlinear Sci. Numer. Simul.* 16 (2011), 1052-1059.
- [20] P. Turchin, I. Hanski, An empirically based model for latitudinal gradient in vole population dynamics, *Am Nat.* 149 (1997), 842-874.
- [21] F.J.F. van Veen, P.D. van Holland, H.C.J. Godfrey, Stable coexistence in experimental insect communities due to density and trait mediated indirect effects, *Ecology* 12 (2005), 241-245.
- [22] Q. Yue, Dynamics of a modified Leslie-Gower predator-prey model with Holling-type II schemes and prey refuge. *Springer Plus* 5 (2016), Article ID 461.