A GEOMETRIC PROGRAMMING APPROACH FOR OPTIMAL RESOURCE ALLOCATION TO CONTROL EPIDEMIC OUTBREAKS IN ARBITRARY NETWORKS

WENWEN ZHANG[1], SHAOKUN LU[1,2], YONGZHEN PEI[1,2,*]

[1]School of Science, Tianjin Polytechnic University, Tianjin, 300387, China

[2]School of Computer Science and Software Engineering, Tianjin Polytechnic University, Tianjin, 300387, China

Communicated by A. Elaiw

Abstract. This paper proposed a control-strategies for nodes to control the spread of an epidemic outbreak in arbitrary directed graphs by optimally allocating their resources throughout the network. Epidemic propagation is well modeled as a networked version of the Susceptible-Exposed-Infected-Susceptible (SEIS) epidemic process. Using the Kolmogorov forward equations and mean-field approximation, we present a mean-field model to describe the spreading dynamics and prove the existence of a necessary and sufficient condition for global exponential stability. Based on this stability condition, we can derive another condition to control the spread of an epidemic outbreak in terms of the eigenvalues of a matrix that depends on the network structure and the parameters of the model. According to different control purposes and conditions, two types of control-theoretic decision can be considered: 1)given a fixed budget, find the optimal resource allocation to achieve the highest level of containment, 2)given a decay rate of epidemic, find the minimum cost to control the spreading process at a desired decay rate. A geometric program can be formulated to solve the optimal problems and the existence of solutions is also proved. Numerical simulations can illustrated our results.

Keywords: Markov processes; arbitrary directed graphs; mean field theory; geometric program.

2010 AMS Subject Classification: 60J20, 05C20.

*Corresponding author

E-mail addresses: yongzhenpei@163.com

1. Introduction

Development of strategies to control spreading processes in networks has brought much attention due to its applications in many relevant fields, including computer virus [1], public health [2-3], and information propagation over social network [4]. The dynamic of spreading processes in networks not only depend on the epidemic model but also the structure of the contact network. In this context, the spread process is modeled by a variant of SIS epidemic model that includes a state of "expose". The individual infected rate and recovery rate can be modified within a feasible range by allocating resources in each node. Based on this model, an efficient convex framework can be formulated to solve the optimal problems.

The dynamic behavior of spreading processes in networks has been widely studied. Mean-field models over arbitrary contact graphs were brought to the forefront in both continuous time [5] and discrete-time. [6] is a paper about the dynamic behavior of spreading processes in arbitrary contact networks for the case of discrete-time dynamic. [7] consider a continuous time SIS model over arbitrary graphs using mean field theory and provide a condition for globally asymptotically stable of the disease-free state. It should be noted that most models are developed base on undirected graphs, however, in practice, directed graphs may be more appropriate to the spread of diseases in human populations. Therefore, we study the spread processes in an arbitrary directed network of heterogeneous nodes.

Designing control strategies of spread processes in networks is a significant problem. Several papers have proposed some methods on different aspects of this problem. In [8], the authors proposed a semidefinite programming (SDP) to find the optimal strategies of resource allocation in an undirected network. [9] uses a linear-fractional optimization program to find the optimal investment on disease awareness in a social network. In [10], Borgs et al. provides a probabilistic analysis for the case of a given contact network to characterize the optimal allocation of a fixed amount of antidote. [11] provided a eigenvalue sensitivity analysis ideas to design optimal strategies of allocation resource to control the spread of a virus. The relationship between the recovery parameters and distributed approach is explored in [12]. Our work is based on [13] and [14], in which a continuous-time time Markov processes, called the N-intertwined model, is used to analyze and control the spread of a SEIV epidemic model.

This paper's layout is as follows. In section 2, some important notation and background need to be introduced. In section 3, we formulate two resource allocation problems for epidemic propagation in the network. In section 4, a convex optimization framework is used to efficiently solve the allocation problems both on strongly connected graphs and general directed graphs (not necessarily strongly connected). In section 5, simulations illustrate our results.

2. Notation and preliminaries

Some graph-theoretical nomenclature and the dynamic spreading model need to be introduced in this section.

2.1 Graph Theory

Let $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ denote an directed graph, where $\mathscr{V} = \{v_1, \cdot, v_n\}$ denote the set of n nodes and $\mathscr{E} \subseteq \mathscr{V} \times \mathscr{V}$ denote the set of ordered pairs of nodes called directed edges. An edge from e_j pointing towards e_i denote as (e_j, e_i). Let $\mathscr{N}_i^{in} = \{j : (e_j, e_i) \in \mathscr{E}\}$ denote the in-neighborhood of node i. A directed graph is strongly connected if and only if there is a directed path from e_j to e_i for every pair of nodes $e_i, e_j \in \mathscr{V}$.

The adjacency matrix of a digraph $\mathscr{G}$, denoted by $U_{\mathscr{G}} = [u_{ij}]$, is an $n \times n$ matrix defined entry-wise as $u_{ij} = 1$ if edge $(e_i, e_j) \in \mathscr{E}$, and $u_{ij} = 0$ otherwise. Hence, the adjacency matrix $U_{\mathscr{G}} = [u_{ij}]$ is always nonnegative. An nonnegative adjacency matrix is irreducible if and only if its associated graph is strongly connected. Given an $n \times n$ matric Z, the sets of eigenvalues and corresponding eigenvectors of Z are defined as $\lambda_1(Z), \cdot, \lambda_n(Z)$ and $\eta_1(Z), \cdot, \eta_n(Z)$, where we order eigenvalues in decreasing order of their real parts, i.e., $\mathfrak{R}(\lambda_1) \geq \mathfrak{R}(\lambda_2) \geq \cdot \geq \mathfrak{R}(\lambda_n)$. $\lambda_1(Z)$ and $\eta_1(Z)$ are called the dominant eigenvalue and eigenvector of Z. $\rho(Z)$ is defined as the spectral radius of Z and equal to the maximum modulus across all eigenvalues of Z.

It should be note that we only consider unweighted digraphs in this paper, hence, the adjacency matrix is always nonnegative.

2.2 Stochastic heterogeneous SEIS model

The Susceptible-Exposed-Infected-Susceptible epidemic model is a variant of the SIS model by including a state of "exposed" which the node has been exposed to the disease and is contagious, but is not aware of the contagion. This model is a continuous-time markov process and each node in the network can be in one out of three possible states. Consider a network of n individuals described by the adjacency matrix $U_{\mathscr{G}} = [u_{ij}]$, and the parameters are defined as follows:

β_i : infection rate that the susceptible node i transitions to the exposed state by contact with its exposed neighbors.

α_i : infection rate that the susceptible node i become exposed by contact with its infected neighbors.

γ_i : the rate at which the exposed node i be infected.

δ_i : the recovery rate of node i.

Let $v_k(\tau) \in \{0,1\}$ and $v_j(\tau) \in \{0,1\}$. Where $v_k(\tau) = 1$ indicates that node k is in the infected sate and other states of node k at time τ denote as $v_k(\tau) = 0$. Similarly, $v_j(\tau) = 1$ indicates that node j is in the exposed sate, and $v_j(\tau) = 0$ otherwise. Three possible types of stochastic transitions during the time interval $[\tau + \Delta\tau)$:

a): Assuming node i is in the susceptible state at time τ. This node can switch to the exposed state during the small time interval $[\tau, \tau + \Delta\tau)$ with a probability that depends on: (i) its infection rates β_i and α_i; (ii) the strength of its incoming connections $\{u_{ij}, \text{for } j \in \mathscr{N}_i^{in}\}$; (iii) the states of its in-neighbors $\{v_j(\tau), \text{for } j \in \mathscr{N}_i^{in}\}$ and $\{v_k(\tau), \text{for } k \in \mathscr{N}_i^{in}\}$. Formally, the probability of this transition is given by

$$P(v_i(\tau + \Delta\tau) = e \mid v_i(\tau) = s, V(\tau)) = \beta_i \sum_{j \in \mathscr{N}_i^{in}} u_{ij} v_j(\tau) \Delta\tau + \alpha_i \sum_{k \in \mathscr{N}_i^{in}} u_{ij} v_k(\tau) \Delta\tau + o(\Delta\tau),$$

where $\Delta\tau > 0$ is an arbitrarily small time interval, and $V(\tau) = (v_i(\tau), i = 1, \cdot, n)$.

b): Assuming node i in exposed state, the probability of i transit to infected in the time interval $[\tau, \tau + \Delta\tau)$ is given by

$$P(v_i(\tau + \Delta\tau) = i \mid v_i(\tau) = e, V(\tau)) = \gamma_i \Delta\tau + o(\Delta\tau).$$

c): Assuming node i is infected, the probability of i recovering back to the susceptible state in the time interval $[\tau, \tau + \Delta\tau)$ is given by

$$P(v_i(\tau + \Delta\tau) = s \mid v_i(\tau) = i, V(\tau)) = \delta_i\Delta\tau + o(\Delta\tau).$$

The Markov process with 3^n states described in the above is very hard to analyze due to the exponential size of the state space. Therefore, we use a mean-field approximation of its dynamics. This approximation is widely used in the field of epidemic analysis and control, since it performs numerically well for many realistic network topologies.

Let h_i and g_i denote the probabilities of node i to be exposed and infected, respectively. Then, the probabilities of node i to be susceptible is $1 - h_i - g_i$. Using the Kolmogorov forward equations and a mean-field approach, one can approximate the dynamics of the epidemic spread using a system of $2n$ ordinary differential equations, as follows:

$$\dot{h}_i = (1 - h_i - g_i)(\beta_i \sum_{j \in \mathcal{N}_i^{in}} u_{ij}h_j + \alpha_i \sum_{j \in \mathcal{N}_i^{in}} u_{ij}g_j) - \gamma_i h_i, \tag{2.1}$$

$$\dot{g}_i = \gamma_i h_i - \delta_i g_i, \tag{2.2}$$

for $i = 1, \cdot, n$.

We can write the mean-field approximation equations in matrix form as

$$\begin{bmatrix} \dot{H} \\ \dot{G} \end{bmatrix} = \begin{bmatrix} BU - R & CU \\ R & -E \end{bmatrix} \begin{bmatrix} H \\ G \end{bmatrix} - \begin{bmatrix} F \\ 0 \end{bmatrix}, \tag{2.3}$$

where $H = (h_1, \cdot, h_n)^T$, $G = (g_1, \cdot, g_n)^T$, $B = diag(\beta_i)$, $C = diag(\alpha_i)$, $R = diag(\gamma_i)$, $E = diag(\delta_i)$, $F = (f_1, \cdot, f_n)^T$, $f_i = (h_i + g_i)(\beta_i \sum_{j \in \mathcal{N}_i^{in}} u_{ij}h_j + \alpha_i \sum_{j \in \mathcal{N}_i^{in}} u_{ij}g_j)$.

Proposition 1 Consider the heterogeneous SEIS epidemic model in (2.3) and assume that $U_{\mathcal{G}} \geq 0$ and $\beta_i, \alpha_i, \gamma_i, \delta_i > 0$. Letting $\mathfrak{L} = \begin{bmatrix} BU - R & CU \\ R & -E \end{bmatrix}$. Then, if the largest real part of the eigenvalues of $\mathfrak{L}$ satisfies

$$\Re[\lambda_1(\mathfrak{L})] \leq -\varsigma, \tag{2.4}$$

for some $\varsigma > 0$, the disease-free equilibrium is globally exponentially stable, and ς is called exponential decay rate.

Proof: Define $\mathscr{X} = [H \ G]^T$ and $\mathscr{F} = [F \ 0]^T$, we are able to rewrite the nonlinear system (2.3) as

$$\dot{\mathscr{X}} = \mathfrak{L}\mathscr{X} - \mathscr{F}.$$

Since $\dot{\mathscr{X}} = \mathfrak{L}\mathscr{X} - \mathscr{F} \leq \mathfrak{L}\mathscr{X}$, the linear dynamic system

$$\dot{\mathscr{X}} = \mathfrak{L}\mathscr{X}$$

upper-bounds the nonlinear dynamical system (2.3). Therefore, the spread rate of the nonlinear system can be bound by controlling the maximum eigenvalue of $\mathfrak{L}$. The spectral result of $\mathfrak{L}$ is a sufficient condition for the mean-field approximation of the SEIS model to be globally exponentially stable.

Lemma 1.1. *Suppose that E is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for all $n \in N$. Suppose further that $\{x_n\}$ and $\{y_n\}$ are sequences of E such that*

$$\limsup_{n \to \infty} \|x_n\| \leq r, \limsup_{n \to \infty} \|y_n\| \leq r,$$

$$\lim_{n \to \infty} \|t_n x_n + (1 - t_n) y_n\| = r$$

hold for some $r \geq 0$, then $\lim_{n \to \infty} \|x_n - y_n\| = 0$.

Lemma 1.2. *Let $\{r_n\}$, $\{s_n\}$ and $\{t_n\}$ be three nonnegative sequences satisfying the following conditions*

$$r_{n+1} \leq (1 + s_n) r_n + t_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n < \infty$ and $\sum_{n=1}^{\infty} t_n < \infty$, then $\lim_{n \to \infty} r_n$ exists.

3. Problem formulation

To control the spread of an epidemic in a given network, the work of proposing an efficient optimization framework to find the optimal resource distribution is very significant. In this paper, we consider two types of resources:

1) preventive resource (e.g. vaccines to reduce infection rate β_i, α_i),

2) corrective resources (e.g. antidotes to help increase recovery rates δ_i).

We can distribute resources to modify the parameters β, α, δ, and γ, within the feasible ranges as follows:

$$0 < \underline{\beta}_i \leq \beta_i \leq \bar{\beta}_i, \quad 0 < \underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i,$$

$$0 < \underline{\gamma}_i \leq \gamma_i \leq \bar{\gamma}_i, \quad 0 < \underline{\delta}_i \leq \delta_i \leq \bar{\delta}_i,$$

where $\underline{\beta}_i$ and $\underline{\alpha}_i$ are the minimum possible infection rates for node i. They can be achieved by allocating a large amount of vaccines at node i. $\bar{\beta}_i$ and $\bar{\alpha}_i$ are the maximal infection rates without any preventive resource for node i. Similarly, the $\underline{\delta}_i$ is a natural recovery rate of node i, and $\bar{\delta}_i$ is the maximal recovery rate which is achieved with enough corrective resources. For convenience, let $\nabla = \{\beta, \alpha, \gamma, \delta\}$ denote as the global set of parameters.

3.1. The cost of preventive and corrective resources

We define three cost functions, the vaccination cost function $f_i(\beta_i)$, $g_i(\alpha_i)$ and treatment cost function $h_i(\delta_i)$. The cost functions are node-dependent and present the following properties:

(1) Assuming the vaccination cost $f_i(\beta_i)$ and $g_i(\alpha_i)$ are monotonically decreasing in the interval $[\underline{\beta}_i, \bar{\beta}_i]$, $[\underline{\alpha}_i, \bar{\alpha}_i]$, respectively. Antidote cost $g_i(\delta_i)$ is monotonically increasing with regard to δ_i.

(2) In the absence of investment, $f_i(\bar{\beta}_i) = g_i(\bar{\alpha}_i) = h_i(\underline{\delta}_i) = 0$.

Apart from the above properties, we assume the cost functions have the following forms to obtain a tractable convex framework.

$$f_i(\beta_i) = \frac{\beta_i^{-1} - \bar{\beta}_i^{-1}}{\underline{\beta}_i^{-1} - \bar{\beta}_i^{-1}}, \quad g_i(\alpha_i) = \frac{\alpha_i^{-1} - \bar{\alpha}_i^{-1}}{\underline{\alpha}_i^{-1} - \bar{\alpha}_i^{-1}}, \quad h_i(\delta_i) = \frac{(1-\delta_i)^{-1} - (1-\underline{\delta}_i)^{-1}}{(1-\bar{\delta}_i)^{-1} - (1-\underline{\delta}_i)^{-1}}.$$

Note that we have normalized these cost functions to have values in the interval $[0, 1]$. Therefore, the cost functions of preventive resource, $f_i(\beta_i)$, $g_i(\alpha_i)$ and the corrective resource cost, $h_i(\delta_i)$, are twice differentiable and satisfies the following constrain:

$$f_i''(\beta_i) \geq -\frac{2}{\beta_i} f_i'(\beta_i),$$

$$g_i''(\alpha_i) \geq -\frac{2}{\alpha_i} g_i'(\alpha_i),$$

$$h_i''(\delta_i) \geq -\frac{2}{1-\delta_i} h_i'(\delta_i).$$

Notice that, since f_i, g_i are monotonically decreasing, h_i is monotonically increasing, we have that $f_i' < 0$, $g_i' < 0$ and $h_i' > 0$. The results implies that $f_i'' > 0$, $g_i'' > 0$ and $h_i'' < 0$. Therefore, the assumption is stronger than convexity.

3.2. Problem statements

In this section we present two types of resource allocation problems for SEIS model:

(1) the budget-constrained allocation problem. Our aim is finding the optimal allocation of vaccination and antidotes to maximize the exponential decay rate ς when given the total budget $T > 0$.

(2) the rate-constrained allocation problem. Given the exponential decay rate ς, find the cost-optimal distribution of vaccines and antidotes to eradicate the disease with decay rate greater than equal to ς.

Problem 1(Budget-constrained allocation) Given the total budget T, this problem can be state as the following optimization problem:

$$\max_{\nabla} \; \varsigma \tag{3.1}$$

$$\text{s.t.} \; \Re[\lambda_1(\mathcal{L})] \leq -\varsigma, \tag{3.2}$$

$$\sum_{i=1}^{n} f_i(\beta_i) + g_i(\alpha_i) + h_i(\delta_i) \leq T, \tag{3.3}$$

$$\underline{\beta}_i \leq \beta_i \leq \bar{\beta}_i, \tag{3.4}$$

$$\underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i, \tag{3.5}$$

$$\underline{\delta}_i \leq \delta_i \leq \bar{\delta}_i,$$

$$\underline{\gamma}_i \leq \gamma_i \leq \bar{\gamma}_i, \tag{3.6}$$

for all $i = 1, \cdot, n$, where (3.3) is the budget constraint.

Problem 2 (Rate-constrained allocation) Given a desire decay rate $\tilde{\varsigma}$, the rate-constrained allocation is formulated as follows:

$$\min_{\nabla} \; \sum_{i=1}^{n} f_i(\beta_i) + g_i(\alpha_i) + h_i(\delta_i) \tag{3.7}$$

$$\text{s.t. } \Re[\lambda_1(\mathfrak{L})] \leq -\tilde{\varsigma}, \tag{3.8}$$

$$\underline{\beta}_i \leq \beta_i \leq \bar{\beta}_i, \tag{3.9}$$

$$\underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i, \tag{3.10}$$

$$\underline{\delta}_i \leq \delta_i \leq \bar{\delta}_i,$$
$$\underline{\gamma}_i \leq \gamma_i \leq \bar{\gamma}_i, \tag{3.11}$$

for all $i = 1, \cdot, n$, where (3.8) constrains the decay rate to $\tilde{\varsigma}$.

In the following section we propose an approach to solve these problems in polynomial time.

4.A convex framework for optimal resource allocation

A convex formulation can be use to solve both the rate-constrained allocation problem and the budget-constrained problem in unweighted, directed networks using GP. We first solve problems in the strongly connected digraphs, then extend the results to general digraphs.

Some important concepts need to be briefly reviewed. Denote n decision variables $\xi = (\xi_1, \cdot, \xi_n)$, $\xi_i > 0$. In the context of geometric programs, a monomial function $h(\xi)$ is defined as a real-valued function of the form $h(\xi) = a\xi_1^{k_1}\xi_2^{k_2} \cdot \xi_n^{k_n}$, where $a > 0$ and $k_i \in \mathbb{R}$ for all $i = 1, \cdot, n$. The sum of monomials is defined as the polynomial function, i.e., $q(\xi) = \sum_{i=1}^{I} a_i\xi_1^{k_1,i}\xi_2^{k_2,i} \cdot \xi_n^{k_n,i}$ with $a_i > 0$ and $k_{j,i} \in \mathbb{R}$ for all $j \in \{1, \cdot, n\}$ and $i \in \{1, \cdot, I\}$.

A geometric problem is an optimization problem of the form:

$$\text{minimize } f(\xi)$$
$$\text{subject to } q_i(\xi) \leq 1, \quad i = 1, \cdot, m, \tag{4.1}$$
$$h_i(\xi) = 1, \ i = 1, \cdot, p,$$

where $f(\xi)$ and $q_i(\xi)$ are polynomial functions, $h_i(\xi)$ are monomials.

Note that polynomials and monomials are convex in log-scale, therefore f is a convex function in log-scale. The quasiconvex optimization problem GP can be transformed to a convex problem based on a logarithmic change of variable $\varphi_i = log\xi_i$, and a logarithmic transformation

of the objective and constraint functions. After this transformation, the GP in (4.1) takes the form as follows:

$$\text{minimize } F(\varphi)$$
$$\text{subject to } Q_i(\varphi) \leq 0, \ i = 1, \cdots, m, \tag{4.2}$$
$$c_i^T \varphi + \log b_i = 0, \ i = 1, \cdot, p,$$

where $F(\varphi) = \log f(\exp \varphi)$ and $Q_i(\varphi) = \log q_i(\exp \varphi)$. Also, assuming that $h_i(\xi) = b_i \xi_1^{c_{1,i}} \xi_2^{c_{2,i}} \cdots \xi_n^{c_{n,i}}$, the equality constraint after the logarithmic change in variable can be obtained with $c_i = (c_{1,i}, \cdot, c_{n,i})^T$. Note that since $f(\xi)$ is a polynomial, therefore, convex in log-scale, $F(\varphi)$ is a convex function. Also, since $q_i(\xi)$ is a polynomial, $Q_i(\varphi)$ is also a convex function. In summary, (4.2) is a convex optimization problem in standard form and can be efficiently solved in polynomial time.

In the following section, we show how to transform our problems into GPs. In our transformation, the theory of nonnegative matrices and the Perron-Frobenius lemma are very useful.

4.1 GP for strongly connected digraphs

For a strongly connected digraph $\mathscr{J}$, its adjacency matrix Z is irreducible. Then the follow lemma holds for the spectral radius of the adjacency matrix of any unweighted, strongly connected digraph.

Lemma 1 (Perron-Frobenius):Z is a nonnegative, irreducible matrix. Then, the following statements about its spectral radius $\rho(Z)$ hold:

(1) $\rho(Z) > 0$ is a simple eigenvalue of Z,

(2) $Z\omega = \rho(Z)\omega, \ $ for some $\omega > 0$,

(3) $\rho(Z) = \inf\{\lambda \in R: \ M\omega \leq \lambda\omega \text{ for } \omega = (\omega_1, \cdot, \omega_n)^T \text{ and } \omega_i > 0\}$.

Based on the above results, we have the following result:

Proposition 2: Consider the $n \times n$ nonnegative, irreducible matrix $Z(x)$ with entries being either polynomials or 0 with domain $\xi \in \Omega$, where Ω is defined as $\Omega = \bigcap_{i=1}^m \{\xi > 0 : f_i(\xi) \leq 1\}$, f_i is

polynomials. Then, we can solve the following GP via minimize $\lambda_1(Z(\xi))$:

$$\text{minimize} \quad \lambda$$

$$\text{subject to} \quad \sum_{j=1}^{n} Z_{ij}(\xi)\omega_j/(\lambda\omega_i) \leq 1, \quad i = 1,\cdot,n, \tag{4.3}$$

$$f_i(\xi) \leq 1, \quad i = 1,\cdot,p.$$

Based on the above results, assuming that the contact graph $\mathscr{J}$ is strongly connected, we can solve both the budget-constrained and the rate-constrained problems.

Theorem 1(Solution to the budget-constrained problem): For strongly connected digraphs, problem 2 can be solved by the following GP

$$\min_{\nabla} \lambda$$

$$\text{subject to} \quad \sum_{j=1}^{n} u_{ij}(\beta_i\eta_j + \alpha_i\omega_j) + (k - \gamma_i)\eta_i \leq \lambda\eta_i,$$

$$\gamma_i\eta_i + \hat{\delta}_i\omega_i \leq \lambda\omega_i,$$

$$\sum_{i=1}^{n} f_i(\beta_i) + g_i(\alpha_i) + h_i(\hat{\delta}_i) \leq T, \tag{4.4}$$

$$\underline{\beta}_i \leq \beta_i \leq \bar{\beta}_i, \ \underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i,$$

$$k - \underline{\delta}_i \leq \hat{\delta}_i \leq k - \bar{\delta}_i, \ \underline{\gamma}_i \leq \gamma_i \leq \bar{\gamma}_i,$$

for all $i \in \{1,\cdot,n\}$, where $\delta_i^* = k - \hat{\delta}_i^*$ and $\lambda_1^*(\mathfrak{L}) \leq \lambda^* - k$.

   Proof: First, note that the matrix $\mathfrak{L}$ is not a nonnegative, we can define a nonnegative matrix from $\mathfrak{L}$ by simply adding a constant $k = \max\{\gamma_i, \bar{\delta}_i\}$ for $i = 1,\cdot,n$. Let $\hat{R} = diag\{k\} - diag\{\gamma_i\} \geq 0$ and $\hat{E} = diag\{k\} - diag\{\delta_i\} \geq 0$. We then obtain

$$\hat{\mathfrak{L}} = \begin{bmatrix} BU + \hat{R} & CU \\ R & \hat{E} \end{bmatrix}.$$

Notice that the matrix $\hat{\mathfrak{L}}$ is nonnegative and $k = \max\{\gamma_i, \bar{\delta}_i\}$ for $i = 1,\cdot,n$. Then, according to proposition 2, we have that maximizing ς in (3.1) is equivalent to minimizing $\lambda_1(\hat{\mathfrak{L}})$ under

budget constraints. We can minimizing $\lambda_1(\hat{\mathfrak{L}})$ by solving the following GP:

$$\min_{\nabla} \quad \lambda$$

$$\text{subject to} \quad \hat{\mathfrak{L}} \begin{bmatrix} \eta \\ \omega \end{bmatrix} \leq \lambda \begin{bmatrix} \eta \\ \omega \end{bmatrix},$$

$$\sum_{i=1}^{n} f_i(\beta_i) + g_i(\alpha_i) + h_i(\delta_i) \leq T,$$

$$\underline{\beta}_i \leq \beta_i \leq \bar{\beta}_i, \ \underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i,$$

$$\underline{\delta}_i \leq \delta_i \leq \bar{\delta}_i, \ \underline{\gamma}_i \leq \gamma_i \leq \bar{\gamma}_i,$$

for all $i \in \{1, \cdot, n\}$. The matrix $\hat{\mathfrak{L}}$ is nonnegative and irreducible if the adjacency matrix $U_{\mathscr{J}}$ corresponding to a strongly connected digraph. Therefore, applying proposition 2, the first constraint can be rewrite as the following two constraints:

$$\sum_{j=1}^{n} u_{ij}(\beta_i \eta_j + \alpha_i \omega_j) + (k - \gamma_i)\eta_i \leq \lambda \eta_i,$$

$$\gamma_i \eta_i + \hat{\delta}_i \omega_i \leq \lambda \omega_i,$$

for all $i \in \{1, \cdot, n\}$, where $\hat{\delta}_i = k - \delta_i$. After the above change of variables, the problem (2) rewrite as a standard GP form (4.1). We can find the optimal resources allocation in a heterogeneous network under the budget constraint.

Theorem 2(Solution to the rate-constrained problem): problem 2 can be solved by solving the following GP:

$$\min_{\nabla} \quad \sum_{i=1}^{n} f_i(\beta_i) + g_i(\alpha_i) + h_i(\hat{\delta}_i)$$

$$\text{subject to} \quad \sum_{j=1}^{n} u_{ij}(\beta_i \eta_j + \alpha_i \omega_j) + (k - \gamma_i)\eta_i \leq \lambda \eta_i,$$

$$\gamma_i \eta_i + \hat{\delta}_i \omega_i \leq \lambda \omega_i, \tag{4.5}$$

$$\underline{\beta}_i \leq \beta_i \leq \bar{\beta}_i, \ \underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i,$$

$$k - \bar{\delta}_i \leq \hat{\delta}_i \leq k - \underline{\delta}_i, \ \underline{\gamma}_i \leq \gamma_i \leq \bar{\gamma}_i,$$

for all $i \in \{1, \cdot, n\}$, where $\delta_i^* = k - \hat{\delta}_i^*$, $k = \max\left\{\underline{\varsigma}, \bar{\delta}_i \ for \ i = 1, \cdot, n\right\}$ and its optimal cost $f_i(\beta_i^*) + g_i(\alpha_i^*) + h_i(\hat{\delta}_i^{\ *})$.

   Proof: The proof is similar to the one for theorem 1, so we omit it.

In this section, we have provided the solutions to problems for strongly digraphs. Then we show how to extend it to general connected digraphs.

4.2 GP for general connected digraphs

The adjacency matrix U is irreducible if and only if its graph is strongly contact. So the Perron-Frobenius lemma is not applicable to digraphs that are not strongly connected. For general digraphs, the statements in the P-F lemma are weaken, as follows:

Lemma 2 (Perron-Frobenius): Z is a $n \times n$ nonnegative matrix. Then, the following statements about its spectral radius $\rho(Z)$ hold:

(1) $\rho(Z) \geq 0$ is an eigenvalue of Z,

(2) $Z\omega = \rho(Z)\omega$, for some $\omega \geq 0$,

(3) $\rho(Z) = \inf\{\lambda \in R : Z\omega \leq \lambda\omega \text{ for } \omega = (\omega_1, \cdot, \omega_n)^T \text{ and } \omega_i > 0\}$.

Note that the components of ω in proposition 2 are strictly positive, however, the components of ω in lemma 2 are nonnegative. So if we want to use GP, this issue must be resolved. Defined the sets $\Theta = \{i : \eta_i \neq 0 \text{ and } \omega_i \neq 0\}$. Hence the variables for $i \notin \Theta$ can be excluded from the GP's in theorems 1 and 2. Hence the allocation problems can be split into two different sets of decision variables.

Rate-Constrained Allocation Problem for General Digraphs: If $i \notin \Theta$, the optimization problem holds for the variables:

$$\min_{\nabla} \quad \Sigma_{i\notin\Theta} f_i(\beta_i) + g_i(\alpha_i) + h_i(\hat{\delta}_i)$$
$$\text{subject to} \quad \underline{\beta}_i \leq \beta_i \leq \bar{\beta}_i,$$
$$\underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i, \ k - \bar{\delta}_i \leq \hat{\delta}_i \leq k - \underline{\delta}_i.$$

Thus, for f_i, g_i decreasing and h_i increasing, it is obviously that the minimum investment for all $i \notin \Theta$ correspond to the optimal infection rates $\bar{\beta}_i$, $\bar{\alpha}_i$ and optimal recovery rate $\underline{\delta}_i$.

Theorem 3: On the other hand, for $i \in \Theta$, the optimal solution can be obtained from the following GP:

$$\min_{\nabla} \quad \sum_{i \in \Theta} f_i(\beta_i) + g_i(\alpha_i) + h_i(\hat{\delta}_i)$$

$$\text{subject to} \quad \sum_{j \in \Theta} u_{ij}(\beta_i \eta_j + \alpha_i \omega_i) + (k - \gamma_i)\eta_i \leq \lambda \eta_i,$$

$$\gamma_i \eta_i + \hat{\delta}_i \omega_i \leq \lambda \omega_i, \tag{4.6}$$

$$\underline{\beta}_i \leq \beta_i \leq \bar{\beta}_i, \ \underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i,$$

$$k - \bar{\delta}_i \leq \hat{\delta}_i \leq k - \underline{\delta}_i, \ \underline{\gamma}_i \leq \gamma_i \leq \bar{\gamma}_i,$$

for $i \in \Theta$, where $\delta_i^* = k - \hat{\delta}_i^*$ and its optimal cost $f_i(\beta_i^*) + g_i(\alpha_i^*) + h_i(\hat{\delta}_i^*)$.

Theorem 4: Budget-Constrained Allocation Problem for General Digraphs: For $i \notin \Theta$, it's easily to find the optimal spreading and recovery rates are $\bar{\beta}_i$, $\bar{\alpha}_i$ and $\underline{\delta}_i$. Since $f_i(\bar{\beta}_i) = g_i(\bar{\alpha}_i) = h_i(\underline{\delta}_i) = 0$, therefore, the total investment equal to $\sum_{i \notin \Theta} f_i(\bar{\beta}_i) + g_i(\bar{\alpha}_i) + h_i(\underline{\delta}_i) = 0$ on each one of the nodes with zero eigenvalue. Thus, we can rewrite the budget-constrained allocation problem as following GP for general digraphs:

$$\min_{\nabla} \quad \lambda$$

$$\text{subject to} \quad \sum_{j \in \Theta} u_{ij}(\beta_i \eta_j + \alpha_i \omega_i) + (k - \gamma_i)\eta_i \leq \lambda \eta_i,$$

$$\gamma_i \eta_i + \hat{\delta}_i \omega_i \leq \lambda \omega_i,$$

$$\sum_{i \in \Theta} f_i(\beta_i) + g_i(\alpha_i) + h_i(\hat{\delta})_i \leq T, \tag{4.7}$$

$$\underline{\beta}_i \leq \beta_i \leq \bar{\beta}_i, \ \underline{\alpha}_i \leq \alpha_i \leq \bar{\alpha}_i,$$

$$k - \bar{\delta}_i \leq \hat{\delta}_i \leq k - \underline{\delta}_i, \ \underline{\gamma}_i \leq \gamma_i \leq \bar{\gamma}_i,$$

for $i \in \Theta$, where $\delta_i^* = k - \hat{\delta}_i^*$ and $\lambda_1^*(\mathfrak{L}) \leq \lambda^* - k$. Theorem 3 and 4 provided the solutions to both two problems for general digraphs.

5. Simulation

We have developed an optimization program for determining optimal-cost parameter distributions such that the desired equilibrium is stabilized. In this section, we present how the geometric programming solve the optimization problem of resource allocation by simulating a

strongly connected digraph with $N = 10$ nodes. For the decision variables we use the lower and upper bounds

$$\underline{\beta} = 0.05, \ \underline{\alpha} = 0.01, \ \underline{\delta} = 0.1, \ \underline{\gamma} = 0.05,$$

$$\bar{\beta} = 0.6, \ \bar{\alpha} = 0.4, \ \bar{\delta} = 0.8, \ \bar{\gamma} = 0.45,$$

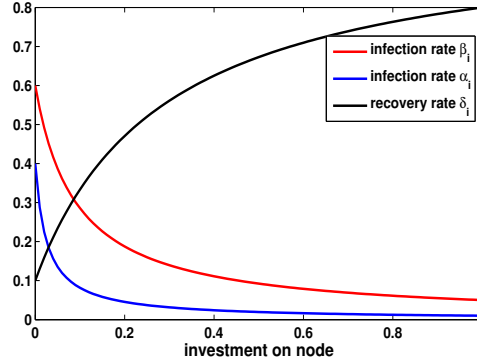for all nodes $i \in 1, \cdot, N$.

FIGURE 1. A plot of infection rate β_i (in red), α_i (in blue) and recovery rate δ_i (in black) achieved at node n_i after the investments on preventive and corrective resources are made on that node.

In Fig.1 we plot the cost functions which are given in subsection 3.1. Here the abscissa is the amount investment on the node n_i and the ordinates are the infection (red and blue line) and recovery (black line) rates achieved by the investment. As we increase the amount invested on protection resources from 0 to 1, the infection rate of that node is reduced from $(\bar{\beta}_i, \bar{\alpha}_i)$ to $(\underline{\beta}_i, \underline{\alpha}_i)$ (red and line). Similarly, as we increase the amount invested on corrective resources at a node n_i, the recovery rate grows from $\underline{\delta}_i$ to $\bar{\delta}_i$ (black line).

Fig.2 demonstrates the performance achieved by using the rate-constrained allocation algorithm from Theorem 2. We have that in the optimal allocation some nodes receive no resources at all; some nodes receive only preventive or corrected resources, and some nodes receive a mixture of preventive and corrective resources.

6. Conclusion

We have presented a convex optimization framework to find the optimal allocation resources of the SEIS epidemic model on arbitrary directed graphs. A necessary and sufficient condition
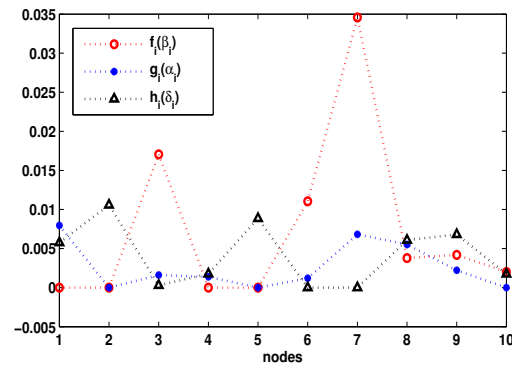
FIGURE 2. The optimal investment on preventive and corrective resources for ten nodes in a strongly connected digraph, where the abscissas are nodes and the ordinates are the amount invested on preventive and corrective resources.

for global exponential stability can be derived from the eigenvalues of a matrix that depends on parameters of the model and the network structure. Furthermore, We have formulated optimization programs for determining optimal resources allocation and reformulating them as geometric programs that can efficiently solve the optimal resource allocation problem. For future work we plan to study the endemic equilibrium which the disease-free equilibrium is not globally asymptotically stable.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] M. Williamson, J.Leveille, An epidemiological model of vorus spread and cleanup, Inf. Infrastr. Lab. 376 (35) (2003), 2389-2393.

[2] R. Pastorsatorras, C. Castellano, P. V. Mieghem, A. Vespignani, Epidemic processes in complex networks, Rev. Mod. Phys. 87 (3) (2014) 120-131.

[3] T. Michele, B. Paolo, P. Chiara, Real-time numerical forecast of global epidemic spreading: case study of 2009 a/h1n1pdm, BMC Med. 10 (2012), Article ID 165.

[4] K. Lerman, R. Ghosh, Information contagion: an empirical study of the spread of news on digg and twitter social networks, Comput. Sci. 52 (2010), 166-176.

[5] P. Van Mieghem, J. Omic, R. Kooij, Virus spread in networks, IEEE Press, 2009.

[6] Y. Wang, D. Chakrabarti, C. Wang, C. Faloutsos, Epidemic spreading in real networks: an eigenvalue viewpoint, International Symposium on Reliable Distributed Systems, 2003. Proceedings, 2003, pp. 25-34.

[7] A. Lajmanovich, J. A. Yorke, A deterministic model for gonorrhea in a nonhomogeneous population, Math. Biosci. 28 (34) (1976), 221-236.

[8] V. M. Preciado, F. D. Sahneh, C. Scoglio, A convex framework for optimal investment on disease awareness in social networks, Global Conference on Signal and Information Processing, 2013, pp. 851-854.

[9] V. M. Preciado, M. Zargham, Traffic optimization to control epidemic outbreaks in metapopulation models, Global Conference on Signal and Information Processing, 2013, pp. 847-850.

[10] C. Borgs, J. Chayes, A. Ganesh, A. Saberi, How to distribute antidote to control epidemics, Random Struct. Algorithms, 37 (2) (2010), 204-222

[11] Y. Wan, S. Roy, A. Saberi, Designing spatially heterogeneous strategies for control of virus spread, IET Syst. Biol. 2 (4) (2008), 184-201.

[12] E. Ramirez-Llanos, S. Martinez, A distributed algorithm for virus spread minimization, American Control Conference, 2014, pp. 184-189.

[13] V. M. Preciado, M. Zargham, C. Enyioha, A. Jadbabaie, Optimal vaccine allocation to control epidemic outbreaks in arbitrary networks, 52nd IEEE Conference on Decision and Control, 2013, pp. 7486-7491.

[14] C. Nowzari, V. M. Preciado, G. J. Pappas, Optimal resource allocation for control of networked epidemic models, IEEE Trans. Control Network Syst. 4 (2) (2017), 159-169..