Available online at http://scik.org
Commun. Math. Biol. Neurosci. 2018, 2018:13
https://doi.org/10.28919/cmbn/3677
ISSN: 2052-2541

STABILITY ANALYSIS OF A STOCHASTIC SIS MODEL WITH DOUBLE EPIDEMIC HYPOTHESIS AND SPECIFIC NONLINEAR INCIDENCE RATE

AMINE EL KOUFI*, MOUHCINE NAIM, ABDELKRIM BENNAR, NOURA YOUSFI

Laboratory of Analysis Modeling and Simulation (LAMS), Faculty of Sciences Ben M'sik, Hassan II University, P.O Box 7955, Sidi Othman, Casablanca, Morocco

Communicated by L. Chen

Abstract. The purpose of this work is to investigate the almost surely exponentially stable of a stochastic SIS model with double epidemic hypothesis and specific nonlinear incidence rate. We establish the global existence and positivity of solution. Furthermore, the stability of the disease-free equilibrium of the model are showed. The analytical results are illustrated by computer simulations.

Keywords: SIS stochastic model; double epidemic; stability.

2010 AMS Subject Classification: 92D30, 47H05.

1. Introduction

Epidemiology is the study of the spread of diseases with the objective to trace factors that are responsible for or contribute to their occurrence. Consequently, it has been investigated by several mathematicians through establishing mathematical models for a long time (see, for

*Corresponding author

E-mail addresses: elkoufiamine1@gmail.com

Received February 12, 2018

example, [1, 2, 3, 4]). Particularly, the susceptible-infected-susceptible SIS epidemic model is often used to model the dynamics of the diseases such as the bacterial diseases and some sexually transmitted diseases where individuals start off susceptible, at some stage catch the disease, and after a short infectious period become susceptible again [5].

In the classic SIS epidemic model, the disease is caused by one virus. In many cases, the disease is frequently not caused by one certain kind of viruses, but two or more kinds of viruses. Recently, the authors of [6, 7, 8] investigated the epidemic model with double epidemic hypothesis which has two epidemic diseases caused by two different viruses. In this paper, we consider a deterministic SI model with double epidemic hypothesis and cure rate described by the following differential system

$$(1)\qquad \begin{cases} \dot{S} = & A - dS - \frac{\beta_1 SI_1}{1+\alpha_1 S+\gamma_1 I_1+\mu_1 SI_1} - \frac{\beta_2 SI_2}{1+\alpha_2 S+\gamma_2 I_2+\mu_2 SI_2} + r_1 I_1 + r_2 I_2, \\ \dot{I}_1 = & \frac{\beta_1 SI_1}{1+\alpha_1 S+\gamma_1 I_1+\mu_1 SI_1} - (d+a_1+r_1) I_1, \\ \dot{I}_2 = & \frac{\beta_2 SI_2}{1+\alpha_2 S+\gamma_2 I_2+\mu_2 SI_2} - (d+a_2+r_2) I_2, \end{cases}$$

where $S(t)$ represents the number of susceptibles at time t, I_1 and I_2 are the total population of the infectives with virus V_1 and V_2 at time t, respectively. A represents the recruitment rate of the population, d is the natural death rate of the population, a_i is the disease-related death rate, r_i is the treatment cure rate, and β_i is the infection coefficient, $i = 1,2$. The incidence rate of disease I_i is modeled by the specifc functional response $\beta_i SI_i/1+\alpha_i S+\gamma_i I_i+\mu_i SI_i$, where $\alpha_i, \gamma_i, \mu_i \geq 0$. This specific functional response was introduced by Hattaf et al [9], and he is becomes the bilinear incidence rate if $\alpha_i = \gamma_i = \mu_i = 0$, the saturated incidence rate if $\alpha_i = \gamma_i = 0$ or $\alpha_i = \mu_i = 0$, the Beddington-DeAngelis functional response [10, 11] if $\mu_i = 0$, and the Crowley-Martin functional response [12] if $\alpha_i\gamma_i = \mu_i$.

In the reality, epidemic systems are inevitably effected by environmental white noise. Therefore it is necessary to study that how the noise influences on the epidemic models. Consequently, many authors have studied stochastic epidemic models, see [13, 14, 15]. For this, we consider the case in which the rates β_i is subject to random fluctuations. namely, $\beta_i dt$ is replaced by $\beta_i dt + \sigma_i dB_i(t)$, where B_i independent standard Brownian motions and σ_i represent the intensities of the white noises of B_i. Therefore, the corresponding stochastic system to (1) can be

described by the Itô's equation

(2)
$$\begin{aligned}
dS &= \left(A - dS - \frac{\beta_1 SI_1}{f_1(S,I_1)} - \frac{\beta_2 SI_2}{f_2(S,I_2)} + r_1 I_1 + r_2 I_2\right)dt - \sigma_1 \frac{SI_1}{f_1(S,I_1)} dB_1(t) - \sigma_2 \frac{SI_2}{f_2(S,I_2)} dB_2(t), \\
dI_1 &= \left(\frac{\beta_1 SI_1}{f_1(S,I_1)} - (d + a_1 + r_1)I_1\right)dt + \sigma_1 \frac{SI_1}{f_1(S,I_1)} dB_1(t), \\
dI_2 &= \left(\frac{\beta_2 SI_2}{f_2(S,I_2)} - (d + a_2 + r_2)I_2\right)dt + \sigma_2 \frac{SI_2}{f_2(S,I_2)} dB_2(t)
\end{aligned}$$

with $f_i(S,I_i) = 1 + \alpha_i S + \gamma_i I_i + \mu_i SI_i$, $i = 1,2$.

The rest of the paper is organized as follows. In the next section , we present the global stability analysis of the disease-free equilibrium for deterministic model (1). In Section 3, we prove that the stochastic system (2) has a unique global positive solution and we give a sufficient condition for the almost sure exponential stability of the disease-free equilibrium. Numerical simulation to illustrate our theoretical result will be presented in Section 4. Finally, we close the paper with discussions and future directions.

2. Deterministic SIS model

For biological reasons, we assume that the initial conditions of system (1) satisfy

$$S(0) = S_0 \geq 0,\ I_1(0) = I_{10} \geq 0,\ I_2(0) = I_{20} \geq 0.$$

Thus the system (1) is positive [16], that is, $S(t) \geq 0$, $I_1(t) \geq 0$ and $I_2(t) \geq 0$ for all $t > 0$. In fact by Proposition 2.1 in [17], we have

$$\begin{cases}
\dot{S} = A + r_1 I_1 + r_2 I_2 \geq 0 & \text{for } S = 0,\ I_1, I_2 \geq 0, \\
\dot{I}_1 = 0 \geq 0 & \text{for } I_1 = 0,\ S, I_2 \geq 0, \\
\dot{I}_2 = 0 \geq 0 & \text{for } I_2 = 0,\ S, I_1 \geq 0.
\end{cases}$$

By summing all the equations of the system (1) we find that the total population size $N(t) = S(t) + I_1(t) + I_2(t)$ satisfies the inequality

$$\dot{N}(t) = A - dN(t) - a_1 I_1(t) - a_2 I_2(t) \leq A - dN(t),$$

which ensures that $\dot{N} < 0$ if $N > \frac{A}{d}$. The standard comparison theorem [18] can be used to deduce that

$$N(t) \leq \frac{A}{d} - \left[\frac{A}{d} - N(0)\right] e^{-dt}.$$

Thus, the feasible solution set of the system equation of the model enters and remains in the region

$$\Gamma = \left\{(S, I_1, I_2) \in \mathbb{R}^3_+ : S + I_1 + I_2 \leq \frac{A}{d}\right\}.$$

Therefore, the model (1) is well posed epidemiologically and mathematically [19]. Hence, it is sufficient to study the dynamics of the model (1) in Γ.

It is easy to see that system (1) has a disease-free equilibrium state $E_0 = (\frac{A}{d}, 0, 0)$. Therefore, the basic reproduction numbers are

$$R_{01} = \frac{\beta_1 A}{(d + \alpha_1 A)(d + a_1 + r_1)}, \quad R_{02} = \frac{\beta_2 A}{(d + \alpha_2 A)(d + a_2 + r_2)}.$$

We mention that the expression of R_{01} and R_{02} can also be obtained by applying the next generation matrix method provided by van den Driessche and Watmough [20].

Biologically, R_{0i} $(i = 1, 2)$ represents the average number of secondary infections that occur when one infectious individual is introduced into a completely susceptible population.

Now, we investigate the local stability of the disease-free equilibrium E_0. The Jacobian matrix of system (1) at the equilibrium E_0 is as follows

$$J = \begin{pmatrix} -d & \frac{-\beta_1 A}{d+\alpha_1 A} + r_1 & \frac{-\beta_2 A}{d+\alpha_2 A} + r_2 \\ 0 & \frac{\beta_1 A}{d+\alpha_1 A} - (d + a_1 + r_1) & 0 \\ 0 & 0 & \frac{\beta_2 A}{d+\alpha_2 A} - (d + a_2 + r_2) \end{pmatrix}.$$

.

The tree eigenvalues of J are $\xi_1 = -d < 0$, $\xi_2 = (d + a_1 + r_1)(R_{01} - 1)$ and $\xi_3 = (d + a_2 + r_2)(R_{02} - 1)$. Hence, the equilibrium E_0 will be locally asymptotically stable if $R_{01} < 1$ and $R_{02} < 1$, and unstable when $R_{01} > 1$ or $R_{02} > 1$.

The following theorem discusses the global stability of the disease-free equilibrium E_0.

Theorem 2.1. *If $R_{01} \leq 1$ and $R_{02} \leq 1$, then the disease-free equilibrium E_0 of (1) is globally asymptotically stable in Γ.*

Proof. *Let V be the Lyapunov function defined as*

$$V(S,I_1,I_2) = (d+\alpha_1 A)I_1 + (d+\alpha_2 A)I_2.$$

Differentiating V along the solutions of positive system (1), we have

$$\begin{aligned}\dot{V}(S,I_1,I_2) = &\left[\frac{\beta_1 (d+\alpha_1 A)S}{1+\alpha_1 S+\gamma_1 I_1+\mu_1 S I_1} - (d+\alpha_1 A)(d+a_1+r_1)\right] I_1 \\ &+ \left[\frac{\beta_2 (d+\alpha_2 A)S}{1+\alpha_2 S+\gamma_2 I_2+\mu_2 S I_2} - (d+\alpha_2 A)(d+a_2+r_2)\right] I_2.\end{aligned}$$

Using the inequalitys

$$\frac{S}{1+\gamma_i I_i+\mu_i S I_i} \leq \frac{A}{d}, \;\; i=1,2$$

we get

$$\frac{(d+\alpha_i A)S}{1+\alpha_i S+\gamma_i I_i+\mu_i S I_i} \leq A, \;\; i=1,2.$$

Thus

$$\dot{V}(S,I_1,I_2) \leq (d+\alpha_1 A)(d+a_1+r_1)I_1(R_{01}-1) + (d+\alpha_2 A)(d+a_2+r_2)I_2(R_{02}-1).$$

Therefore, $R_{01} \leq 1$ and $R_{02} \leq 1$ ensures that $\dot{V}(S,I_1,I_2) \leq 0$. Furthermore, it is easy to verify that the singleton $\{E_0\}$ is the largest compact invariant set in $\left\{(S,I_1,I_2) \in \Gamma : \dot{V}(S,I_1,I_2) = 0\right\}$, and hence by the LaSalle's invariance principle [21]*, every solution to equations of system (1), with initial conditions in Γ, approaches E_0 as $t \to \infty$. Thus E_0 is globally asymptotically stable.*

3. Stochastic SIS model

Let $(\Omega, \mathscr{F}, \{\mathscr{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathscr{F}_0$ contains all P-null sets). Next, we consider the d-dimensional stochastic system :

$$dx(t) = f(x(t),t)dt + g(x(t),t)dB(t) \tag{3}$$

where $f(x,t)$ is a function defined in $\mathbb{R}^d \times [t_0, +\infty)$ and $g(x,t)$ is an $d \times m$ matrix, f and g are locally Lipschitz functions in x. $\{B(t)\}_{t\geq 0}$ is an d-dimensional standard Wiener process defined on the above probability space.

Let us suppose that $f(t,0) = g(t,0) = 0$ for all $t \geq 0$. We assume that $x = 0$ is a solution of the system (3).

Definition 3.1. [22] *The trivial solution $x = 0$ of system (3) is said to be almost surely exponentially stable if for all $x(0) = x_0 \in \mathbb{R}^d$:*

$$\limsup_{t\to\infty} \frac{1}{t} \ln |x(t,x_0)| < 0, \ \ almost \ \ surely \ \ (briefly \ \ a.s.).$$

Denote by $\mathscr{C}^{2,1}(\mathbb{R}^d \times [t_0, +\infty); \mathbb{R}_+)$ the family of all nonnegative functions $U(x,t)$ defined on $\mathbb{R}^d \times [t_0, +\infty)$ such that they are continuously twice differentiable in x and once in t. The differential operator $\mathscr{L}$ [23] associated with (3) is defined by

$$\mathscr{L} = \frac{\partial}{\partial t} + \sum_{i=1}^{d} f_i(x,t).\frac{\partial}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{d} [g^T(x,t)g(x,t)]_{ij}.\frac{\partial}{\partial x_i \partial x_j}.$$

If the differential operator $\mathscr{L}$ acts on a function $U \in \mathscr{C}^{2,1}(\mathbb{R}^d \times [t_0, +\infty); \mathbb{R}_+)$, then

$$\mathscr{L}U = U_t(x,t) + U_x(x,t)f(x,t) + \frac{1}{2}Trac[g(x,t)^T U_{xx}(x,t)g(x,t)]$$

where $U_t(x,t) = \frac{\partial U}{\partial t}$, $U_x(x,t) = \left(\frac{\partial U}{\partial x_1}, ..., \frac{\partial U}{\partial x_d}\right)$, $U_{xx}(x,t) = \left(\frac{\partial^2 U}{\partial x_i \partial x_j}\right)$.

3.1. Existence and uniqueness of the global positive solution

The following theorem shows that the solution of our system (2) is global and positive.

Theorem 3.1. *For any initial value $(S_0, I_{10}, I_{20}) \in \Gamma$, there is a unique solution $(S(t), I_1(t), I_2(t))$ to (2) on $t \geq 0$, and this solution remains in Γ with probability one.*

Proof. *Since the coefficients of system (2) are locally Lipschitz continuous, then for any initial value $(S_0, I_{10}, I_{20}) \in \Gamma$ there is a unique local solution $(S(t), I_1(t), I_2(t))$ on $t \in [0, \tau_e)$, where τ_e is the explosion time. To show that this solution is global, we only need to prove $\tau_e = \infty$ a.s.*

Define the stopping time

$$\tau = \inf\{t \in [0,\tau_e) : \ S(t) \leq 0 \ \text{ or } \ I_1(t) \leq 0 \ \text{ or } \ I_2(t) \leq 0\}.$$

We set $\inf\emptyset = \infty$*, as usual* $\emptyset$ *denotes the empty set. We have* $\tau \leq \tau_e$*, if* $\tau = \infty$ *a.s., then* $\tau_e = \infty$ *a.s., and* $(S(t), I_1(t), I_2(t)) \in \Gamma$ *for all* $t \geq 0$*. In addition, to complete the proof we only need to prove* $\tau = \infty$*. Assume that* $\tau < \infty$*, then there exists a* $T > 0$ *and* $\varepsilon \in (0,1)$ *such that*

$$\mathbb{P}(\tau < T) > \varepsilon.$$

Define a $\mathscr{C}^2$*-function*

$$U(S, I_1, I_2) = \ln(SI_1I_2).$$

Calculating the differential of U *along the solution trajectories of system (2) using Itô's formula, we get*

$$\begin{aligned}
dU =& \frac{1}{S}\left[A - dS - \frac{\beta_1 SI_1}{f_1(S,I_1)} - \frac{\beta_2 SI_2}{f_2(S,I_2)} + r_1I_1 + r_2I_2\right]dt + \frac{1}{I_1}\left[\frac{\beta_1 SI_1}{f_1(S,I_1)} - (d + a_1 + r_1)I_1\right]dt \\
&+ \frac{1}{I_2}\left[\frac{\beta_2 SI_2}{f_2(S,I_2)} - (d + a_2 + r_2)I_2\right]dt - \frac{\sigma_1^2}{2}\left(\frac{I_1^2 + S^2}{f_1(S,I_1)}\right) - \frac{\sigma_2^2}{2}\left(\frac{I_2^2 + S^2}{f_2(S,I_2)}\right)dt \\
&+ \sigma_1\frac{S - I_1}{f_1(S,I_1)}dB_1(t) + \sigma_2\frac{S - I_2}{f_2(S,I_2)}dB_2(t) \\
&\geq \left(-d - \beta_1 I_1 - \beta_2 I_2 - (d + a_1 + r_1) - (d + a_2 + r_2) - \frac{\sigma_1^2}{2}I_1^2 - \frac{\sigma_1^2}{2}S^2 - \frac{\sigma_2^2}{2}I_2^2 - \frac{\sigma_2^2}{2}S^2\right)dt \\
&+ \sigma_1\frac{S - I_1}{f_1(S,I_1)}dB_1(t) + \sigma_2\frac{S - I_2}{f_2(S,I_2)}dB_2(t).
\end{aligned}$$

Integrating the above inequality, we obtain

$$\begin{aligned}
U(S,I_1,I_2) \geq U(S_0,I_{10},I_{20}) &+ \int_0^t F(S(s),I_1(s),I_2(s))ds + \int_0^t \sigma_1\frac{S(s) - I_1(s)}{f_1(S(s),I_1(s))}dB_1(s) \\
&+ \int_0^t \sigma_2\frac{S(s) - I_2(s)}{f_2(S(s),I_2(s))}dB_2(s),
\end{aligned} \tag{4}$$

with $F(S,I_1,I_2) = -d - \beta_1 I_1 - \beta_2 I_2 - (d + a_1 + r_1) - (d + a_2 + r_2) - \frac{\sigma_1^2}{2}I_1^2 - \frac{\sigma_1^2}{2}S^2 - \frac{\sigma_2^2}{2}I_2^2 - \frac{\sigma_2^2}{2}S^2$.
Then,

$$\lim_{t\to\tau}\ln(S(t)I_1(t)I_2(t)) = -\infty.$$

Letting $t \to \tau$ in (4), we have

$$-\infty \geq U(S_0, I_{10}, I_{20}) + \int_0^\tau F(S(s), I_1(s), I_2(s))ds + \int_0^\tau \sigma_1 \frac{S(s) - I_1(s)}{f_1(S(s), I_1(s))} dB_1(s) + \int_0^\tau \sigma_2 \frac{S(s) - I_2(s)}{f_2(S(s), I_2(s))} dB_2(s) > -\infty.$$

Which is a contradiction. Thus, $\tau = \tau_e = +\infty$ a.s., which completes the proof.

3.2. Exponentially stability

The goal of this subsection is to establish a sufficient conditions for the exponentially stability of the disease-free equilibrium. For this, we consider

$$\Psi := \Psi(S, I_1, I_2) = (\frac{A}{d} - S) + I_1 + I_2, \quad V(S, I_1, I_2) = \ln(\Psi(S, I_1, I_2)).$$

Theorem 3.1. *$\Psi(S(t), I_1(t), I_2(t))$ converges exponentially to zero if the following condition holds*

$$\limsup_{t \to \infty} \mathscr{L}V(S(t), I_1(t), I_2(t)) < 0 \ \ a.s.$$

Proof. By Itô's formula, we have

$$dV(S(t), I_1(t), I_2(t)) = \mathscr{L}V(S(t), I_1(t), I_2(t))dt + 2\sigma_1 \frac{S(t)I_1(t)}{\Psi f_1(S(t), I_1(t))} dB_1(t) + 2\sigma_2 \frac{S(t)I_2(t)}{\Psi f_2(S(t), I_2(t))} dB_2(t).$$

Integrating both sides from 0 to t yields that

$$V(S(t), I_1(t), I_2(t)) = V(S_0, I_{10}, I_{20}) + \int_0^t \mathscr{L}V(S(s), I_1(s), I_2(s))ds + \int_0^t 2\sigma_1 \frac{S(s)I_1(s)}{\Psi f_1(S(s), I_1(s))} dB_1(s) + \int_0^t 2\sigma_2 \frac{S(s)I_2(s)}{\Psi f_2(S(s), I_2(s))} dB_2(s),$$

with $M_i(t) = \int_0^t 2\sigma_i \frac{S(s)I_i(s)}{\Psi f_i(S(s), I_i(s))} dB_i(s)$ is a continuous local martingale and $M(0) = 0$. Moreover, whose quadratic variation is

$$< M_i, M_i >_t = 4\sigma_i^2 \int_0^t \left(\frac{S(s)I_i(s)}{\Psi f_i(S(s), I_i(s))} \right)^2 ds \leq ct.$$

So, the strong law of large number for local martingales [23] implies that

$$\lim_{t\to\infty}\frac{1}{t}\int_0^t 2\sigma_i\frac{S(s)I_i(s)}{\Psi f_i(S(s),I_i(s))}dB_i(s)=0 \quad a.s.,$$

and

$$\lim_{t\to\infty}\frac{1}{t}V(S_0,I_{10},I_{20})=0.$$

It follows that

$$\limsup_{t\to\infty}\frac{1}{t}V(S(t),I(t)_1,I(t)_2)=\limsup_{t\to\infty}\frac{1}{t}\int_0^t \mathscr{L}V(S(s),I_1(s),I_2(s))ds \quad a.s.,$$

the proposition is proved.

In order to establish the conditions for the exponentially stability of the disease-free equilibrium of system (2), we need the following lemma [24].

Lemma 3.1. *For $k\in N$, let $X(t)=(X_1(t),X_2(t),..,X_k(t))$ be a bounded $\mathbb{R}^k$-valued function. Let $(t_{0,n})$ be any increasing unbounded sequence of positive real numbers. Then there is a family of sequences $(t_{s,n})$ such that for each $s\in\{1,2,\ldots,k\}$, $(t_{s-1,n})$ is a subsequence of $(t_{s,n})$ and the sequence $X_s(t_{s,n})$ converges to the largest limit point of the sequence $X_s(t_{s-1,n})$.*

Let

$$\mathscr{R}_1^*=\frac{2\sigma_2^2\left(\frac{A}{d\delta_2}\right)^2+\frac{\delta_1^2\beta_1}{2\sigma_1^2}}{d+a_1+r_1}, \quad \mathscr{R}_2^*=\frac{2\sigma_1^2\left(\frac{A}{d\delta_1}\right)^2+\frac{\delta_2^2\beta_2}{2\sigma_2^2}}{d+a_2+r_2}$$

with $\delta_i=1+\alpha_i\frac{A}{d}+\gamma_i\frac{A}{d}+\mu_i(\frac{A}{d})^2$, $i=1,2$.

Theorem 3.2. *Assume that $\mathscr{R}_i^*<1$, $i=1,2$ and*

$$d>2\left(\frac{A}{d\delta_2}\right)^2\sigma_2^2+2\left(\frac{A}{d\delta_1}\right)^2\sigma_1^2,$$

then the disease-free equilibrium of system (2) is almost surely exponentially stable.

Proof. By Itô's formula, we have

$$\begin{aligned}\mathscr{L}V((S,I_1,I_2)) &= \frac{1}{\Psi}\Bigg[-A+dS+\frac{2\beta_1 SI_1}{f_1(S,I_1)}+\frac{2\beta_2 SI_2}{f_2(S,I_2)}-(d+a_1+r_1)I_1 \\ &\quad -(d+a_2+r_2)I_2\Bigg]-4\sigma_1^2\left(\frac{SI_1}{\Psi f_1(S,I_1)}\right)^2-4\sigma_2^2\left(\frac{SI_2}{\Psi f_2(S,I_2)}\right)^2 \\ &\leq -d\frac{\left(\frac{A}{d}-S\right)}{\Psi}+\frac{2\beta_1 SI_1}{\Psi f_1(S,I_1)}-(d+a_1+r_1)\frac{I_1}{\Psi}+\frac{2\beta_2 SI_2}{\Psi f_2(S,I_2)} \\ &\quad -(d+a_2+r_2)\frac{I_2}{\Psi}-4\sigma_1^2\left(\frac{SI_1}{\Psi f_1(S,I_1)}\right)^2-4\sigma_2^2\left(\frac{SI_2}{\Psi f_2(S,I_2)}\right)^2. \qquad (5)\end{aligned}$$

For every sample path w of the process $B(t)$, there exists an unbounded increasing sequence t_n' of positive time values for which

$$\lim_{n\to\infty}\mathscr{L}V(S(t_n'),I_1(t_n'),I_2(t_n'))=\limsup_{t\to\infty}\mathscr{L}V(S(t),I_1(t),I_2(t)).$$

Then by Lemma 3.1, there exists a subsequence t_n for which the following limits exists

$$\lim_{n\to\infty}\frac{\left(\frac{A}{d}-S(t_n)\right)}{\Psi}=x,\quad \lim_{n\to\infty}\frac{I_i(t_n)}{\Psi}=y_i,\quad \lim_{n\to\infty}S(t_n)=s,$$

and as well the conditions $x+y_1+y_2=1$, $s\leq\frac{A}{d}$ holds. Let us write

$$\Theta=\lim_{n\to\infty}\mathscr{L}V(S(t_n),I_1(t_n),I_2(t_n)).$$

Then from (5), we obtain

$$\begin{aligned}\Theta &\leq -dx+2\beta_1 sy_1-(d+a_1+r_1)y_1+2\beta_2 sy_2-(d+a_2+r_2)y_2 \\ &\quad -4\sigma_2^2\left(\frac{sy_1}{\delta_1}\right)^2-4\sigma_2^2\left(\frac{sy_2}{\delta_2}\right)^2 \qquad (6)\end{aligned}$$

Therefore, by substituting the following inequality

$$\begin{aligned}\frac{-4\sigma_1^2 s^2 y_1^2}{\delta_1^2} &\leq \frac{-2\sigma_1^2 s^2 y_1^2}{\delta_1^2}\\ &= \frac{-2\sigma_1^2 s^2 y_1}{\delta_1^2}(1-x-y_2)\\ &= \frac{-2\sigma_1^2 s^2 y_1}{\delta_1^2} + \frac{2\sigma_1^2 s^2 y_1 x}{\delta_1^2} + \frac{2\sigma_1^2 s^2 y_1 y_2}{\delta_1^2}\\ &\leq \frac{-2\sigma_1^2 s^2}{\delta_1^2} y_1 + \frac{2\sigma_1^2\left(\frac{A}{d}\right)^2}{\delta_1^2} x + \frac{2\sigma_1^2\left(\frac{A}{d}\right)^2}{\delta_1^2} y_2,\end{aligned}$$

and

$$\frac{-4\sigma_2^2 (s y_2)^2}{\delta_2^2} \leq -\frac{2\sigma_2^2 s^2}{\delta_2^2} y_2 + \frac{2\sigma_2^2\left(\frac{A}{d}\right)^2}{\delta_2^2} x + \frac{2\sigma_2^2\left(\frac{A}{d}\right)^2}{\delta_2^2} y_1.$$

In 6, we get

$$\begin{aligned}\Theta \leq &-\left(d - 2\left(\frac{A}{d\delta_1}\right)^2 \sigma_1^2 - 2\left(\frac{A}{d\delta_2}\right)^2 \sigma_2^2\right) x - \left((d+a_1+r_1) - 2\sigma_2^2\left(\frac{A}{d\delta_2}\right)^2\right) y_1\\ &- \left((d+a_2+r_2) - 2\sigma_1^2\left(\frac{A}{d\delta_1}\right)^2\right) y_2 - 2\sigma_1^2 \frac{s^2}{\delta_1^2} y_1 + 2\beta_1 s y_1\\ &\frac{-2\sigma_2^2 s^2}{\delta_2^2} y_2 + 2\beta_2 s y_2.\end{aligned}$$

Since,

$$\frac{-2\sigma_i^2 s^2}{\delta_i^2} + 2\beta_i s \leq \delta_i^2 \frac{\beta_i}{2\sigma_i^2}, \;\; i = 1,2,$$

we deduce that

$$\begin{aligned}\Theta \leq &-\left(d - 2\sigma_1^2\left(\frac{A}{d\delta_1}\right)^2 - 2\sigma_2^2\left(\frac{A}{d\delta_2}\right)^2\right) x - \left((d+a_1+r_1) - 2\sigma_2^2\left(\frac{A}{d\delta_2}\right)^2 - \delta_1^2\frac{\beta_1}{2\sigma_1^2}\right) y_1\\ &- \left((d+a_2+r_2) - 2\sigma_1^2\left(\frac{A}{d\delta_1}\right)^2 - \delta_2^2\frac{\beta_2}{2\sigma_2^2}\right) y_2.\end{aligned}$$

The coefficient of x, y_1 and y_2 are nonnegative. Furthermore, x and y_i cannot all be zero since $x + y_1 + y_2 = 1$. Thus $\Theta < 0$. This completes the proof.

4. Numerical examples and simulations

In this section, we give some numerical simulations in order to illustrate our theoretical results.

Example 4.1. *As a numerical example, for*

$$A = 0.5, \quad d = 0.3, \quad \beta_1 = 0.9, \quad \beta_2 = 0.5, \quad a_1 = 0.2, \quad a_2 = 0.4, \quad r_1 = 0.3,$$
$$r_2 = 0.4, \quad \sigma_1 = 0, \quad \sigma_2 = 0, \quad \alpha_1 = 3, \quad \gamma_1 = 1.5, \quad \mu_1 = 2, \quad \alpha_2 = 3, \quad \gamma_2 = 1, \quad \mu_2 = 2.$$

By calculation, we have $R_{01} = 0.3125 < 1$ *and* $R_{02} = 0.1263 < 1$*, From Theorem 2.1, we deduce that the disease dies out, Figure (a) illustrates this result.*

Example 4.2. *In this example parameters and the initial conditions are the same as in Example 5.1. unless* $\sigma_1 = 0,013$*,* $\sigma_2 = 0,014$*, the conditions of Theorem 3.2 hold. Therefore, the disease-free equilibrium of system (2) is almost surely exponentially stable. Figure (b) illustrates this result.*

Example 4.3. $A = 1, \quad d = 0.1, \quad \beta_1 = 1.2, \quad \beta_1 = 1.5 \quad a_1 = 0.2, \quad a_2 = 0.4, \quad r_1 = 0.3, \quad r_2 = 0.3, \quad \alpha_1 = 1, \quad \gamma_1 = 1.5, \quad \mu_1 = 1, \quad \alpha_2 = 2, \gamma_2 = 1, \quad \mu_2 = 1, \quad \sigma_1 = 0.2, \quad \sigma_2 = 0.2.$
In this case, the conditions of Theorem 3.2 is not satisfied (see Figure (c)).

3. Conclusion

In this work, we have proposed and analyzed a new stochastic SIS with double epidemic hypothesis and specific functional response by introducing random perturbations of white noise type directly to β_i. Firstly, in absence of noise we have proved that the disease dies out if the basic reproduction numbers $R_{0i} \leq 1$ (Theorem 2.1). Next, we have proved the global existence and positivity of solution for our stochastic model (Theorem 3.1). In addition, we give a sufficient conditions for exponentially stable of our stochastic model (Theorem 3.2). What is more, we can also investigate the stochastic SIS epidemic model with two infectious diseases with Markov chain. We will investigate these cases in our future work.

Conflict of Interests

The authors declare that there is no conflict of interests.

REFERENCES

[1] A. Korobeinikov, Lyapunov functions and global stability for *SIR* and *SIRS* epidemiologcal models with non-linear transmission, Math. Biol. 30(4) (2006), 615-626, .

[2] X.B. Liu, L.J. Yang, Stability analysis of an *SEIQV* epidemic model with saturated incidence rate, Nonlinear Anal., Real World Appl. 13 (6) (2012), 2671-2679.

[3] L.H. Zhou, M. Fan, Dynamics of an *SIR* epidemic model with limited medical resources revisited, Nonlinear Anal., Real World Appl. 13 (2012), 312-324.

[4] J. Li and Z. Ma, Qualitative analyses of *SIS* epidemic model with vaccination and varying total population size, Math. Comput. Model. 35(11-12) (2002), 1235-1243.

[5] A. Gray, D. Greenhalgh, L. Hu, X.Mao, and J. Pan, A stochastic differential equation *SIS* epidemic model, SIAM J. Appl. Math. 71(3) (2011), 876-902.

[6] X. Meng, S. Zhao, T. Feng, and T. Zhang, Dynamics of a novel nonlinear stochastic *SIS* epidemic model with double epidemic hypothesis, J. Math. Anal. Appl. 433(1) (2016), 227-242.

[7] A. Miao, X. Wang, T. Zhang, W. Wang, and B. Sampath Aruna Pradeep, Dynamical analysis of a stochastic *SIS* epidemic model with nonlinear incidence rate and double epidemic hypothesis, Adv. Differ. Equ. (2017), 2017:226.

[8] X.Z. Meng, Stability of a novel stochastic epidemic model with double epidemic hypothesis, Appl. Math. Comput. 217(2) (2010), 506-515.

[9] K. Hattaf, N. Yousfi, A. Tridane, Stability analysis of a virus dynamics model with general incidence rate and two delays, Appl. Math. Comput. 221 (2013), 514-521.

[10] J.R. Beddington, Mutual interference between parasites or predators and its effect on searching efficiency, J. Animal Ecol. 44 (1975), 331-341.

[11] D.L. DeAngelis, R.A. Goldsten, R. Neill, A model for trophic interaction, Ecology, 56 (1975), 881–892.

[12] P. H. Crowley, E. K. Martin, Functional responses and interference within and between year classes of a dragonfly population, J. North Amer. Benth. Soc. 8 (1989), 211-221.

[13] F. Li, X. Meng, and Y. Cui, Nonlinear stochastic analysis for a stochastic *SIS* epidemic model, J. Nonlinear Sci. Appl. 10(09) (2017), 5116-5124.

[14] Q. Liu, Q. Chen, D. Jiang, The threshold of a stochastic delayed *SIR* epidemic model with temporary immunity, Physica A, 450 (2016), 115-125.

[15] Yanan Zhao, Daqing Jiang, Dynamics of stochastically perturbed *SIS* epidemic model with vaccination, Abstr. Appl. Anal. 2013 (2013), Article ID 517439, 12 pages.

[16] L. Benvenuti, L. Farina, Positive and compartmental systems, IEEE Trans. Autom. Control, 47 (2002), 370-373.

[17] W.M. Haddad, V. Chellaboina, and Q. Hui, Nonnegative and compartmental dynamical systems, Princeton, 2010.

[18] H. L. Smith, P. Waltman, The Theory of the Chemostat, Cambridge University Press, 1995.

[19] H.W. Hethcote, Themathematics of infectious diseases, SIAM Review, 42(4) (2000), 599-653.

[20] P. van den Driessche and J.Watmough, Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission, Math. Biosci. 180 (2002), 29-48.

[21] J. La Salle, S. Lefschetz, Stability by Liapunov's direct method with applications, Academic Press, New York, 1961.

[22] X. Mao, Stochastic differential equations and applications, 2nd Edition, Horwood, 2007.

[23] B. Oksendal, Stochastic differential equations: an introduction with applications, Springer, Heidelberg, 2000.

[24] P.J. Witbooi, Stability of a stochastic model of an *SIR* epidemic with vaccination, Acta Biotheor. 65 (2017), 151-165.

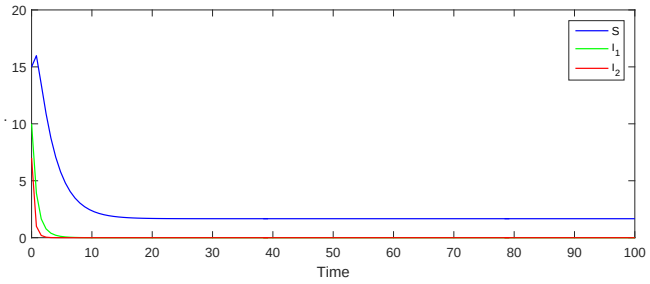

Figure (a)

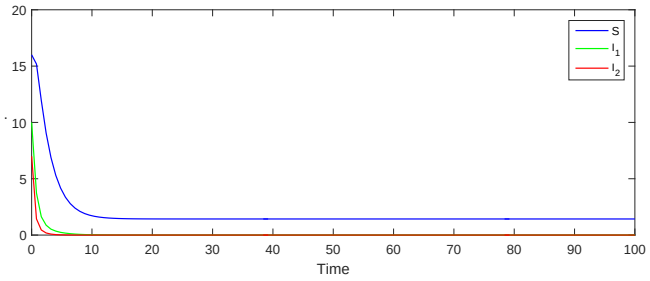

Figure (b)

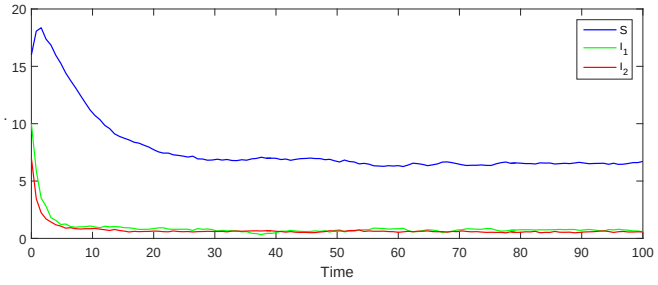

Figure (c)