Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2018, 2018:7
https://doi.org/10.28919/cmbn/3636
ISSN: 2052-2541

GLOBAL STABILITY AND HOPF BIFURCATION OF A DELAYED EPIDEMIOLOGICAL MODEL WITH LOGISTIC GROWTH AND DISEASE RELAPSE

YUGUANG MU*, RUI XU

Institute of Applied Mathematics, Shijiazhuang Mechanical Engineering College, Shijiazhuang 050003, P.R. China

Communicated by A. Elaiw

Abstract. In this paper, an SIRI epidemiological model with relapse and a time delay describing the latent period of the disease is investigated. In the model, it is assumed that the susceptible population is subject to logistic growth in the absence of the disease. We show that the dynamic of the model are determined by the basic reproduction number. If the basic reproduction number is less than unity, then the disease-free equilibrium is globally asymptotically stable. If the basic reproduction number is greater than unity, Hopf bifurcation occurs as the time delay passes through a critical value. Numerical simulations are carried out to support our theoretical conclusion.

Keywords: SIRI epidemic model; disease relapse; latent period; time delay; stability; Hopf bifurcation.

2010 AMS Subject Classification: 00A71.

1. Introduction

Mathematical models describing the population dynamics of infectious disease have played an important role in better understanding epidemiological patterns and disease control for a long

*Corresponding author
E-mail address: muyuguang2016mec@163.com
Received January 2, 2018

time. For some diseases, recovered individuals may relapse with reactivation of latent infection and revert back to the infective class. This recurrence of disease is an important feature of some animal and human diseases, for example, tuberculosis, including human and bovine [1, 7], and herpes [1, 8]. For human tuberculosis, incomplete treatment can lead to relapse, but relapse can also occur in patients who took a full course of treatment and were declared cured. Most tuberculosis in human adults (caused by Mycobacterium tuberculosis) in the USA results from reactivation of latent infection [1].

Let $S(t)$, $I(t)$ and $R(t)$ be the numbers of individuals in the susceptible, infective and the recovered classes, respectively, with the total population $N(t) = S(t) + I(t) + R(t)$, in [2], van den Driessche and Zou considered the following infectious disease model with relapse:

$$\begin{aligned}
\dot{S}(t) =& d - \lambda S(t)I(t) - dS(t),\\
\dot{I}(t) =& -(d+\gamma)I(t) + \lambda I(t)\left[1 - I(t) - \int_0^t \gamma I(\xi)e^{-d(t-\xi)}P(t-\xi)d\xi\right]\\
&- \int_0^t \gamma I(\xi)e^{-d(t-\xi)}d_tP(t-\xi)d\xi,\\
R(t) =& \int_0^t \gamma I(\xi)e^{-d(t-\xi)}P(t-\xi)d\xi,
\end{aligned} \tag{1.1}$$

where $d > 0$ is the birth rate and death rate constants, and $\lambda > 0$ is the average number of effective contacts of an infectious individual per unit time (a fraction S/N are with susceptible). Denote by $P(t)$ the fraction of recovered individuals remaining in the recovered class t time units after recovery. By the meaning of $P(t)$, it is reasonable to assume the following properties:

(A) $P : [0,\infty) \to [0,\infty)$ is differentiable (hence continuous) except at possibly finitely many points where it may have jump discontinuities, non-increasing and satisfies $P(0) = 1$, $\lim_{t\to\infty} P(t) = 0$ and $\int_0^\infty P(u)du$ is positive and finite.

For $P(t) = e^{-\alpha\tau}$, system (1.1) reduces to the following ODE system:

$$\begin{aligned}
\dot{S}(t) =& d - \lambda S(t)I(t) - dS(t),\\
\dot{I}(t) =& -(d+\gamma)I(t) + \lambda I(t)\left[1 - I(t) - R(t)\right]\\
&+ \alpha R(t),\\
\dot{R}(t) =& -(d+\alpha)R(t) + \gamma I(t).
\end{aligned} \tag{1.2}$$

This model is given in Diekmann and Heesterbeek [9]. For system (1.2), the basic reproduction number was identified and its threshold property was discussed in Tudor [11]. The epidemiological model with disease relapse above has been discussed by some authors(see, for example, [3-6] and the references cited therein).

An important aspect of the mathematical study of epidemiology is the time delay describing a latent period. In order to study the effect of the latent period on the transmission dynamics of a disease, in [10], Liu et al. considered the following SIR epidemic model described by delay differential equations:

$$
\begin{aligned}
\dot{S}(t) =& \Lambda - \mu S(t) - \beta e^{-\mu\tau} S(t) G(I(t-\tau)),\\
\dot{I}(t) =& \beta e^{-\mu\tau} S(t) G(I(t-\tau)) - (\mu+\gamma+\alpha) I(t),\\
\dot{R}(t) =& \gamma I(t) - \mu R(t),
\end{aligned}
\tag{1.3}
$$

where Λ is the recruitment rate of the population, μ represents the natural death rate of the population, β is the transmission rate between compartments S and I, γ denotes the recovery rate of infectives, α stands for the disease-caused death rate of infectious individuals, the constant $\tau > 0$ is the incubation time. The authors assumed that the force of infection at any time t is given by $\beta e^{-\mu\tau} S(t) G(I(t-\tau))$, since those infected at time $t-\tau$ become infectious at time τ latter. The term $0 < e^{-\mu\tau} \leq 1$ denotes the survival of vector population in which the time taken to become infectious is τ. The parameters Λ, μ, β, γ, α are all positive constants. For more details about epidemiological model with latent period, we refer to [12-14] going in this direction.

Motivated by the works of Liu et al. [10], van den Driessche and Zou [2], in this paper, we are concerned with the joint effects of disease relapse, saturation incidence and time delay describing latent period on the global dynamics of infectious diseases. To this end, we consider the following delayed differential equations:

$$
\begin{aligned}
\dot{S}(t) &= rS(t)\left(1 - \frac{S(t)}{K}\right) - \frac{\beta S(t) I(t)}{S(t)+I(t)},\\
\dot{I}(t) &= \frac{\beta e^{-\mu\tau} S(t-\tau) I(t-\tau)}{S(t-\tau)+I(t-\tau)} - (\mu+\gamma+\alpha) I(t) + \delta R(t),\\
\dot{R}(t) &= \gamma I(t) - (\mu+\delta) R(t).
\end{aligned}
\tag{1.4}
$$

where the parameters β, γ, μ and α are the same as that defined in model (1.3). $\tau \geq 0$ represents a time delay describing the latent period of the disease, the term $\beta e^{-\mu\tau}S(t-\tau)$ $I(t-\tau)/(S(t-\tau)+I(t-\tau))$ represents the individuals surviving in the latent period τ and becoming infective at time t. The parameter $r>0$ is the intrinsic growth rate of the susceptible population, $K>0$ is the carrying capacity of the environment of the area. $\delta>0$ is a constant representing the rate at which an individual in the recovered class reverts to the infective class. The initial conditions for system (1.4) are of the form

$$\begin{aligned} &S(t)=\phi_1(\theta),\ I(t)=\phi_2(\theta),\ R(t)=\phi_3(\theta),\\ &\phi_i(\theta)\geq 0,\ \theta\in[-\tau,0],\ \phi_i(0)>0\ (i=1,2,3), \end{aligned} \tag{1.5}$$

where $(\phi_1(\theta),\ \phi_2(\theta),\ \phi_3(\theta))\in C([-\tau,0],\ \mathbb{R}^3_{+0})$, the space of continuous functions mapping the interval $[-\tau,0]$ into $\mathbb{R}^3_{+0}$, where $\mathbb{R}^3_{+0}=\{(x_1,x_2,x_3): x_i\geq 0,\ i=1,2,3\}$.

It is well known by the fundamental theory of functional differential equations [15], system (1.4) has a unique solution $(S(t),I(t),R(t))$ satisfying the initial condition (1.5). It is easy to show that all solutions of system (1.4) with initial condition (1.5) are defined on $[0,+\infty)$ and remain positive for all $t\geq 0$.

The organization of this paper is as follows. In the next section, the basic reproduction number is calculated, and the existence of equilibrium is discussed. In Section 3, we discuss the local stability of a disease-free equilibrium and an endemic equilibrium respectively and establish the existence of Hopf bifurcations at the endemic equilibrium of system (1.4). In Section 4, by means of Lyapunov functionals and LaSalle's invariance principle, we prove that if the basic reproduction number is less than unity, the disease-free equilibrium is globally asymptotically stable; if the basic reproduction number is greater than unity, sufficient conditions are defined for the global asymptotic stability of the endemic equilibrium. In Section 5, numerical simulations are given to support our results. A brief remark is given in Section 6 to conclude this work.

2. Local stability and Hopf bifurcation

In this section, we investigate local asymptotic stability of equilibrium and the existence of Hopf bifurcation for system (1.4). System (1.4) always has an disease-free equilibrium $E_0 = (S_0, 0, 0)$, where $S_0 = K$. Let

$$\mathscr{R}_0 = \frac{\beta e^{-\mu\tau}(\mu+\delta)}{(\mu+\alpha)(\mu+\delta)+\mu\gamma}. \tag{2.1}$$

R_0 represents the average number of secondary transmissions of a single infectious individual in a fully susceptible population.

A direct calculation shows that if $\mathscr{R}_0 > 1$, in addition to the disease-free equilibrium $E_0(K,0,0)$, system (1.4) has a unique endemic equilibrium $E^*(S^*, I^*, R^*)$, where

$$\begin{aligned} S^* &= \frac{K}{r}[r-\beta+(\mu+\alpha+\frac{\mu\gamma}{\mu+\delta})e^{\mu\tau}], \\ I^* &= (\mathscr{R}_0-1)S^*, \quad R^* = \frac{\gamma}{\mu+\delta}I^*. \end{aligned} \tag{2.2}$$

Theorem 2.1. If $\mathscr{R}_0 < 1$, the disease-free equilibrium E_0 of system (1.4) is locally asymptotically stable; If $\mathscr{R}_0 > 1$, E_0 becomes unstable.

Proof. The characteristic equation at E_0 is

$$(\lambda+r)[(\lambda-\beta e^{-\mu\tau-\lambda\tau}+\mu+\gamma+\alpha)(\lambda+\mu+\delta)-\delta\gamma] = 0. \tag{2.3}$$

Obviously, $\lambda = -r$ is a negative real root of equation (2.3), and other roots are determined by the following equation:

$$(\lambda-\beta e^{-\mu\tau-\lambda\tau}+\mu+\gamma+\alpha)(\lambda+\mu+\delta)-\delta\gamma = 0. \tag{2.4}$$

Let

$$f(\lambda) = (\lambda-\beta e^{-\mu\tau-\lambda\tau}+\mu+\gamma+\alpha)(\lambda+\mu+\delta)-\delta\gamma.$$

If $\mathscr{R}_0 > 1$, then we have

$$f(0) = [(\mu+\delta)(\mu+\alpha)+\mu\gamma](1-\mathscr{R}_0) < 0, \quad \lim_{\lambda\to+\infty} f(\lambda) = +\infty.$$

So there must be a $\lambda_0 > 0$ such that $f(\lambda_0) = 0$, which implies that Equation (2.4) has at least one positive root. Therefore, the disease-free equilibrium $E_0(S_0,0,0)$ is unstable when $\mathscr{R}_0 > 1$.

If $\mathscr{R}_0 < 1$, then we have

$$f(0) = [(\mu+\delta)(\mu+\alpha)+\mu\gamma](1-\mathscr{R}_0) > 0$$

and

$$f'(\lambda) = (\mu+\alpha+\frac{\mu\gamma}{\mu+\delta})(1-\mathscr{R}_0 e^{-\lambda\tau}) + \frac{\delta\gamma}{\mu+\delta} + 2\lambda + \delta + (\mu+\delta+\lambda)\tau\beta e^{-\mu\tau-\lambda\tau} > 0$$

for all $\lambda \geq 0$ and $\tau \geq 0$. Hence, Equation (2.4) has no positive real root.

Suppose there is a complex root $\lambda = x+iy$ with $y \geq 0$. Filling in to Equation (2.4), and separating the real and imaginary parts, we obtain

$$\begin{aligned} (x+\mu+\gamma+\alpha)(x+\mu+\delta)-y^2 &= (x+\mu+\delta)\beta e^{-\mu\tau-x\tau}\cos y\tau + y\beta e^{-\mu\tau-x\tau}\sin y\tau, \\ y(2x+2\mu+\gamma+\alpha+\delta) &= y\beta e^{-\mu\tau-x\tau}\cos y\tau - (x+\mu+\delta)\beta e^{-\mu\tau-x\tau}\sin y\tau. \end{aligned} \tag{2.5}$$

Squaring and adding the two equations of Equation (2.5), we have

$$y^2 = [(\mu+\alpha+\frac{\mu\gamma}{\mu+\delta})(\mathscr{R}_0 e^{-x\tau}-1) - \frac{\gamma\delta}{\mu+\delta} - x](\beta e^{-\mu\tau-x\tau} + x+\mu+\gamma+\alpha) < 0.$$

Therefore, all roots of Equation (2.4) have negative real parts. Hence, the disease-free equilibrium E_0 is locally asymptotically stable if $\mathscr{R}_0 < 1$. This completes the proof.

Next, we investigate stability and Hopf bifurcation of the endemic equilibrium for system (1.4). The characteristic equation at E^* is

$$P(\lambda,\tau) + Q(\lambda,\tau)e^{-\lambda\tau} = 0, \tag{2.6}$$

where

$$\begin{aligned} P(\lambda,\tau) &= \lambda^3 + A_2(\tau)\lambda^2 + A_1(\tau)\lambda + A_0(\tau), \\ Q(\lambda,\tau) &= B_2(\tau)\lambda^2 + B_1(\tau)\lambda + B_0(\tau), \end{aligned}$$

and

$$\begin{aligned} A_0(\tau) &= [-r + 2rS^*/K + \beta(1-1/\mathscr{R}_0)^2][(\mu+\alpha)(\mu+\delta)+\mu\gamma], \\ A_1(\tau) &= (2\mu+\gamma+\alpha+\delta)[-r+2rS^*/K+\beta(1-1/\mathscr{R}_0)^2] + (\mu+\alpha)(\mu+\delta)+\mu\gamma, \\ A_2(\tau) &= 2\mu+\gamma+\alpha+\delta-r+2rS^*/K+\beta(1-1/\mathscr{R}_0)^2, \\ B_0(\tau) &= [r-2rS^*/K][(\mu+\alpha)(\mu+\delta)+\mu\gamma]/\mathscr{R}_0, \\ B_1(\tau) &= [r-2rS^*/K-(\mu+\delta)](\mu+\alpha+\mu\gamma/(\mu+\delta))/\mathscr{R}_0, \\ B_2(\tau) &= -(\mu+\alpha+\mu\gamma/(\mu+\delta))/\mathscr{R}_0. \end{aligned}$$

when $\tau = 0$, Equation (2.6) becomes

$$\lambda^3 + (A_2 + B_2)\lambda^2 + (A_1 + B_1)\lambda + A_0 + B_0 = 0. \tag{2.7}$$

Let

$$H_1 = A_2 + B_2, \;\; H_2 = (A_2 + B_2)(A_1 + B_1) - (A_0 + B_0).$$

After some tedious computations, it is clear that $H_1 > 0,\ H_2 > 0$. By the Routh-Hurwitz criterion, all roots of Equation (2.7) have negative real parts. Hence, the interior equilibrium E^* is locally asymptotically stable.

Now, for $\tau > 0$, let $\tau_{\max}$ is the maximum of τ under the condition that the equilibrium point E_1 exists. In order to apply the methods proposed by Kuang and Beretta in [16], we need to verify that the following conditions are established for $\tau \in [0, \tau_{\max})$:

(i) $P(0,\tau) + Q(0,\tau) \neq 0 \;\; \forall \tau \in \mathbb{R}_{+0}$, *i.e.*, $\lambda = 0$ is not a characteristic root of (2.6);

(ii) if $\lambda = i\omega,\ \omega \in \mathbb{R}$, then $P(i\omega,\tau) + Q(i\omega,\tau) \neq 0$, for $\tau > 0$;

(iii) $\limsup \left\{ \left| Q(\lambda,\tau)/P(\lambda,\tau) \right| : |\lambda| \to \infty, \mathrm{Re}\lambda \geq 0 \right\} < 1$ for any τ;

(iv) $F(\omega,\tau) = |P(i\omega,\tau)|^2 - |Q(i\omega,\tau)|^2$ for each τ has at most a finite number of real zeroes;

(v) Each positive root $\omega(\tau)$ of $F(\omega,\tau) = 0$ is continuous and differentiable in τ whenever it exists.

Here, $P(\lambda,\tau)$ and $Q(\lambda,\tau)$ are the same as those in (2.6). Because of $P(0,\tau) + Q(0,\tau) = A_0(\tau) + B_0(\tau) \neq 0$, the condition (i) is established clearly. Note that

$$\begin{aligned} P(i\omega,\tau) + Q(i\omega,\tau) =& - i\omega^3 - [A_2(\tau) + B_2(\tau)]\omega^2 + i[A_1(\tau) + B_1(\tau)]\omega \\ &+ A_0(\tau) + B_0(\tau) \\ =&[A_0(\tau) + B_0(\tau) - (A_2(\tau) + B_2(\tau))\omega^2] \\ &+ i\omega[A_1(\tau) + B_1(\tau) - \omega^2] \\ \neq& 0. \end{aligned} \tag{2.8}$$

Therefore, the condition (ii) is established. From Equation (2.6), we obtain

$$\lim_{|s|\to\infty} \left| \frac{Q(\lambda,\tau)}{P(\lambda,\tau)} \right| = \lim_{|s|\to\infty} \left| \frac{B_2(\tau)\lambda^2 + B_1(\tau)\lambda + B_0(\tau)}{\lambda^3 + A_2(\tau)\lambda^2 + A_1(\tau)\lambda + A_0(\tau)} \right| = 0, \tag{2.9}$$

This completes the proof of (iii). To justify (iv), we have

$$
\begin{aligned}
|P(i\omega,\tau)|^2 =& \omega^6+[A_2^2(\tau)-2A_1(\tau)]\omega^4+[A_1^2(\tau)-2A_0(\tau)A_2(\tau)]\omega^2+A_0^2(\tau),\\
|Q(i\omega,\tau)|^2 =& B_2^2(\tau)\omega^4+[B_1^2(\tau)-2B_0(\tau)B_2(\tau)]\omega^2+B_0^2(\tau),
\end{aligned} \tag{2.10}
$$

Thus, we obtain that

$$
F(\omega,\tau)=\omega^6+a_1(\tau)\omega^4+a_2(\tau)\omega^2+a_3(\tau), \tag{2.11}
$$

where

$$
\begin{aligned}
a_1(\tau) =& A_2^2(\tau)-2A_1(\tau)-B_2^2(\tau),\\
a_2(\tau) =& A_1^2(\tau)+2B_0(\tau)B_2(\tau)-2A_0(\tau)A_2(\tau)-B_1^2(\tau),\\
a_3(\tau) =& A_0^2(\tau)-B_0^2(\tau).
\end{aligned}
$$

Obviously, the condition (iv) is established. According to the Implicit Function Theorem, the condition (v) is also established.

Let $\lambda=i\omega_1$ ($\omega_1>0$) be a root of Equation (2.6). Substituting it into Equation (2.6) and separating real and imaginary parts, we obtain

$$
\begin{cases}
(B_2(\tau)\omega_1^2-B_0(\tau))\cos\omega_1\tau-B_1(\tau)\omega_1\sin\omega_1\tau=A_0(\tau)-A_2(\tau)\omega_1^2,\\
(B_2(\tau)\omega_1^2-B_0(\tau))\sin\omega_1\tau+B_1(\tau)\omega_1\cos\omega_1\tau=\omega_1^3-A_1(\tau)\omega_1,
\end{cases} \tag{2.12}
$$

which gives

$$
\begin{cases}
\sin(\omega_1\tau) =& \dfrac{B_2(\tau)\omega_1^5+[A_2(\tau)B_1(\tau)-B_0(\tau)-A_1(\tau)B_2(\tau)]\omega_1^3}{B_2^2(\tau)\omega_1^4+[B_1^2(\tau)-2B_0(\tau)B_2(\tau)]\omega_1^2+B_0^2(\tau)}\\
&+\dfrac{[A_1(\tau)B_0(\tau)-A_0(\tau)B_1(\tau)]\omega_1}{B_2^2(\tau)\omega_1^4+[B_1^2(\tau)-2B_0(\tau)B_2(\tau)]\omega_1^2+B_0^2(\tau)},\\
\cos(\omega_1\tau) =& \dfrac{[B_1(\tau)-A_2(\tau)B_2(\tau)]\omega_1^4-A_0(\tau)B_0(\tau)}{B_2^2(\tau)\omega_1^4+[B_1^2(\tau)-2B_0(\tau)B_2(\tau)]\omega_1^2+B_0^2(\tau)}\\
&+\dfrac{[A_0(\tau)B_2(\tau)+B_0(\tau)A_2(\tau)-A_1(\tau)B_1(\tau)]\omega_1^2}{B_2^2(\tau)\omega_1^4+[B_1^2(\tau)-2B_0(\tau)B_2(\tau)]\omega_1^2+B_0^2(\tau)}.
\end{cases} \tag{2.13}
$$

Assume that $I\subseteq\mathbb{R}_{+0}$ is the set where $\omega_1(\tau)$ is a positive root of $F(\omega_1,\tau)=|P(i\omega_1,\tau)|^2-|Q(i\omega_1,\tau)|^2$ and for $\tau\notin I$, $\omega_1(\tau)$ is not definite. According to [16], we can define the angle

$\theta(\tau) \in [0, 2\pi]$, as the solution of (2.13)

$$\begin{cases} \sin\theta(\tau) = \mathrm{Im}\dfrac{P(i\omega_1,\tau)}{Q(i\omega_1,\tau)}, \\ \cos\theta(\tau) = -\mathrm{Re}\dfrac{P(i\omega_1,\tau)}{Q(i\omega_1,\tau)}, \end{cases} \tag{2.14}$$

and introduce the functions $I \to \mathbb{R}$

$$S_n(\tau) := \tau - \frac{\theta(\tau) + 2n\pi}{\omega_1(\tau)}, \quad \tau \in I, \quad n \in \mathbb{N}_0. \tag{2.15}$$

Using Theorem 2.1 in [16], we have the following theorem.

Theorem 2.2. Assume that $\omega_1(\tau)$ is a positive real root of $F(\omega_1, \tau) = 0$ defined for $\tau \in I$, $I \subseteq \mathbb{R}_{+0}$ and at some $\tau^* \in I$,

$$S_n(\tau^*) = 0, \ \ for\ some\ n \in \mathbb{N}_0, \tag{2.16}$$

Then a pair of simple conjugate pure imaginary roots $\lambda = \pm i\omega(\tau^*)$ of (2.6) exists at $\tau = \tau^*$ which crosses the imaginary axis from left to right if $\delta(\tau^*) > 0$ and crosses the imaginary axis from right to left if $\delta(\tau^*) < 0$, where

$$\delta(\tau^*) = sign\left\{ \frac{dRe\lambda(\tau)}{d\tau}\bigg|_{\tau=\tau^*} \right\} = sign\left\{ F'_{\omega}(\omega_1(\tau^*), \tau^*) \right\} sign\left\{ \frac{dS_n(\tau)}{d\tau}\bigg|_{\tau=\tau^*} \right\}. \tag{2.17}$$

3. Global stability

Theorem 3.1. If $\mathscr{R}_0 < 1$, then the disease-free equilibrium E_0 is globally asymptotically stable.

Proof. Define

$$V_1(t) = I(t) + \frac{\delta}{\mu+\delta}R(t) + e^{-\mu\tau}U^-(t), \tag{3.1}$$

where

$$U^-(t) = \int_{t-\tau}^{t} \frac{\beta S(\theta)I(\theta)}{S(\theta)+I(\theta)} \mathrm{d}\theta. \tag{3.2}$$

Then the time derivative of $V_1(t)$ along system (1.4) satisfies

$$\begin{aligned}\frac{d}{dt}V_1(t) =&\frac{\beta e^{-\mu\tau}S(t-\tau)I(t-\tau)}{S(t-\tau)+I(t-\tau)}-(\mu+\gamma+\alpha)I(t)+\delta R(t)\\&+\frac{\delta}{\mu+\delta}(\gamma I(t)-(\mu+\delta)R(t))\\&+\frac{\beta e^{-\mu\tau}S(t)I(t)}{S(t)+I(t)}-\frac{\beta e^{-\mu\tau}S(t-\tau)I(t-\tau)}{S(t-\tau)+I(t-\tau)}\\=&\frac{\beta e^{-\mu\tau}S(t)I(t)}{S(t)+I(t)}-\frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta}I(t)\\=&\frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta}\left(\mathscr{R}_0\frac{S(t)I(t)}{S(t)+I(t)}-I(t)\right)\\=&\frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta}\left((\mathscr{R}_0-1)\frac{S(t)I(t)}{S(t)+I(t)}-\frac{I^2(t)}{S(t)+I(t)}\right).\end{aligned}\tag{3.3}$$

It follows that $\dot{V}_1(t)\le 0$ if $\mathscr{R}_0<1$. By Theorem 5.3.1 in [15], solutions limit to $\mathscr{M}_0$, the largest invariant subset of $\{\dot{V}_1(t)=0\}$. Clearly, it follows from (3.3) that $\dot{V}_1(t)=0$ iff $I=0$. Noting that $\mathscr{M}_0$ is invariant, for each element in $\mathscr{M}_0$, we have $I=0$, $\dot{I}(t)=0$. We therefore obtain from the second equation of system (1.4) that $0=\dot{I}(t)=\delta R(t)$, which yields $R=0$. It follows from the first equation and $\dot{V}_1(t)=0$ that $\dot{S}(t)=rS(t)(1-S(t)/K)$, which yields $\lim_{t\to\infty}S(t)=K$. Therefore, $\dot{V}_1(t)=0$ iff $(S,I,R)=(K,0,0)$. Note that if $\mathscr{R}_0<1$, the equilibrium E_0 is locally asymptotically stable. Hence, the global asymptotic stability of $E_0(K,0,0)$ follows from LaSalle's invariance principle for delay differential systems. This completes the proof.

Lemma 3.1. There is a constant $H_0>0$ such that $I(t)\le H_0$ for any positive solution $(S(t),I(t),R(t))$ of system (1.4).

Proof. It is clear that all solutions of system (1.4) with initial condition (1.5) remain positive for all $t\ge 0$. Let

$$V(t)=e^{-\mu\tau}S(t-\tau)+I(t)+R(t).$$

Then the time derivative of $V(t)$ along system (1.4) satisfies

$$\begin{aligned}\frac{d}{dt}V(t) =&e^{-\mu\tau}\left(2rS(t-\tau)-r\frac{S^2(t-\tau)}{K}\right)-re^{-\mu\tau}S(t-\tau)\\&-(\mu+\alpha)I(t)-\mu R(t)\\\le&-hV(t)+e^{-\mu\tau}\left(2rS(t-\tau)-r\frac{S^2(t-\tau)}{K}\right),\end{aligned}$$

where $h=\min\{r,\mu+\alpha,\mu\}$. Let $G(s)=2rs-\frac{rs^2}{K}$, obviously, $G(s)$ gets the maximum $G_{\max}(s)=rK$ at $s=K$. Therefore

$$\frac{d}{dt}V(t) \leq -hV(t)+e^{-\mu\tau}rK. \tag{3.4}$$

It follow that

$$\lim_{t\to+\infty}\sup V(t) \leq \frac{rK}{he^{\mu\tau}} := N_0.$$

Thus, there is a constant $T_1>0$ such that $I(t)\leq N_0$ for any $\varepsilon>0$ when $t>T_1$.

Lemma 3.2. There is an $H_1>0$ so that for any positive solution $(S(t),I(t),R(t))$ of the system (1.4), when t is sufficiently large, there are $S(t)\geq H_1$ holds.

Proof. By the proof of Lemma 3.1, when $t>T_1$, there are $I(t)\leq N_0$. From the first equation of the system (1.4)

$$\frac{d}{dt}S(t)=rS\left(1-\frac{S}{K}\right)-\frac{\beta SI}{S+I}\geq -\frac{rS^2}{K}+\left(r-\frac{\beta H_0}{S_{\max}+H_0}\right)S. \tag{3.5}$$

Let

$$H(s)=-\frac{rs^2}{K}+\left(r-\frac{\beta H_0}{S_{\max}+H_0}\right)s,$$

Clearly, $H(s)=0$ has the only positive root

$$s^*=\frac{K}{r}\left(r-\frac{\beta H_0}{S_{\max}+H_0}\right),$$

From the equation of (3.5)

$$\lim_{t\to+\infty}\inf S(t)\geq s^*,$$

Thus, there is a constant $T_2>T_1$ such that $S(t)\geq s^*+\varepsilon_1 := H_1$ for any $\varepsilon_1>0$ when $t>T_2$.

Theorem 3.2. If $\mathscr{R}_0>1$ and $H_1\geq K/2$, then the endemic equilibrium E^* is globally asymptotically stable.

Proof. Let $(S(t),I(t),R(t))$ be any positive solution of system (1.4) with initial conditions (1.5). Define

$$\begin{aligned}U(t)=&e^{-\mu\tau}\left(S(t)-S^*-\int_{S^*}^{S(t)}\frac{\theta+I^*}{S^*+I^*}\frac{S^*}{\theta}d\theta\right)+(I(t)-I^*-I^*\ln\frac{I(t)}{I^*})\\&+\frac{\delta}{\mu+\delta}(R(t)-R^*-R^*\ln\frac{R(t)}{R^*}).\end{aligned} \tag{3.6}$$

Calculating the derivative of $U(t)$ along positive solutions of system (1.4), we have

$$\begin{aligned}\frac{d}{dt}U(t)=&e^{-\mu\tau}\left(1-\frac{S(t)+I^*}{S^*+I^*}\frac{S^*}{S(t)}\right)\dot{S}(t)+\left(1-\frac{I^*}{I(t)}\right)\dot{I}(t)+\frac{\delta}{\mu+\delta}\left(1-\frac{R^*}{R(t)}\right)\dot{R}(t)\\=&e^{-\mu\tau}\left(1-\frac{S(t)+I^*}{S^*+I^*}\frac{S^*}{S(t)}\right)\left(rS(t)\left(1-\frac{S(t)}{K}\right)-\frac{\beta S(t)I(t)}{S(t)+I(t)}\right)\\&+\left(1-\frac{I^*}{I(t)}\right)\left(\frac{\beta e^{-\mu\tau}S(t-\tau)I(t-\tau)}{S(t-\tau)+I(t-\tau)}-(\mu+\gamma+\alpha)I(t)+\delta R(t)\right)\\&+\frac{\delta}{\mu+\delta}\left(1-\frac{R^*}{R(t)}\right)(\gamma I(t)-(\mu+\delta)R(t)).\end{aligned}\tag{3.7}$$

Substituting $rS^*\left(1-S^*/K\right)-\beta S^*I^*/(S^*+I^*)=0,\ \beta e^{-\mu\tau}S^*I^*/(S^*+I^*)-(\mu+\gamma+\alpha)I^*+\delta R^*=0$, and $\gamma I^*-(\mu+\delta)R^*=0$ into (3.7) yields

$$\begin{aligned}\frac{d}{dt}U(t)=&e^{-\mu\tau}\left(1-\frac{S(t)+I^*}{S^*+I^*}\frac{S^*}{S(t)}\right)\left(-rS^*\left(1-\frac{S^*}{K}\right)+rS(t)\left(1-\frac{S(t)}{K}\right)+\frac{\beta S^*I^*}{S^*+I^*}\right.\\&\left.-\frac{\beta S(t)I(t)}{S(t)+I(t)}\right)+\frac{\mu+\gamma+\alpha}{\gamma}\left(1-\frac{R^*}{R(t)}\right)(\gamma I(t)-(\mu+\delta)R(t))\\&+\left(1-\frac{I^*}{I(t)}\right)\left(\frac{\beta e^{-\mu\tau}S(t-\tau)I(t-\tau)}{S(t-\tau)+I(t-\tau)}-(\mu+\gamma+\alpha)I(t)+\delta R(t)\right)\\=&\frac{r(S(t)-S^*)^2}{e^{\mu\tau}(S^*+I^*)S(t)}\left(1-\frac{S^*+S(t)}{K}\right)-\frac{\beta e^{-\mu\tau}S^*I^*}{S^*+I^*}\frac{S(t)+I^*}{S^*+I^*}\frac{S^*}{S(t)}\\&+\frac{\beta e^{-\mu\tau}S(t)I(t)}{S(t)+I(t)}\frac{S(t)+I^*}{S^*+I^*}\frac{S^*}{S(t)}+\frac{\beta e^{-\mu\tau}S^*I^*}{S^*+I^*}-\frac{\beta e^{-\mu\tau}S(t)I(t)}{S(t)+I(t)}\\&+\frac{\beta e^{-\mu\tau}S(t-\tau)I(t-\tau)}{S(t-\tau)+I(t-\tau)}-(\mu+\gamma+\alpha)I(t)-\frac{\beta e^{-\mu\tau}S(t-\tau)I(t-\tau)}{S(t-\tau)+I(t-\tau)}\frac{I^*}{I(t)}\\&+(\mu+\gamma+\alpha)I^*-\delta\frac{I^*}{I(t)}R(t)+\frac{\delta\gamma}{\mu+\delta}I(t)-\frac{\delta\gamma}{\mu+\delta}I(t)\frac{R^*}{R(t)}+\delta R^*\\=&\frac{r(S(t)-S^*)^2}{e^{\mu\tau}(S^*+I^*)S(t)}\left(1-\frac{S^*+S(t)}{K}\right)-\frac{\beta e^{-\mu\tau}S(t)I(t)}{S(t)+I(t)}+\frac{\beta e^{-\mu\tau}S(t-\tau)I(t-\tau)}{S(t-\tau)+I(t-\tau)}\\&+\frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta}I^*\left(1-\frac{S(t)+I^*}{S^*+I^*}\frac{S^*}{S(t)}+\frac{S(t)I(t)}{S(t)+I(t)}\frac{S(t)+I^*}{S(t)I^*}\right)\\&+\frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta}I^*\left(1-\frac{I(t)}{I^*}-\frac{(S^*+I^*)S(t-\tau)I(t-\tau)}{S^*I^*(S(t-\tau)+I(t-\tau))}\frac{I^*}{I(t)}\right)\\&+\delta R^*\left(2-\frac{I(t)}{I^*}\frac{R^*}{R(t)}-\frac{I^*}{I(t)}\frac{R(t)}{R^*}\right).\end{aligned}\tag{3.8}$$

Let $V_2(t) = U(t) + \frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta} I^* U^+(t)$, where

$$\begin{aligned} U^+(t) = \int_{t-\tau}^{t} \Bigg[& \frac{(\mu+\delta)e^{-\mu\tau}\beta S(\theta)I(\theta)}{((\mu+\alpha)(\mu+\delta)+\mu\gamma)I^*(S(\theta)+I(\theta))} - 1 \\ & - \ln \frac{(\mu+\delta)e^{-\mu\tau}\beta S(\theta)I(\theta)}{((\mu+\alpha)(\mu+\delta)+\mu\gamma)I^*(S(\theta)+I(\theta))} \Bigg] d\theta. \end{aligned} \tag{3.9}$$

Calculating the derivative of $U^+(t)$,

$$\begin{aligned} \frac{d}{dt}U^+(t) = & \frac{(\mu+\delta)e^{-\mu\tau}\beta S(t)I(t)}{((\mu+\alpha)(\mu+\delta)+\mu\gamma)I^*(S(t)+I(t))} \\ & - \frac{(\mu+\delta)e^{-\mu\tau}\beta S(t-\tau)I(t-\tau)}{((\mu+\alpha)(\mu+\delta)+\mu\gamma)I^*(S(t-\tau)+I(t-\tau))} \\ & + \ln \frac{S(t-\tau)I(t-\tau)(S(t)+I(t))}{(S(t-\tau)+I(t-\tau))S(t)I(t)}. \end{aligned} \tag{3.10}$$

Using equality

$$\begin{aligned} \ln \frac{S(t-\tau)I(t-\tau)(S(t)+I(t))}{(S(t-\tau)+I(t-\tau))S(t)I(t)} = & \ln \left(\frac{(S^*+I^*)S(t-\tau)I(t-\tau)}{S^*I^*(S(t-\tau)+I(t-\tau))} \frac{I^*}{I(t)} \right) \\ & + \ln \frac{(S(t)+I^*)S^*}{(S^*+I^*)S(t)} + \ln \frac{S(t)+I(t)}{S(t)+I^*}, \end{aligned} \tag{3.11}$$

we obtain

$$\begin{aligned} \frac{d}{dt}V_2(t) = & \frac{d}{dt}U(t) + \frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta} I^* \frac{d}{dt}U^+(t) \\ = & \frac{r(S(t)-S^*)^2}{e^{\mu\tau}(S^*+I^*)S(t)} \left(1 - \frac{S^*+S(t)}{K}\right) \\ & + \frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta} I^* \ln \frac{S(t-\tau)I(t-\tau)(S(t)+I(t))}{(S(t-\tau)+I(t-\tau))S(t)I(t)} \\ & + \frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta} I^* \left(1 - \frac{S(t)+I^*}{S^*+I^*}\frac{S^*}{S(t)} + \frac{S(t)I(t)}{S(t)+I(t)}\frac{S(t)+I^*}{S(t)I^*}\right) \\ & + \frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta} I^* \left(1 - \frac{I(t)}{I^*} - \frac{(S^*+I^*)S(t-\tau)I(t-\tau)}{S^*I^*(S(t-\tau)+I(t-\tau))}\frac{I^*}{I(t)}\right) \\ & + \delta R^* \left(2 - \frac{I(t)}{I^*}\frac{R^*}{R(t)} - \frac{I^*}{I(t)}\frac{R(t)}{R^*}\right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{r(S(t)-S^*)^2}{e^{\mu\tau}(S^*+I^*)S(t)}\left(1-\frac{S^*+S(t)}{K}\right)\\
&+\frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta}I^*\left(1-\frac{S(t)+I^*}{S^*+I^*}\frac{S^*}{S(t)}+\ln\frac{(S(t)+I^*)S^*}{(S^*+I^*)S(t)}\right)\\
&+\frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta}I^*\left(1-\frac{(S^*+I^*)S(t-\tau)I(t-\tau)}{S^*I^*(S(t-\tau)+I(t-\tau))}\frac{I^*}{I(t)}\right.\\
&\left.+\ln\left(\frac{(S^*+I^*)S(t-\tau)I(t-\tau)}{S^*I^*(S(t-\tau)+I(t-\tau))}\frac{I^*}{I(t)}\right)\right) \qquad (3.12)\\
&+\frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta}I^*\left(1-\frac{S(t)+I(t)}{S(t)+I^*}+\ln\frac{S(t)+I(t)}{S(t)+I^*}\right)\\
&-\frac{(\mu+\alpha)(\mu+\delta)+\mu\gamma}{\mu+\delta}\frac{S(t)(I(t)-I^*)^2}{(S(t)+I(t))(S(t)+I^*)}\\
&+\delta R^*\left(2-\frac{I(t)}{I^*}\frac{R^*}{R(t)}-\frac{I^*}{I(t)}\frac{R(t)}{R^*}\right).
\end{aligned}
$$

Since the function $H(t)=1-f(t)+\ln f(t)$ is always non-positive for any function $f(t)>0$, and $H(t)=0$ iff $f(t)=1$. From Lemma 3.2, when t is sufficiently large, there are $S(t)\geq H_1$ holds. Therefore, when $H_1\geq K/2$,

$$1-\frac{S^*+S(t)}{K}\leq 0.$$

Thus, when $\mathscr{R}_0>1$ and $H_1\geq K/2$, there are $\frac{d}{dt}V_2(t)\leq 0$ holds. And $\frac{d}{dt}V_2(t)=0$ iff $I(t)=I^*, R(t)=R^*, S(t)=S^*$. Then, the largest invariant set $\mathscr{M}_1\subseteq\mathscr{M}=\left\{(S,I,R)\,\middle|\,\frac{d}{dt}V_2(t)=0\right\}$ is the singleton $\{E^*\}$. By LaSalle's Invariance Principle, the endemic equilibrium E^* is globally asymptotically stable.

4. Numerical simulations

In this section, we perform numerical simulations to illustrate the existence of Hopf bifurcation for system (1.4). In the following, we choose a set of parameters $r=1$, $K=5$, $\beta=0.8$, $\mu=0.3$, $\gamma=0.2$, $\alpha=0.1$, $\delta=0.1$, $\tau=0.4$, where "day" is used as the unit of time. By calculation, we obtain that $\mathscr{R}_0\approx 1.29>1$. At this point, system(1.4) has Hopf bifurcation at endemic equilibrium E^*.

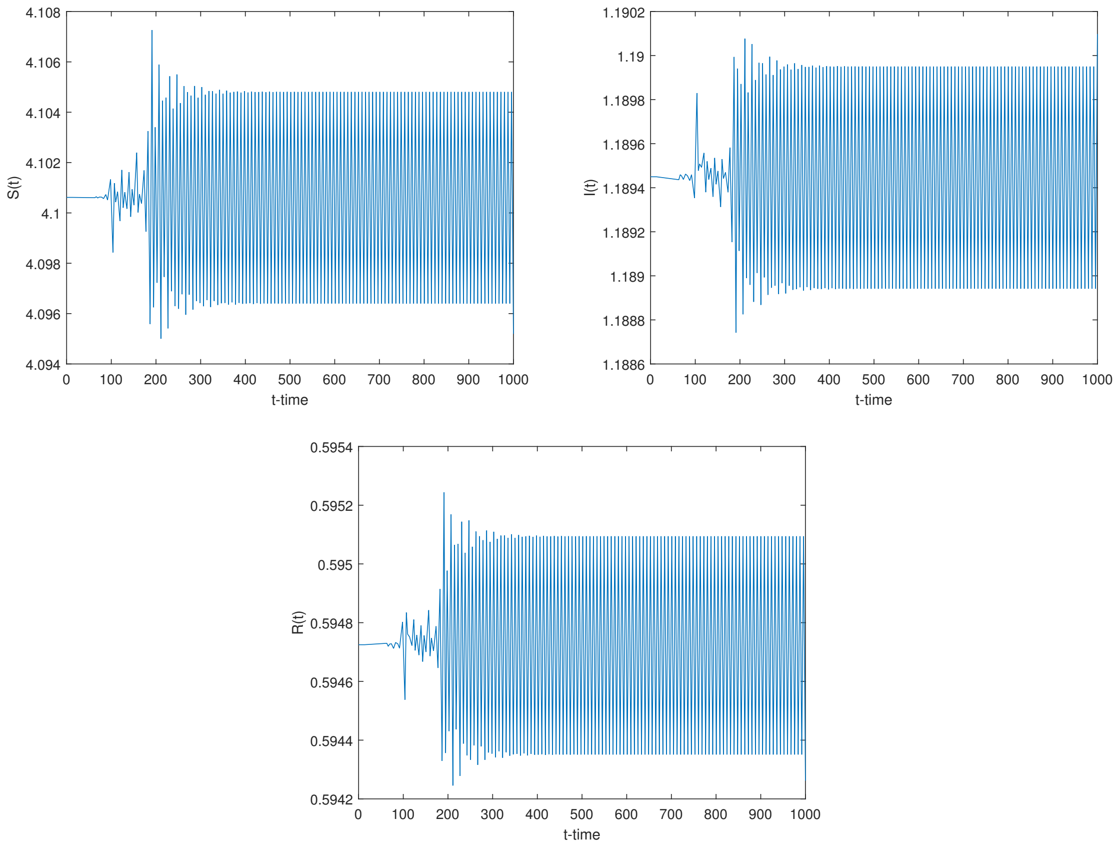

FIGURE 1. There are periodic solutions bifurcated from the endemic equilibrium E^* of system (1.4) when $\tau = 0.4$.

5. Conclusions

In this paper, we have considered an epidemiological model with saturation incidence, a time delay describing the latent period and relapse. We have calculated the basic reproduction number $\mathscr{R}_0$. It has shown that if $\mathscr{R}_0 < 1$, then the disease-free equilibrium E_0 is globally asymptotically stable. If $\mathscr{R}_0 > 1$, stability switch is shown to occur, and periodic solutions bifurcated from the endemic equilibrium. The establishment of global stability is crucial to our study, since local stability cannot rule out the possibility of periodic solutions far away from equilibria.

From the expression of $\mathscr{R}_0$, we see that the natural death rate μ, the disease-induced death rate α, the recovery rate γ, the constant rate δ at which an individual in the recovered class reverts to the infective class, the average contact rate β, and the latent period τ do affect the value of the basic reproduction number. Nevertheless, the intrinsic growth rate r have no effect on the value of the basic reproduction number. To control the disease, a strategy should reduce the reproduction number to below unity. Clearly, deceasing the parameters β, δ, and increasing the parameters μ, α, γ are helpful in controlling bovine by reducing the basic reproduction number.

Competing interests

The authors declare that have no competing interests.

Authors' contributions

The manuscript was written through contributions of all authors. All authors have given approval to the final version of the manuscript.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (No. 11371368) and the Natural Science Foundation of Hebei Province of China (No. A2014506015).

REFERENCES

[1] J. Chin, Control of Communicable Diseases Manual, American Public Health Association, Washington, 1999.

[2] P. van den Driessche, X. Zou, Modeling relapse in infectious diseases, Math. Biosci. 207 (2007) 89-103.

[3] P. van den Driessche, L. Wang, X. Zou, Modeling diseases with latency and relapse, Math. Biosci. Eng. 4 (2007) 205-219.

[4] P. Raja Sekhara Rao, M. Naresh Kumar, A dynamic model for infectious diseases: The role of vaccination and treatment, Chaos, Solitons Fractals. 75 (2015) 34-49.

[5] S. Liu, S. Wang, L. Wang, Global dynamics of delay epidemic models with nonlinear incidence rate and relapse, Nonlinear Anal., Real World Appl. 12 (2011) 119-127.

[6] F.B. Agusto, Mathematical model of Ebola transmission dynamics with relapse and reinfection, Math. Biosci. 283 (2017) 48-59.

[7] S.W. Martin, Livestock Disease Eradication: Evaluation of the Cooperative State-Federal Bovine Tuberculosis Eradication Program, National Academy Press, Washington, 1994.

[8] K.E. VanLandingham, H.B. Marsteller, G.W. Ross, F.G. Hayden, Relapse of herpes simplex encephalitis after conventional acyclovir therapy, J. Amer. Med. Assoc. 259 (1988) 1051-1053.

[9] O. Diekmann, J.A.P. Heesterbeek, Mathematical Epidemiology of Infectious Diseases: Model Building, Analysis and Interpretation, Wiley, New York, 2000.

[10] Q. Liu, D. Jiang, N. Shi, Asymptotic behaviors of a stochastic delayed SIR epidemic model with nonlinear incidence, Commun. Nonlinear Sci. Numer. Simul. 40 (2016) 89-99.

[11] D. Tudor, A deterministic model for herpes infections in human and animal populations, SIAM Rev. 32 (1990) 136.

[12] R. Xu, Z. Ma, Global stability of a SIR epidemic model with nonlinear incidence rate and time delay, Nonlinear Anal., Real World Appl. 10 (2009) 3175-3189.

[13] C.C. McCluskey, Global stability for an SIR epidemic model with delay and nonlinear incidence, Nonlinear Anal., Real World Appl. 11 (2010) 3106-3109.

[14] Y. Enatsu, E. Messina, Y. Muroya, Stability analysis of delayed SIR epidemic models with a class of nonlinear incidence rates, Appl. Math. Comput. 218 (2012) 5327-5336.

[15] J.K. Hale, S. Verduyn Lunel, Introduction to Functional Differential Equations, Springer, New York, 1993.

[16] E. Beretta, Y. Kuang, Geometric stability switch criteria in delay differential systems with delay dependent parameters, SIAM J. Math. Anal. 33 (2002) 1144-1165.