Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2018, 2018:11
https://doi.org/10.28919/cmbn/3643
ISSN: 2052-2541

LOCAL DYNAMICS OF A THREE-DIMENSIONAL HOST-PARASITE MODEL

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Communicated by Z. He

Abstract. In this paper, local dynamics of a three-dimensional host parasite model as a discrete dynamical system has been studied. The existence of fixed points and stability behaviour near these points are investigated. By use of centre manifold theorem we describe the stability of non-hyperbolic fixed points. Some numerical simulations explain our theoretical results in better way.

Keywords: discrete dynamical systems; invariant manifold; host-parasite model; local dynamics.

2010 AMS Subject Classification: 37N25, 34C45.

1. Introduction

The behavior of population in nature and the interaction between species have been a subject of interest over many years. In recent years, many researchers paid more attention to population models as a discrete dynamical systems [2,3,8]. Discrete-time models are actually more reasonable than the continuous time models when populations have non-overlapping generations. Also, by development in mathematical softwares, more accurate numerical results can be obtained from the discrete model, related to the continuous one, since numerical simulation of

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Received January 6, 2018

continuous models are obtained by discretization [5,14,16]. A host-parasite model describes the dynamics of two insects: parasitoid and its host. This model is named after the two researchers, Nicholson and Bailey, who developed the model [1]. In this model, parasitoids randomly search for hosts population that grows exponentially in the absence of parasitoids. Moreover, both parasitoids and hosts are supposed to be distributed in a non-contiguous method in the environment [15]. After Nicholson-Bailey model, many models have been introduced to describe the relationship between host and parasitoid. Citing some examples: Nicholson and Bailey [15] introduced discrete host- parasitoid model of the form

$$H_{n+1} = rH_n e^{-\gamma P_n},$$
$$P_{n+1} = sH_n(1-e^{-\gamma P_n}).$$

Beddington et al. [6] improved the Nicholson-Bailey model by adding the effect of carrying capacity

$$H_{n+1} = H_n e^{r(1-\frac{H_n}{K})-\gamma P_n},$$
$$P_{n+1} = sH_n(1-e^{-\gamma P_n}).$$

Khan and Qureshi [11] assumed that the host has bounded dynamics in the absence of parasitoid and they studied dynamics of their system. Atabaigi and Akrami [3] investigated stability and bifurcations of host-parasite model

$$H_{n+1} = rH_n(1-H_n)e^{-\gamma P_n},$$
$$P_{n+1} = H_n(1-e^{-\gamma P_n}).$$

Interested readers can go through the references [1,2,4,8,12] for more details.

However, most of the papers in this fields investigated two dimensional models, i.e. a population model with two species. But higher dimensional models are less considered in literature. Moreover, the host species may be attacked in different developmental stages by a range of parasitoids. For example, Hassell and Waage discussed a winter moth that is parasitized by egg, larval, and pupal parasitoids [10]. Zwölfer [18] modelled the interaction of two species of eurytomid parasitoids, Eurytoma serratulae and E. Robusta, attacking a common host species, the knapweed gall fly (Urophora cardui), on creeping thistle (Cirsium arvense).

Motivated by the above discussion, in this paper, we extend the Nicholson- Bailey host-parasitoid model including one host and two parasitoid:

$$
\begin{aligned}
H_{n+1} &= rH_n e^{-aP_n - bQ_n}, \\
P_{n+1} &= H_n(1 - e^{-aP_n}), \\
Q_{n+1} &= H_n e^{-aP_n}(1 - e^{-bQ_n}),
\end{aligned}
\tag{1.1}
$$

where H_n is host population and P_n, Q_n are the two parasitoid populations at n-th generation respectively, r is the intrinsic growth rate of host and a, b are the searching efficiency of the parasitoids. The underlying biological assumptions in this model are as follows:

- There is one host and two parasitoids which parasitize the same host.
- Suppose the parasitoid P acts first, followed by Q that acts on the surviving hosts.

Our paper is organized as follows: in the next section we prepare some analytical tools needed in the sequence. Section 3 is devoted to the local dynamic of the model. In this section we introduce our system as a three-dimensional host-parasite model. Then, we focus to determine the fixed points and corresponding stability nature. In the rest, some discussions are presented to check whether a fixed point is stable, asymptotically stable or unstable under some parameter conditions. In the last Section, some numerical simulation are presented to explain our work in better way

2. Local dynamics of 3 dimensional maps

In this section, we provide a brief review of the local stability of fixed points in three dimensional discrete dynamical systems. Here, consider nonlinear system

$$
\begin{aligned}
x &\mapsto f_1(x, y, z), \\
y &\mapsto f_2(x, y, z), \\
z &\mapsto f_3(x, y, z).
\end{aligned}
\tag{2.1}
$$

For simplicity, let us write this system as $\mathbf{x} \mapsto \mathbf{F}(\mathbf{x})$, where $\mathbf{x} = (x, y, z)$ and $\mathbf{F} = (f_1, f_2, f_3)$. First, we obtain fixed points of (2.1), i.e. $\mathbf{x}^* = \mathbf{F}(\mathbf{x}^*)$. We can determined the local stability of fixed point $\mathbf{x}^*$ by linearization of system (2.1) near $\mathbf{x}^*$. In this way, the Jacobian matrix of system

(2.1) calculated by

$$\mathbf{JF}(\mathbf{x}) = \left.\begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{pmatrix}\right|_{\mathbf{x}=\mathbf{x}^*}.$$

The fixed point $\mathbf{x}^*$of F is locally asymptotically stable if all the eigenvalues of the Jacobian, $\mathbf{JF}(\mathbf{x}^*)$ lie inside the unit disk and if at least one of the eigenvalue lies out of unit disk, the fixed point is unstable. These conditions can be expressed in terms of the roots of characteristic polynomials of $\mathbf{JF}(\mathbf{x}^*)$. Following lemma can be useful in the proof of the local stability of $\mathbf{x}^*$.

Lemma 2.1. [13, appendix A.1.2] *Let the equation be*

$$c_0\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0, \tag{2.2}$$

where $c_i \in \mathbb{R}$, $i = 0,1,2,3$. *The roots* $\lambda_1, \lambda_2, \lambda_3$ *of equation (2.2) satisfy that* $|\lambda_i| < 1$ *for* $i = 1,2,3$, *if and only if the following conditions are fulfilled.*

$$\begin{cases} c_0 + c_1 + c_2 + c_3 > 0, \\ c_0 - c_1 + c_2 - c_3 > 0, \\ c_0 + c_3 > 0, \\ c_0 - c_3 > 0, \\ c_0(c_0 + c_2) - c_3(c_1 + c_3) > 0, \\ c_0(c_0 - c_2) + c_3(c_1 - c_3) > 0. \end{cases} \tag{2.3}$$

Note that, if some of the eigenvalues of $\mathbf{JF}(\mathbf{x}^*)$ lie on the unit disk, then the above lemma is failed. In this case, $\mathbf{x}^*$ is called non-hyperbolic fixed point. For determining the stability of non-hyperbolic fixed points we can use the center manifold theory. By suitable change of variable, one can transfer the $\mathbf{x}^*$ on the origin and without loss of generality, system (2.1) can be written as

$$\begin{aligned} \mathbf{y} &\mapsto A\mathbf{y} + f(\mathbf{y},\mathbf{z}), \\ \mathbf{z} &\mapsto B\mathbf{z} + g(\mathbf{y},\mathbf{z}), \end{aligned} \tag{2.4}$$

where $(\mathbf{y},\mathbf{z}) \in \mathbb{R}^t \times \mathbb{R}^s$, $t+s=3$, and all eigenvalues of A lie on the unit disk and all of the eigenvalues of B are off the unit disk.

The following theorem asserts the existence of a centre manifold, on which the dynamics of system (2.4) is given by the map on the centre manifold [17].

Theorem 2.2. [17] *There is a C^r-center manifold for system (2.4) that can be represented locally as*

$$M_c = \{(\mathbf{y},\mathbf{z}) \in \mathbb{R}^t \times \mathbb{R}^s \ \mathbf{z} = h(\mathbf{y}), |\mathbf{y}| < \delta, h(0) = 0, Dh(0) = 0\},$$

for a sufficiently small δ. Furthermore, the dynamics restricted to M_c are given locally by the map

$$\mathbf{y} \mapsto A\mathbf{y} + f(\mathbf{y}, h(\mathbf{y})). \tag{2.5}$$

The next theorem states that the dynamics on the center manifold Mc deter- mines completely the dynamics of system (2.4).

Theorem 2.3. [17] *If the fixed point $\mathbf{y}^* = 0$ of Equation (2.5) is stable, asymptotically stable, or unstable, then the fixed point $(0,0)$ of System (2.4) is stable, asymptotically stable, or unstable, respectively.*

3. Local dynamics of the model

Consider the following host-parasite model

$$\begin{aligned} x_{n+1} &= r x_n e^{-ay_n - bz_n}, \\ y_{n+1} &= x_n(1 - e^{-ay_n}), \\ z_{n+1} &= x_n e^{-ay_n}(1 - e^{-bz_n}), \end{aligned} \tag{3.1}$$

where $r, a, b > 0$. Here, we investigate fixed points of the model and theirs stabilities. In the following, we provide the boundary fixed points of the three-dimensional map (3.1).

3.1. Boundary fixed points

There are three boundary fixed points $(0,0,0)$, $(\frac{r\ln r}{a(r-1)}, \frac{\ln r}{a}, 0)$ and $(\frac{r\ln r}{b(r-1)}, 0, \frac{\ln r}{b})$. The Jacobian matrix associated with (3.1) is given by

$$J_{|(x,y,z)} = \begin{pmatrix} r\mathrm{e}^{-ay-bz} & -rxa\mathrm{e}^{-ay-bz} & -rxb\mathrm{e}^{-ay-bz} \\ 1-\mathrm{e}^{-ay} & xa\mathrm{e}^{-ay} & 0 \\ \mathrm{e}^{-ay}\left(1-\mathrm{e}^{-bz}\right) & -xa\mathrm{e}^{-ay}\left(1-\mathrm{e}^{-bz}\right) & x\mathrm{e}^{-ay}b\mathrm{e}^{-bz} \end{pmatrix}. \tag{3.2}$$

At $(0,0,0)$ we have

$$J_{|(0,0,0)} = \begin{pmatrix} r & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

We observe that from (3.3), $\lambda_1 = r$ and $\lambda_2 = \lambda_3 = 0$. Therefore, the origin is asymptotically stable if $0 < r < 1$ and unstable if $r > 1$. According to Lemma 2.1, for determining the local stability of nontrivial fixed points we can state the following proposition.

Proposition 3.1. *Fixed point $(\frac{r\ln r}{a(r-1)}, \frac{\ln r}{a}, 0)$ is stable if and only if the following conditions are fulfilled.*

$$\frac{b}{a} < \frac{(r-1)^2}{r(\ln r)^2},$$
$$\frac{\ln r}{r-1} > \frac{a}{a+b}\left(\frac{b^3}{a^3} + 2\frac{b^4}{a^4} - 1\right),$$
$$\frac{\ln^3 r}{(r-1)^3} > \left(\frac{a}{b} + \frac{b^4}{a^4}\right)\left(\frac{a^2}{b(a+b)}\right).$$

Proof. At $(\frac{r\ln r}{a(r-1)}, \frac{\ln r}{a}, 0)$ we have

$$J|_{\left(\frac{r\ln r}{a(r-1)}, \frac{\ln r}{a}, 0\right)} = \begin{pmatrix} 1 & -\frac{r\ln r}{r-1} & -\frac{br\ln r}{a(r-1)} \\ \frac{r-1}{r} & \frac{\ln r}{r-1} & 0 \\ 0 & 0 & \frac{b\ln r}{a(r-1)} \end{pmatrix}. \tag{3.4}$$

The characteristic polynomial of (3.4) is

$$p(\lambda) = \lambda^3 - \frac{(a+b)\ln r + a(r-1)}{a(r-1)}\lambda^2 + \frac{(b(r-1) + b\ln r + ar(r-1))\ln r}{a(r-1)^2}\lambda - \frac{(\ln r)^2 br}{a(r-1)^2}.$$

Now, its suffice to apply conditions of Lemma 2.1 to $p(\lambda)$. Since $r>1$ we have

$$c_0+c_1+c_2+c_3=\frac{(a(r-1)-b\ln(r))\ln(r)}{a(r-1)}\Longrightarrow\frac{b}{a}<\frac{r-1}{\ln r}. \tag{3.5}$$

$$c_0-c_1+c_2-c_3=\frac{\big(2(r-1)+(r+1)\ln r\big)\big(b\ln r+a(r-1)\big)}{a(r-1)^2}>0.$$

$$c_0+c_3=\frac{a(r-1)^2-br\ln^2 r}{a(r-1)^2}\Longrightarrow\frac{b}{a}<\frac{(r-1)^2}{r(\ln r)^2}. \tag{3.6}$$

Note that condition (3.6) implies condition (3.5).

$$c_0(c_0+c_2)-c_3(c_1+c_3)=1+\frac{\ln r(ar+b-b\ln r)}{a(r-1)}-\frac{b(a+b)r(\ln r)^3}{a^2(r-1)^3}-\frac{b^2r^2(\ln r)^4}{a^2(r-1)^4}. \tag{3.7}$$

Now by using (3.5) and (3.6) in (3.7) we obtain

$$c_0(c_0+c_2)-c_3(c_1+c_3)>1+\frac{\ln r}{r-1}\left(\frac{a+b}{a}\right)-\frac{b^3(a+b)}{a^4}-\frac{b^4}{a^4}.$$

Therefore, $c_0(c_0+c_2)-c_3(c_1+c_3)>0$ iff

$$\frac{\ln r}{r-1}>\frac{a}{a+b}\left(\frac{b^3}{a^3}+2\frac{b^4}{a^4}-1\right). \tag{3.8}$$

Finally,

$$c_0(c_0-c_2)+c_3(c_1-c_3)=1-\frac{\ln r(ar+b-br\ln r)}{a(r-1)}+\frac{b(a+b)r(\ln r)^3}{a^2(r-1)^3}-\frac{b^2r^2(\ln r)^4}{a^2(r-1)^4}. \tag{3.9}$$

Similarly, by using (3.5) and (3.6) in (3.9) we have $c_0(c_0-c_2)+c_3(c_1-c_3)>0$ iff

$$\frac{\ln^3 r}{(r-1)^3}>\left(\frac{a}{b}+\frac{b^4}{a^4}\right)\left(\frac{a^2}{b(a+b)}\right).$$

This completes the proof.

Proposition 3.2. *Fixed point* $(\frac{r\ln r}{b(r-1)},0,\frac{\ln r}{b})$ *is stable if and only if the following conditions are fulfilled.*

$$\frac{a}{b}<(\frac{r-1}{r\ln r})^2,$$
$$\frac{r\ln r}{r-1}>\frac{b^2}{a}+\frac{b}{a^2}(b-a),$$
$$\frac{r^2\ln r)^3}{(r-1)^3}>\left(\frac{a}{b}+\frac{b^2}{a^2}\right)\left(\frac{b^2}{a(a+b)}\right).$$

Proof.

At $(\frac{r\ln r}{b(r-1)}, 0, \frac{\ln r}{b})$ we have

$$J|_{\left(\frac{r\ln r}{b(r-1)},0,\frac{\ln r}{b}\right)} = \begin{pmatrix} 1 & -\frac{ar\ln r}{b(r-1)} & -\frac{r\ln r}{r-1} \\ 0 & \frac{ar\ln r}{b(r-1)} & 0 \\ \\ \frac{r-1}{r} & -\frac{a\ln r}{b} & \frac{\ln r}{r-1} \end{pmatrix}. \tag{3.10}$$

The characteristic polynomial of (3.10) is

$$p^*(\lambda) = \lambda^3 - \frac{(b+ar)\ln r + b(r-1)}{b(r-1)}\lambda^2 + \frac{(b+a)(r-1)r\ln r + ar(\ln r)^2}{b(r-1)^2}\lambda - \frac{a(r\ln r)^2}{b(r-1)^2}.$$

Similar to the previous proposition, its suffice to apply conditions of Lemma 2.1 to $p^*(\lambda)$. So,

$$c_0 + c_1 + c_2 + c_3 = \frac{(b(r-1) - ar\ln(r))\ln(r)}{b(r-1)} \Longrightarrow \frac{a}{b} < \frac{r-1}{r\ln r}. \tag{3.11}$$

$$c_0 - c_1 + c_2 - c_3 = \frac{\big(2(r-1) + (r+1)\ln r\big)\big(ar\ln r + b(r-1)\big)}{b(r-1)^2} > 0.$$

$$c_0 + c_3 = \frac{b(r-1)^2 - ar^2\ln^2 r}{b(r-1)^2} \Longrightarrow \frac{a}{b} < \frac{(r-1)^2}{(r\ln r)^2}. \tag{3.12}$$

Note that condition (3.12) implies condition (3.11).

$$c_0(c_0 + c_2) - c_3(c_1 + c_3) = 1 + \frac{r\ln r(b + a - \ln r)}{b(r-1)} - \frac{ar^2(\ln r)^3(ar+b)}{b^2(r-1)^3} - \frac{a^2r^4(\ln r)^4}{b^2(r-1)^4}. \tag{3.13}$$

by using (3.11) and (3.12) in (3.13) we obtain

$$c_0(c_0 + c_2) - c_3(c_1 + c_3) > 1 + \frac{r\ln r}{r-1}\left(\frac{a+b}{b}\right) - b\left(\frac{a+b}{a}\right) - \frac{b^2}{a^2}.$$

Hence, $c_0(c_0 + c_2) - c_3(c_1 + c_3) > 0$ iff

$$\frac{r\ln r}{r-1} > \frac{b^2}{a} + \frac{b}{a^2}(b-a).$$

For the last condition we obtain

$$\begin{aligned} c_0(c_0 - c_2) + c_3(c_1 - c_3) =& 1 - \frac{r\ln r(b + a - \ln r)}{b(r-1)}, \\ & + \frac{ar^2(\ln r)^3(ar+b)}{b^2(r-1)^3} - \frac{a^2r^4(\ln r)^4}{b^2(r-1)^4}, \\ >& 1 - \frac{b + a - \ln r}{a} + \frac{ar^2(\ln r)^3(ar+b)}{b^2(r-1)^3} - \frac{b^2}{a^2}, \\ >& \frac{r^2\ln r)^3}{(r-1)^3}\frac{a(a+b)}{b^2} - \frac{b^2}{a^2}. \end{aligned}$$

Hence, $c_0(c_0 - c_2) + c_3(c_1 - c_3) > 0$ iff

$$\frac{r^2 \ln r)^3}{(r-1)^3} > \left(\frac{a}{b} + \frac{b^2}{a^2}\right)\left(\frac{b^2}{a(a+b)}\right).$$

This completes the proof.

3.2. Positive fixed points

In this subsection, we search for positive fixed point(s). So, we suppose that $xyz \neq 0$, and have to solve the following system

$$1 = re^{-ay-bz},$$

$$y = x(1-e^{-ay}),$$

$$z = xe^{-ay}(1-e^{-bz}).$$

By solving the above system we obtain

$$z^* = \frac{\ln r - ay^*}{b}, \quad x^* = \frac{y^*}{1-e^{-ay^*}},$$

and y^* is the positive solution

$$\frac{\ln r - ay^*}{b} = \frac{y^*}{1-e^{-ay^*}}\left(e^{-ay^*} - \frac{1}{r}\right), \quad 0 < y^* \leq \frac{\ln r}{a}.$$

Let

$$f_1(y) = \frac{\ln r - ay}{b}, \quad f_2(y) = \frac{y}{1-e^{-ay}}\left(e^{-ay} - \frac{1}{r}\right), \quad 0 < y \leq \frac{\ln r}{a}.$$

Then, the positive fixed point is the positive intersection point(s) of the functions f_1 and f_2. Moreover, we have $f_1(0) = \frac{\ln r}{b}$ and $\lim_{y\to 0^+} f_2(y) = \frac{r-1}{ar}$.

By simple calculation we have

$$f_2'(y) = \frac{ay + r + 1 - ary - re^{-ay} - e^{ay}}{re^{ay}(1-e^{-ay})^2} := \frac{h(y)}{re^{ay}(1-e^{-ay})^2}.$$

It is clear that h(0) = 0 and

$$h'(y) = ae^{-ay}(e^{ay} + r)(1 - e^{ay}) < 0,$$

therefore $f_2'(y) < 0$ for $y_2 \in (0, \frac{\ln r}{a}]$. We also have

$$f_2''(y) = \frac{a(r-1)(aye^{-ay} + ay + 2e^{-ay} - 2)}{r(1-e^{-ay})^3}.$$

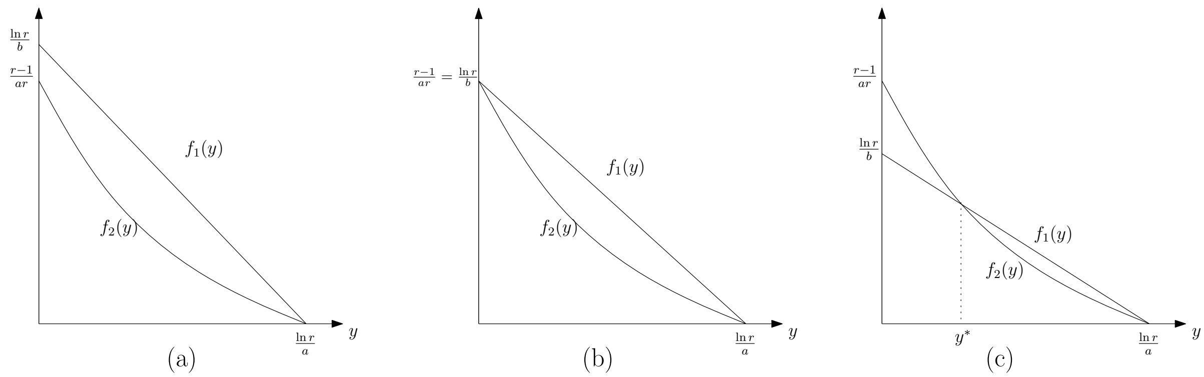


FIGURE 1. Graph of functions $f_1(y)$ and $f_2(y)$ for: (a) $a > \frac{b(r-1)}{r\ln r}$, the system has no positive fixed point, (b) $a = \frac{b(r-1)}{r\ln r}$, the system has no positive fixed point and (c) $a < \frac{b(r-1)}{r\ln r}$, the system has a unique positive fixed point.

By similar argument we can show that $f_2''(y) > 0$ for $y \in (0, \frac{\ln r}{a}]$. It is easy to see that functions f_1 and f_2 connect to each other at the terminal point of the interval, i.e. $f_1(\frac{\ln r}{a}) = f_2(\frac{\ln r}{a}) = 0$. Therefore, with this information we consider three cases:

(1) If $a > \frac{b(r-1)}{r\ln r}$ the function f_1 lies above f_2 for $y \in [0, \frac{\ln r}{a})$, so the system has no positive fixed point (Figure 1 (a)).
(2) If $a = \frac{b(r-1)}{r\ln r}$ the function f_1 lies above f_2 for $y \in (0, \frac{\ln r}{a})$, and two functions reach to each other at the endpoints. So, the system has no positive fixed point (Figure 1 (b)).
(3) If $a < \frac{b(r-1)}{r\ln r}$ two functions have a unique positive intersection at y^*. So, the system has a unique positive fixed point (Figure 1 (c)).

Hence, we summarize the above discussion in the following proposition.

Proposition 3.3. *If $a < \frac{b(r-1)}{r\ln r}$ the system (3.1) has a unique positive fixed point $p^* = (x^*, y^*, z^*)$, where*

$$x^* = \frac{y^*}{1 - e^{-ay^*}}, \quad z^* = \frac{\ln r - ay^*}{b},$$

and y^* is a unique positive solution of

$$\frac{\ln r - ay^*}{b} = \frac{y^*}{1 - e^{-ay^*}} \left(e^{-ay^*} - \frac{1}{r} \right), \quad 0 < y^* < \frac{\ln r}{a}.$$

Now, we need to determine the stability of the positive fixed point. The Jacobian matrix at p^* is

$$J^* = \begin{pmatrix} 1 & -ax^* & -bx^* \\ \frac{y^*}{x^*} & a(x^*-y^*) & 0 \\ \frac{z^*}{x^*} & -az^* & \frac{bx^*}{r} \end{pmatrix},$$

the characteristic polynomial of J^* is

$$p(\lambda) = \lambda^3 - \frac{(bx^*+r+arx^*-ary^*)}{r}\lambda^2 + \frac{brz^*+bx^*+abx^{*^2}-abx^*y^*+arx^*}{r}\lambda - \frac{abx^*(x^*+rz^*)}{r}.$$

Now, we can apply Lemma 1 to prove the following proposition:

Proposition 3.4. *The fixed point p^* in Proposition 3.3 is stable if and only if the following conditions are fulfilled.*

$$1) ay^*(r-bx^*)+brz^*(1-ax^*) > 0 \ or \ r\ln r(1-ax^*)+ax^*y^*(ar-b) > 0,$$

$$2) \frac{abx^*(x^*+rz^*)}{r} < 1,$$

$$3) 1+\frac{r\ln r - ray^*+bx^*+abx^{*^2}-abx^*y^*+arx^*}{r} - \left(\frac{abx^{*^2}+arx^*\ln r - a^2rx^*y^*}{r}\right) \times \left(\frac{bx^*+r+arx^*-ary^*}{r} + \frac{abx^{*^2}+arx^*\ln r - a^2rx^*y^*}{r}\right) > 0,$$

$$4) 1-\frac{r\ln r - ray^*+bx^*+abx^{*^2}-abx^*y^*+arx^*}{r} - \left(\frac{abx^{*^2}+arx^*\ln r - a^2rx^*y^*}{r}\right) \times \left(-\frac{bx^*+r+arx^*-ary^*}{r} + \frac{abx^{*^2}+arx^*\ln r - a^2rx^*y^*}{r}\right) > 0.$$

Remark. *Since the positive fixed point coordinates are not explicitly obtained and there are many parameters in Proposition 3.4, it's not easy to have an analyti- cally proof for it. By numerically computing for a set of different parameters, we obtain that the positive fixed point is unstable. In the other hand, by Gauses competitive exclusion principle [9], one of the parasite may be excluded from the system. So, we expected the positive fixed point to be an unstable fixed point.*

3.3. Non-hyperbolic cases

In this subsection, we state some conditions that the system (3.1) has the non- hyperbolic fixed point. • **Case (1):**

If $b=\frac{a(r-1)}{\ln r}$, then $p_1^*=(\frac{r\ln r}{a(r-1)},\frac{\ln r}{a},0)$ is a non-hyperbolic fixed point. First, we bring the fixed point p_1^* to the origin by linear transformation, i.e. we have

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix}\mapsto\begin{pmatrix}1-\frac{r\ln r}{r-1} & -r & \\ \frac{r-1}{r} & \frac{\ln r}{r-1} & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix}+\begin{pmatrix}\frac{ar\ln r}{2(r-1)}\tilde{y}^2+\frac{r(r-1)^2}{2\ln r}\tilde{z}^2-a\tilde{x}\tilde{y}-\frac{a(r-1)}{\ln r}\tilde{x}\tilde{z}+O(3)\\ \frac{a}{r}\tilde{x}\tilde{y}-\frac{a\ln r}{2(r-1)}\tilde{y}^2+O(3)\\ -\frac{r-1}{2\ln r}\tilde{z}^2-a\tilde{z}\tilde{y}+\frac{a(r-1)}{r\ln r}\tilde{x}\tilde{z}+O(3)\end{pmatrix}. \tag{3.14}$$

The eigenvalues of linear part of system (3.14), are $\lambda_1=1$ and $\lambda_{2,3}=\alpha\pm i\beta$, where

$$\Delta=(\ln r+r-1)^2-4r(r-1)\ln r,\quad \alpha=\frac{\ln r+r-1}{2(r-1)},\quad \beta=\frac{\sqrt{-\Delta}}{2(r-1)}.$$

Let

$$\sigma_1=\frac{r(\ln r-r+1)}{(r-1)\ln r},\quad \sigma_2=\frac{\sqrt{-\Delta}}{2r\ln r},\quad \sigma_3=\frac{r-1-\ln r}{2r\ln r},$$

then by using linear transformation

$$\begin{pmatrix}\tilde{x}\\ \tilde{y}\\ \tilde{z}\end{pmatrix}=\begin{pmatrix}\sigma_1 & 1 & \sigma_2\\ -\frac{r-1}{\ln r} & \sigma_3 & 0\\ 1 & 0 & 0\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}, \tag{3.15}$$

we can write this system in Jordan form as:

$$\begin{pmatrix}u\\ v\\ w\end{pmatrix}\mapsto\begin{pmatrix}1 & 0 & 0\\ 0 & \alpha & -\beta\\ 0 & \beta & \alpha\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}+\begin{pmatrix}\psi_1(u,v,w)\\ \psi_2(u,v,w)\\ \psi_3(u,v,w)\end{pmatrix}, \tag{3.16}$$

where,

$$
\begin{aligned}
\psi_1(u,v,w) =& \frac{r(r-1)(1-a)}{2\ln r}u^2 + \left(\frac{a\sigma_3^2 r\ln r}{2(r-1)} - a\sigma_3\right)v^2 - a\sigma_1\sigma_3 uv \\
& - a\sigma_2\sigma_3 vw + O(3), \\
\psi_2(u,v,w) =& -\frac{a(r-1)}{2r\ln r}(r+2\sigma_1)u^2 + \left(\frac{a\sigma_3}{r} - \frac{a\sigma_3^2\ln r}{2(r-1)}\right)v^2 + \frac{a\sigma_2\sigma_3}{r}vw \\
& + \frac{1}{r}(ar\sigma_3 - \frac{a(r-1)}{\ln r} + a\sigma_1\sigma_3)uv + -\frac{a\sigma_2(r-1)}{r\ln r}uw + O(3), \\
\psi_3(u,v,w) =& \frac{r-1}{\ln r}\left(a - \frac{1}{2} + \frac{a\sigma_1}{r}\right)u^2 + \left(\frac{a(r-1)}{r\ln r} - a\sigma_3\right)uv \\
& + \frac{a\sigma_2(r-1)}{r\ln r}uw + O(3).
\end{aligned}
$$

We will apply center manifold theory to this problem. The center manifold can locally be represented as follows

$$M_c = \{(u,v,w) \in \mathbb{R}^3 | v = h_1(u), w = h_2(u), h_i(0) = 0, Dh_i(0) = 0,\ i = 1,2\},$$

for sufficiently small u. We assume a center manifold of the form

$$h(u) = \begin{pmatrix} h_1(u) \\ h_2(u) \end{pmatrix} = \begin{pmatrix} a_2u^2 + a_3u^3 + O(u^4) \\ b_2u^2 + b_3u^3 + O(u^4) \end{pmatrix}.$$

By some calculation, the center manifold is given by the graph of $h(u)$, where

$$
\begin{aligned}
h_1(u) &= \frac{\beta k_2 + (\alpha-1)k_1}{-\beta^2 + (\alpha-1)^2}u^2 + o(u^3), \\
h_2(u) &= \frac{\beta k_1 + (\alpha-1)k_2}{\beta^2 - (\alpha-1)^2}u^2 + o(u^3),
\end{aligned}
$$

and $k_1 = -\frac{a(r-1)}{2r\ln r}(r+2\sigma_1)$, $k_2 = \frac{r-1}{\ln r}\left(a - \frac{1}{2} + \frac{a\sigma_1}{r}\right)$. The map on the center manifold is given by

$$u \mapsto u + \frac{r(r-1)(1-a)}{2\ln r}u^2 - (a\sigma_1\sigma_3\frac{\beta k_2 + (\alpha-1)k_1}{-\beta^2 + (\alpha-1)^2})u^3 + o(u^4). \tag{3.17}$$

Hence, we can summarize above discussion in the following proposition.

Proposition 3.5. *Suppose* $b = \frac{a(r-1)}{\ln r}$, *then* $p_1^* = (\frac{r\ln r}{a(r-1)}, \frac{\ln r}{a}, 0)$ *is a non-hyperbolic fixed point. If* $a \neq 1$, *then* p_1^* *is stable, else if* $6a\sigma_1\sigma_3\frac{\beta k_2+(\alpha-1)k_1}{-\beta^2+(\alpha-1)^2} < 0$ *then* p_1^* *is unstable and if* $6a\sigma_1\sigma_3\frac{\beta k_2+(\alpha-1)k_1}{-\beta^2+(\alpha-1)^2} > 0$ *then* p_1^* *is asymptotically stable.*

• **Case (2):**

If $b=\frac{ar\ln r}{(r-1)}$, then $p_2^*=(\frac{r\ln r}{b(r-1)},0,\frac{\ln r}{b})$ is a non-hyperbolic fixed point. First, we bring the fixed point p_1^* to the origin by linear transformation, i.e. we have

$$\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix}\mapsto\begin{pmatrix}1 & -1 & -\frac{r\ln r}{r-1}\\ 0 & 1 & 0\\ \frac{r-1}{r} & -\frac{r-1}{r} & \frac{\ln r}{r-1}\end{pmatrix}\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix}+\begin{pmatrix}\frac{1}{2}a\hat{y}^2+\frac{ar^2\ln^r}{2(r-1)^2}\hat{z}^2-a\hat{x}\hat{y}-\frac{ar\ln r}{r-1}\hat{z}(\hat{x}-\hat{y})+O(3)\\ a\hat{x}\hat{y}-\frac{1}{2}a\hat{y}^2+O(3)\\ \frac{ar\ln^r}{2(r-1)^2}\hat{z}^2+\frac{a(r-1)}{2r}\hat{y}^2+\frac{a\ln r}{r-1}\hat{z}(\hat{x}-\hat{y})-\frac{a(r-1)}{r}\hat{x}\hat{y}+O(3)\end{pmatrix}. \tag{3.18}$$

The eigenvalues of linear part of system (3.18), are $\lambda_1=1$ and $\lambda_{2,3}=\alpha\pm i\beta$, where

$$\Delta=(\ln r+r-1)^2-4r(r-1)\ln r,\quad \alpha=\frac{\ln r+r-1}{2(r-1)},\quad \beta=\frac{\sqrt{-\Delta}}{2(r-1)}.$$

Let

$$\sigma_1=\frac{r(\ln r-r+1)}{(r-1)\ln r},\quad \sigma_2=\frac{\sqrt{-\Delta}}{2r\ln r},\quad \sigma_3=\frac{r-1-\ln r}{2r\ln r},$$

then using linear transformation

$$\begin{pmatrix}\hat{x}\\ \hat{y}\\ \hat{z}\end{pmatrix}=\begin{pmatrix}\delta_1 & 1 & 0\\ \delta_2 & 0 & 0\\ 1 & \delta_3 & \delta_4\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}, \tag{3.19}$$

where

$$\delta_1=-\frac{r(\ln r-r+1)}{(r-1)^2},\quad \delta_2=-\frac{r\ln r}{r-1},$$
$$\delta_3=\frac{r-1-\ln r}{2r\ln r},\quad \delta_4=-\frac{\sqrt{-\Delta}}{2r\ln r},$$

we can write this system in the Jordan form as follow.

$$\begin{pmatrix}u\\ v\\ w\end{pmatrix}\mapsto\begin{pmatrix}1 & 0 & 0\\ 0 & \alpha & -\beta\\ 0 & \beta & \alpha\end{pmatrix}\begin{pmatrix}u\\ v\\ w\end{pmatrix}+\begin{pmatrix}\phi_1(u,v,w)\\ \phi_2(u,v,w)\\ \phi_3(u,v,w)\end{pmatrix}, \tag{3.20}$$

where,

$$\begin{aligned}
\phi_1(u,v,w) =& \left(-\frac{a\delta_1 \ln r}{r-1} - \frac{a(r+1)\ln^2 r}{2(r-1)} - a\delta_1\delta_2\right)u^2 + \frac{ar^2\delta_4^2\ln^2 r}{2(r-1)^2}w^2 \\
&+ \left(\frac{ar^2\delta_3^2\ln^2 r}{2(r-1)^2} - \frac{ar\delta_3\ln r}{r-1}\right)v^2 + \left(\frac{ar^2\delta_3\delta_4\ln^2 r}{(r-1)^2} - \frac{ar\delta_4\ln r}{r-1}\right)vw \\
&- \frac{ar\delta_1\delta_4}{r-1}uw - \left(a\delta_2 + \frac{ar(\delta_1\delta_3+1)\ln r}{r-1}\right)uv + O(3), \\
\phi_2(u,v,w) =&(a\delta_1 - \frac{1}{2}\delta_2^2)u^2\delta_2 uv + O(3), \\
\phi_3(u,v,w) =& \left(\frac{ar\ln^2 r}{2(r-1)^2} + \frac{a\delta_1\ln r}{r-1} + a\delta_1\ln r + \frac{ar\ln^2 r}{2(r-1)}\right)u^2 - \frac{ar^2\delta_4^2\ln^2 r}{2(r-1)^2}w^2 \\
&+ \left(\frac{a\delta_3\ln r}{r-1} - \frac{a\delta_3 r\ln^2 r}{2(r-1)^2}\right)v^2 + \left(a\ln r + \frac{a(\delta_1\delta_3+1)}{r-1}\right)uv \\
&+ \left(\frac{a\delta_4\ln r}{r-1} - \frac{ar\delta_3\delta_4\ln^2 r}{(r-1)}\right)vw + \frac{a\delta_1\delta_4\ln r}{r-1}uw + O(3).
\end{aligned}$$

Similar to **Case (1)**, we will apply center manifold theory to this problem. So, the center manifold can locally be represented as follows

$$M_c = \{(u,v,w) \in \mathbb{R}^3 | v = h_1(u), w = h_2(u), h_i(0) = 0, Dh_i(0) = 0,\ i = 1,2\},$$

for sufficiently small u. We consider a center manifold of the form

$$h(u) = \begin{pmatrix} h_1(u) \\ h_2(u) \end{pmatrix} = \begin{pmatrix} a_2u^2 + a_3u^3 + O(u^4) \\ b_2u^2 + b_3u^3 + O(u^4) \end{pmatrix}$$

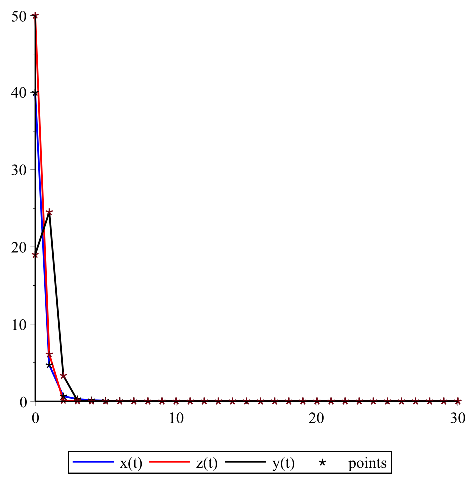
By some calculation, the center manifold is given by the graph of $h(u)$, where

$$\begin{aligned}
h_1(u) &= \frac{\beta k + (\alpha-1)(\frac{1}{2}\delta_2^2 - a\delta_1)}{\beta^2 + (\alpha-1)^2}u^2 + o(u^3), \\
h_2(u) &= -\frac{\beta(\frac{1}{2}\delta_2^2 - a\delta_1) + (\alpha-1)k}{\beta^2 + (\alpha-1)^2}u^2 + o(u^3),
\end{aligned}$$

where, $k = \frac{ar\ln^2 r}{2(r-1)^2} + \frac{a\delta_1\ln r}{r-1} + a\delta_1\ln r + \frac{ar\ln^2 r}{2(r-1)}$. The map on the center manifold is given by

$$u \mapsto u + \left(-\frac{a\delta_1\ln r}{r-1} - \frac{a(r+1)\ln^2 r}{2(r-1)} - a\delta_1\delta_2\right)u^2 + o(u^3).$$

Proposition 3.6. *suppose $b = \frac{ar\ln r}{(r-1)}$, then $p_2^* = (\frac{r\ln r}{b(r-1)}, 0, \frac{\ln r}{b})$ is a non-hyperbolic fixed point. If $\left(-\frac{a\delta_1\ln r}{r-1} - \frac{a(r+1)\ln^2 r}{2(r-1)} - a\delta_1\delta_2\right) \neq 0$, then this fixed point is unstable.*



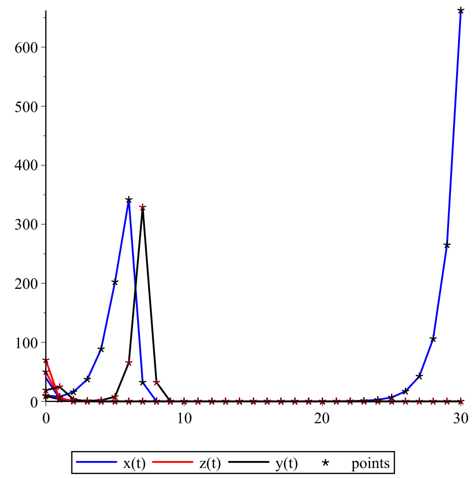


FIGURE 2. Some simulations of model (3.1): (left) $r = 0.5, a = 0.05, b = 0.01$ and $(x_0, y_0, z_0) = (40, 19, 50)$. (right) $r = 2.5, a = 0.05, b = 0.01$ and $(x_0, y_0, z_0) = (10, 9, 70)$.

4. Numerical simulation

In this section, we present some phase portrait of model (3.1) to explain the above theoretical analysis.

Figure 2 illustrates the dynamics of model (3.1) for some values of r. When $r < 1$, three populations get very close to zero. In the other words, the origin is asymptotically stable. When $r > 1$, the host population increase in amplitude. In this case, two parasite populations $y(t)$ and $z(t)$ extinct but the host population $x(t)$ goes to infinity.

Let $r = 2.5$, $a = 0.05$ and $b = 0.01$. In this case, there are three fixed points $p_0^* = (0,0,0)$, $p_1^* \simeq (30.54302440, 18.32581464, 0)$ and $p_2^* \simeq (152.7151220, 0, 91.62907319)$. By simple calculation, one can see that fixed points p_1^* and p_2^* are unstable. We choose an initial value close to fixed points and consider the behaviour of three populations. Figure 3, shows that orbits left the neighbourhood of the fixed points.

Let $r = 2.5$, $a = 0.05$ and $b = \frac{a(r-1)}{\ln(r)}$, then there are fixed points $p_0^* = (0,0,0)$, $p_1^* \simeq (30.54302440, 18.32581464, 0)$ and $p_2^* \simeq (46.64381697, 0, 27.98629018)$. In this case, p_1^* is a nonhyperbolic fixed point which is unstable (Figure 4-left).

Let $r = 2.5$, $a = 0.05$ and $b = \frac{ar\ln r}{r-1}$, then there are fixed points $p_0^* = (0,0,0)$, $p_2^* = (20, 0, 12)$ and $p_1^* \simeq (30.54302440, 18.32581464, 0)$. In this case, p_2^* is a nonhyperbolic fixed point which

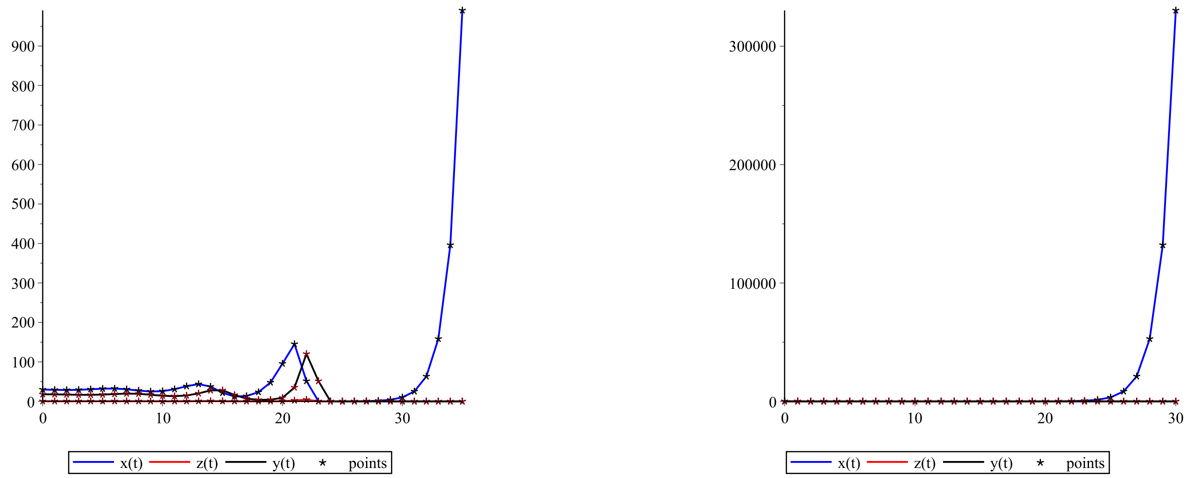


FIGURE 3. Some simulations of model (3.1) when $r = 0.5, a = 0.05, b = 0.01$ and (left) $(x_0, y_0, z_0) = (30, 18, 0.5)$. (right) $(x_0, y_0, z_0) = (152, 0.5, 91)$.

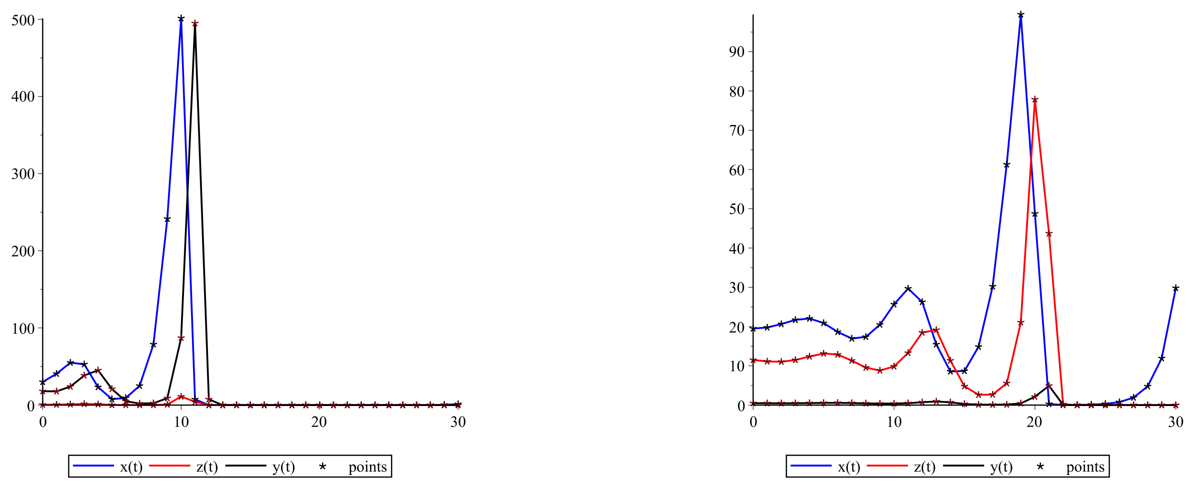


FIGURE 4. Some simulations of model (3.1) when $r = 0.5, a = 0.05$ and (left) $b = \frac{a(r-1)}{\ln(r)}$, $(x_0, y_0, z_0) = (30, 18, 0.5)$. (right) $b = \frac{ar\ln r}{r-1}$, $(x_0, y_0, z_0) = (19.5, .5, 11)$.

is unstable (Figure 4-right). Moreover, we proved that if $a < \frac{b(r-1)}{r\ln(r)}$, then the model has a unique positive fixed point. Now, we take $r = 3, b = 0.01$ and $a = 0.005968261510$, so the model has a unstable positive fixed point $p^* = (183.4678407, 30.88162833, 91.43026553)$ (Figure 5-left). But if we change a to 0.01606826151, then the model has no positive fixed point (Figure 5-right).

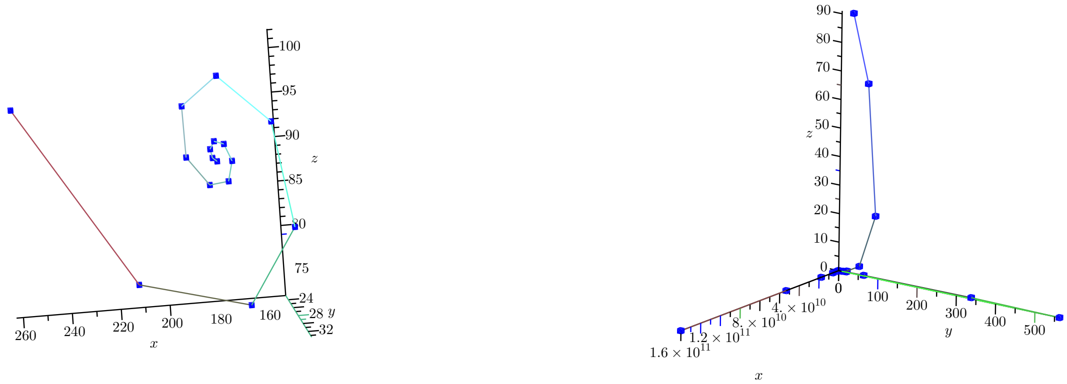


FIGURE 5. Some phase portrait of the model (left) $a < \frac{b(r-1)}{r\ln(r)}$, $(x_0, y_0, z_0) = (183, 30, 91)$. The model has a unique positive fixed point and the solutions are far from the unstable positive fixed point $p^* = (183 : 4678407; 30 : 88162833; 91 : 43026553)$. (right) $a > \frac{b(r-1)}{r\ln(r)} \frac{ar\ln r}{r-1}$, $(x_0, y_0, z_0) = (183, 30, 91)$. The model not any positive fixed point.

Conflict of Interests

The author declares that there is no conflict of interests.

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